Non-Linear Vibration
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Module - 4
Stability and Bifurcation Analysis
of Nonlinear Responses
Lecture - 3
Saddle-Node, Pitch Fork, Trans Critical
and Hopf bifurcation

Welcome to today class of non-linear vibration. So, last class we have discussed on these static and dynamic static bifurcations and there we have studied about the stability and bifurcation analysis of non-linear fixed point responses. Today, we will make more discussions on this saddle node, pitchfork, and trans critical, bifurcations and also on Hopf bifurcation, which is a dynamic bifurcation. So, the static bifurcations or saddle node pitch fork and trans critical and dynamic bifurcations is hopf bifurcation.

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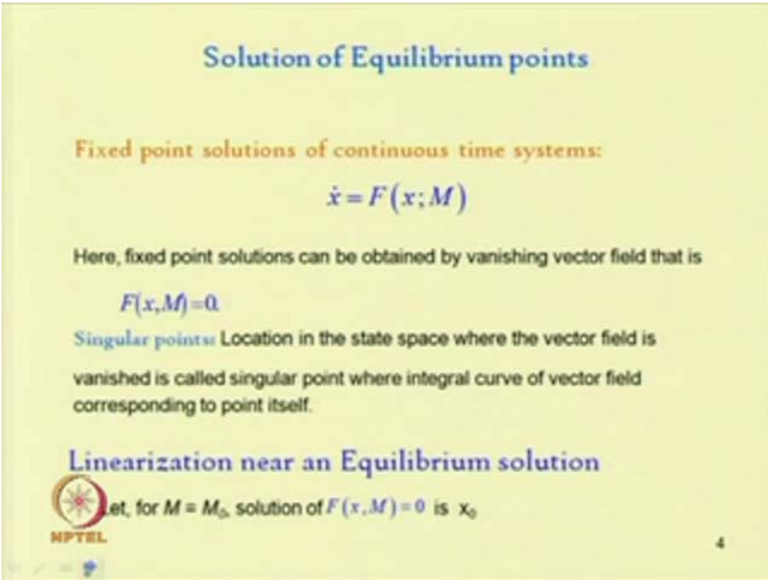

So, before discussing that thing, let us review what we have studied before. So, we studied about the solution of the equilibrium points. So, equilibrium points are the fixed points, for example in this equation x dot equal to F x M, where M is the control parameter and x is the state variables. So, to find the fixed point, we are first equating the terms with time derivatives equal to 0. That means, x dot equal to 0 and if it is x double also, if x double dot term, that is acceleration terms are also there, then we may quote that thing also equal to 0. So, in this first order equation, so we can write this x dot equal to 0, so it will reduce to F x M equal to 0, from which we will get this value of x, which will correspond to the equilibrium positions. So, whether these equilibrium positions are stable or unstable, so that thing we can solve by linearization, near the equilibrium position. So, or we can perturb the solution near this equilibrium position x equal to x 0 and we can find the Jacobian matrix and finding the Eigen value of the Jacobian matrix, we can tell whether the system is stable or unstable.

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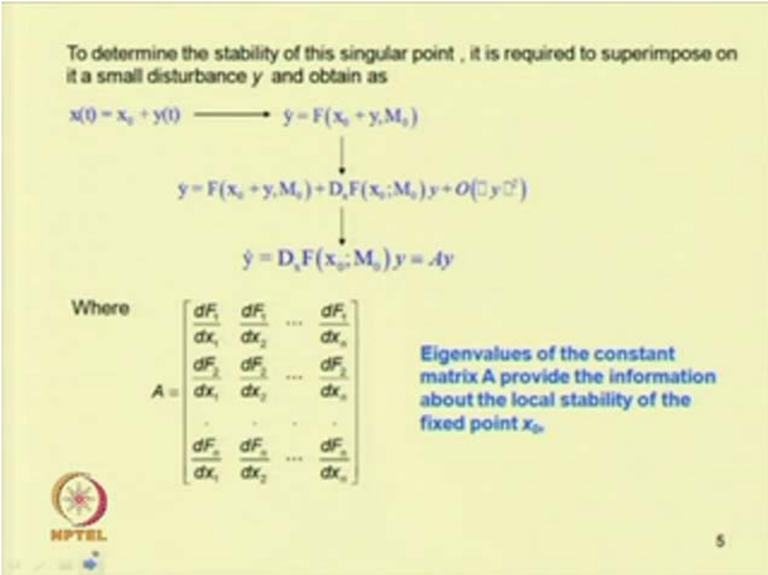

So, for that purpose, so we can perturb the solution. Let x 0 is the equilibrium solution, then we have given a small perturbation y t to the solution and we can write this x t equal to x 0 plus y t. Then, this equation, substituting this equation in the original equation x dot equal to F x M, so we can write this y dot equal to F x 0 plus y M 0. So, we can write this y dot equal to d x F x 0 M y or A y, where A is the Jacobian matrix of a system. So now, finding this, finding the Eigen value of this Jacobian matrix, so we can study the stability of the system. So, in this case, already we have discussed how to find the stability.

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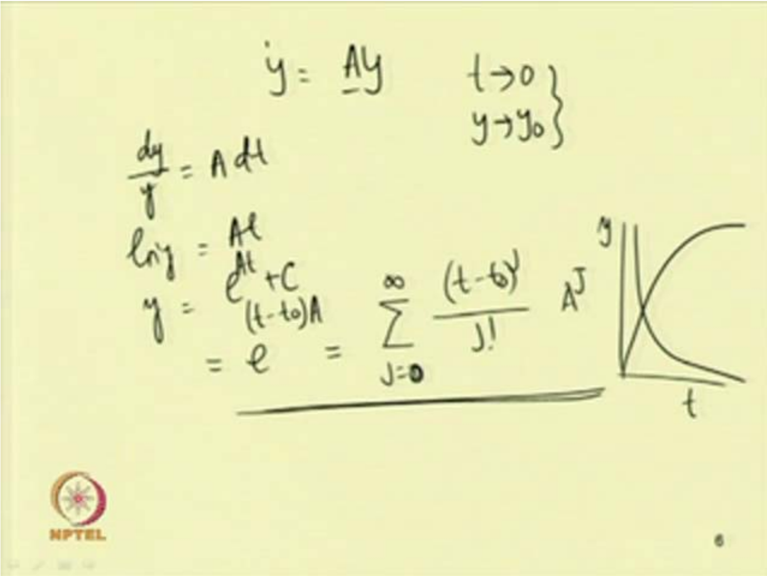

So now, by perturbing this thing, so as we have this equation y dot equal to A y, so as t tends to, so here as t tends to 0, so y will tends to y 0 and in this case, so we have this. We can write this d y by d t, so this is d y by d t equal to a or d y by y will be equal to a d t or if we integrate this thing, so we can have this l n y and l n y will be equal to A t or we can write this y equal to e to the power A t plus the constant or this solution, we can write it equal to e to the power, we can write this thing equal to e to the power t minus t 0 into a or this thing can be written as J equal to 0 to infinity t minus t 0 to the power J j factorial into a to the power J.

So, this is the solution of the, so one can get this solution from this equation. So, if the real part of this Eigen value is negative, then exponentially it will decrease and one can obtain, so if one plot y versus, so if one plot this y versus t curve, so it will exponentially decrease. So, if this real part becomes negative and if real parts is positive, so then the response will grow exponentially and one can get a unstable solution. So, to get stable solution, the real part of the solution should be, the real part of the Eigen value of A should be negative. So now, we obtain the solution y equal to e to the power t minus t 0 a, so which is equal to J equal to 0 to infinity t minus t 0 to the power J by J factorial into a to the power J. So, if the Eigen values lambda I of the matrix A are distinct, then there exist a matrix d.

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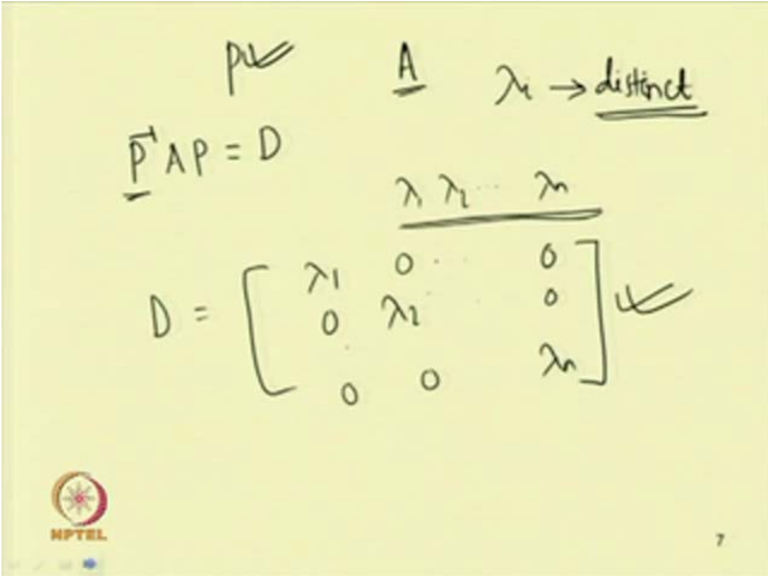

So, then there exist a matrix p, so that is known as the model matrix. This is equivalent to the model matrix. So, there exists a matrix p, such that this p transpose, p inverse or one can write this p inverse. You can take this p in such way that this p inverse A P, so this is a diagonal matrix, which is d. So, we can find a matrix p. We can select a matrix p, such that, so p inverse A P will be equal to d. So, this will happen if all the Eigen value of this matrix lambda I are distinct. So, if you have a distinct Eigen value, if all the Eigen values of A, if all the Eigen values of A which are lambda I are distinct, then we can have a matrix p, such that we can write this p inverse A P equal to d.

So, in this case, we can write let lambda 1 lambda 2 or lambda n are the distinct n distinct Eigen values. Then, we can write this d matrix equal to, so this is d 1 0 0, so 0 lambda 2. All 0's. Similarly, we can have 0 0 lambda n. So, we can obtain a matrix p, such that you can write this d equal p inverse A P equal to d, where d is this matrix.

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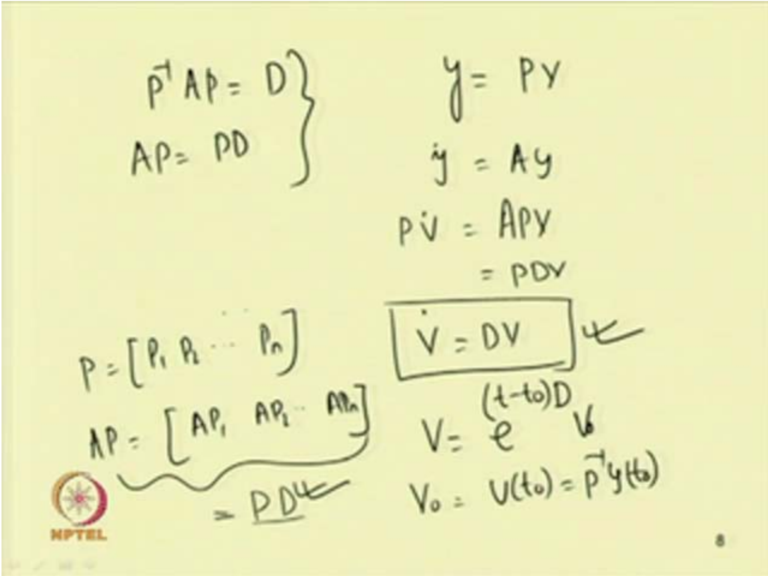

So, now we can write, so this y equal to, so we can write y equal to, we have this p, so we are taking p inverse A P equal to d or we can write, as we can take this y equal to, let us take y equal to p into v, then this y dot equal to A y will reduce to p v dot equal to A P v. Or, we can write this A P v, so here, I can write from this equation, I can write this A P will be equal to P D. So, as A P equal to P D, so I can write this p v equal to P D V or we can write this V dot equal to D V. So, we can write this V dot equal to D V and from this also we can get this distinct, as lambda are distinct, so we can write the solution in a distinct form.

So, by using this V dot equal to D V, so we can write the solutions and we can, so in this case, if the Eigen values are complex, then the matrix P will also be complex. The column of matrix P are the right Eigen vectors of p 1 p 2 p n. So, this p can be written as p 1 p 2 p n. So, we can write this p matrix equal to p 1 p 2 and p n. So, this A P can be written as, A into P can be written as A P 1 A P 2 A P n. So, we can write, so from this way we can get this A P and already we have shown that this A P will be equal to P D, p into d, so d is the diagonal matrix, which is in this form lambda 1 0. So, the diagonal terms contains all the Eigen values. So, in this way, we can write this d dot equal to this. So, as our d dot equal to d v, so the solution v will be equal to, similar to the previous case, so the solution d will be equal to e to the power t minus t 0 into d v 0. So, we can write, so where v 0 equal to v at t equal to t 0, so this can be written also equal to p inverse y t 0.

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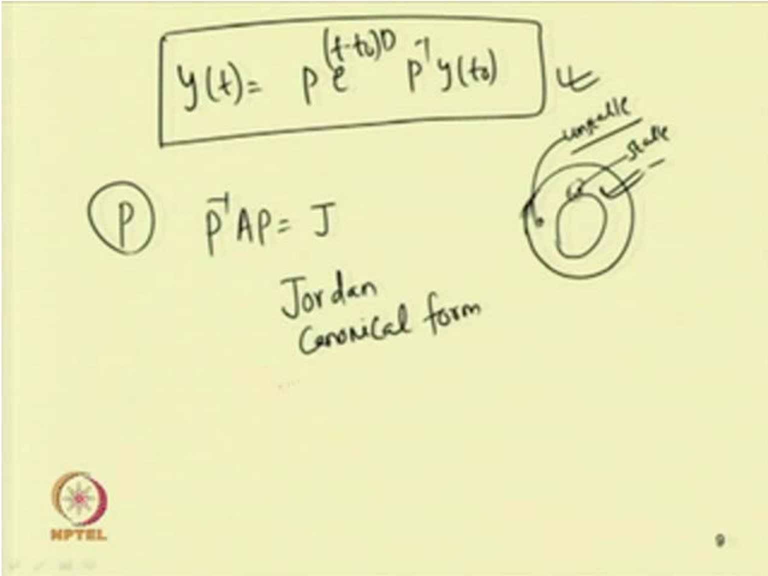

So, we can write substituting this in our original equation. So, you can write this y t equal to p e to the power t minus t 0 d into p inverse y t 0. So, the advantage of using this equation is that, d is a diagonal term. So, the matrix e to the power t minus t 0 d is a diagonal matrix with entries e to the power t minus t 0 lambda I. Hence, the Eigen values of A are also called the characteristic exponents associated with F. So, in this way, we can find the solution y t. So, y is the perturbation we have taken. So, if this perturbation grows, then the system should be unstable and according to our (( )) principle, so if it remains within a bounded region, then the solution will be stable.

So, let this is the equilibrium solution. So, if we are perturbing this thing and it is remaining within the circle or in this bounded region, then the systems become stable and if it grows, so let we have taken a point and it grows, so if it grows with time, then it becomes unstable. So, for the system to be stable, so it should remain within this region. So, if you perturb it or if you start the solution nearer to this point, always it should come to this solution point. If it takes infinite time to come to this solution, then it is asymptotically stable.

But, if it remains within this state, it is within the circle, and then it is stable. So, we have stable and unstable solutions. In this case, we have seen, when we have taken the distinct Eigen values, so instead of distinct Eigen values, if some of the Eigen values are not distinct, then there exist a matrix p, such that, so we can find a matrix p, such that this p inverse A P will be equal to J, which is known as Jordan Canonical form. So, this is Jordan Canonical form.

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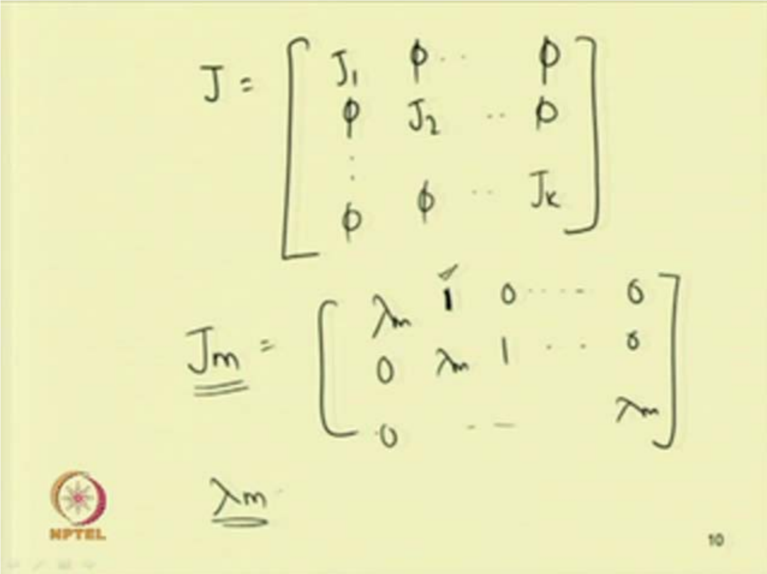

So, we can write the system using this Jordan Canonical form, where this J equal to J 1 phi. So, this is phi. So, let us assume there are k distinct Eigen values. So, we can write this phi and finally, it will be phi J k. So here, we are assuming that we have k distinct Eigen values. So, here this J M can be written as lambda M 1 0 0 0 lambda M 1 0. So, all are 0. So here, we have lambda M. So here, one can find this J M will contain this one above the diagonal terms. So you have, for example, in this second diagonal, so this is lambda M. So, above that thing, so this is lambda. So, above this, so we have a term that is 1. So, this J m matrix is not diagonal, as we have these entries 1 just above the diagonal terms. So here, we are assuming that we have k distinct Eigen values and let us also assume that the multiplicity of these mth Eigen value, that is J m, mth Eigen value lambda M, let the multiplicity of m th Eigen value lambda M be n M.

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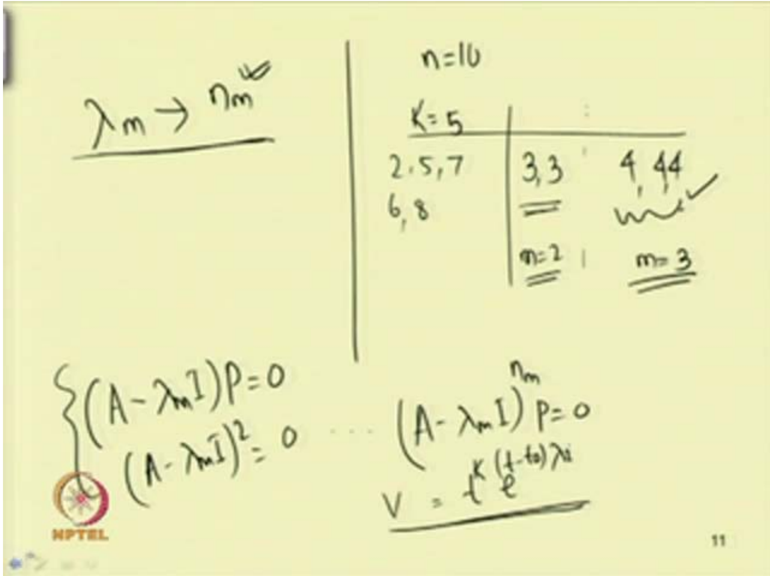

For example, let us take we have total 15 Eigen values. So, out of these 15 Eigen values, let us, we have 10. So, k will be equal to 10. So, let us have k 10 distinct Eigen values. So for example, distinct Eigen values will be 2 5 7 6 8. So, we can have, so let n equal to 10. Let us take example where we have total 10 number of Eigen values. Out of which 5 for, so 5 are distinct. So, 5 distinctive 2 5 6 2 5 7 6 8, so these are distinct Eigen values. So here, k equal to 5. Now, let us take, so the first, so let us take this is 3 and this is another Eigen value is 3 3. So, this is repeated, so here it is repeated twice and another Eigen value let us take 4 4 4. So here, it is repeated 3 times. So here, M equal to, so in this case M equal to 2 and in this case, M equal to 3. So, the multiplicity in this case equal to 3. That means, these Eigen value, that lambda equal to 4 lambda M equal to, so I can put m equal to, here lambda 1 equal to, so this is 3 and 3. So, this is repeated. Here also it is repeated. So, this Eigen value is repeated 2 times and this Eigen value is repeated 3 times.

So, in this case, let us assume that this lambda M Eigen value of this multiplicity n M. So, if lambda M Eigen value has multiplicity n M, for example, in this case, the 6th Eigen value has multiplicity 2 and the 7th Eigen value has the multiplicity 3. So, we have k distinct Eigen values and in addition to that, we will have some other Eigen values, which has repeated roots. So, in that case, let n M be the multiplicity of this lambda M. So, the matrix J M corresponding to the Eigen value lambda M, so differs from the distinct Eigen value because we have this entry 1 just above the diagonal term.

So, in this case, the column p I of the matrix p are called generalized Eigen vectors corresponding to the Eigen value lambda I of matrix a. So, there are n M. So, we will have n M generalized Eigen vectors corresponding to the Eigen values lambda M. So, these Eigen vectors are the non 0 solutions of, so they will be non 0 solution of a minus lambda M I p equal to 0. Then, a minus lambda M I square p equal to 0. Similarly, it will continue, a minus lambda M I n m p equal to 0.

So, by solving these equations, so we can get these values. So, the corresponding v will be, so in this case, the corresponding v we can get. So, it will be in this form t to the power k e to the power t minus t 0 lambda M. So, the corresponding value of v will be equal to t to the power k e to the power t minus t 0 into lambda I, where the integer k, so t to the power k what we have taken, so this integer depends on the multiplicity n i of the Eigen value lambda. So for example, let us take a simple example.

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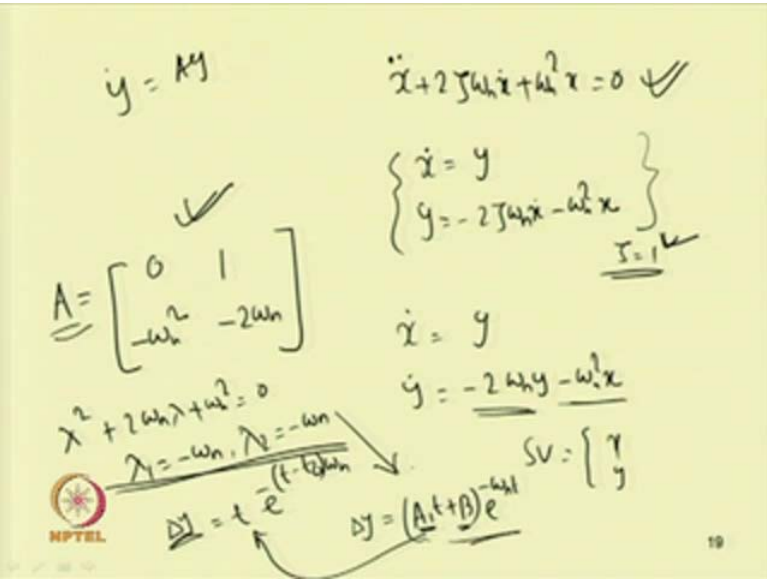

In case of a simple example, so let us take the simple single degree of freedom system. So, our equation x double dot plus 2 zeta omega n x dot plus omega n square x equal to 0. So, this equation I can write in this form, x dot equal to y and y dot equal to minus 2 zeta omega n x dot minus omega n square x. So, taking zeta equal to, let zeta equal to 1, so in this case we know, we will have repeated roots. So, our equation will reduce to x dot equal to y and y dot becomes minus 2 omega n y minus omega n square x.

So here, our state vector are x and y. So, these are the state vector. So, from this equation, so I can write this Jacobian matrix A. So, Jacobian matrix A will be equal to, so this equation can be, so you can differentiate this thing. As x term is not there, so first derivative, so this becomes 0 and this is 1 and in this case, so this becomes, this is minus omega n square x. So, differentiating with respect to x will give minus omega n square and differentiating this thing will give you minus 2 omega n. So, a becomes 0 1 minus omega n square minus 2 omega n. So, our auxiliary equation or Eigen value too, we can obtain the Eigen value by taking the determinant of a minus lambda I equal to 0.

So, by taking that, so what we got? So, you got this lambda square plus 2 omega n lambda plus omega n square equal to 0. So, from this we have seen that lambda 1 equal to minus omega n and lambda 1 and lambda 2 also equal to minus omega n. So, according to this, our solution becomes, so that is our y solution becomes, so the solution we can write or x we can write equal to t into e to the power minus t minus t 0 omega n. So, the formula what we have derived before, so this is, so using that formula or directly also, from this case we can write the solution will be, so the solution we can write, so the solution y will be equal to, so we can write the solution y equal to c 1 e to the power, so we can write in 2 different forms. So, we can write this thing also A 1 t plus b into e to the power minus omega n t.

Here we have 2 constants, A 1 and b or we can write that thing in this form also. So, this is our perturbation. So, if our perturbation, so this is x dot equal to y and y dot equal to x, so we can write this perturbation and that is our v. So, this v will be equal to, so in this form, so v will be equal to A 1 t plus b e to the power minus omega n t. Or, it can be written in this form, the solution, t e to the power of minus t minus t 0 omega n. So, this is perturbation. So, the perturbation delta y, so if I am writing this equation, this equation, whole equation as y dot equal to A y, then the perturbation will; this is the Jacobian matrix. So, from this Jacobian matrix, we can find the solution and from the solution, so this will be our delta y. If delta is the perturbation of y, then this becomes delta y. delta y or I can take it v also. delta y becomes this.

So, in this way one can find, so this is the perturbation. If this perturbation grows with time, then the system becomes unstable and if it remains within the limit, then it becomes stable. So, in this way one can find the stability of the system by perturbing the given differential equation. So here, we have taken a simple second order differential equation. Then, we have converted the second order differential to a set of first order differential equation and just by perturbing this thing, we can obtain the Jacobian matrix and this Jacobian matrix can be solved. That is, a minus lambda I. So, if we put a minus lambda I, determinant of a minus lambda I equal to 0, where lambda is the Eigen value of the system. Then, we can get the solution and if we have repeated roots, then the solution will be, so for that, the solution will be this and if this perturbation grows, then the system will be unstable. Otherwise, it will be stable.

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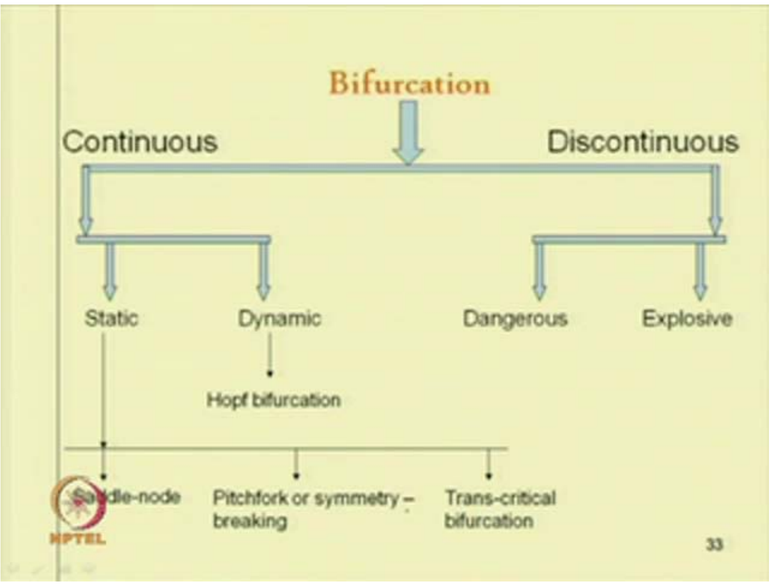

Again, last class we have discussed about different bifurcation points. So, these bifurcation points are static bifurcation point and dynamic bifurcation point. In case of static bifurcation point, we have seen the saddle node bifurcation point, pitch fork bifurcation point and trans critical bifurcation points.

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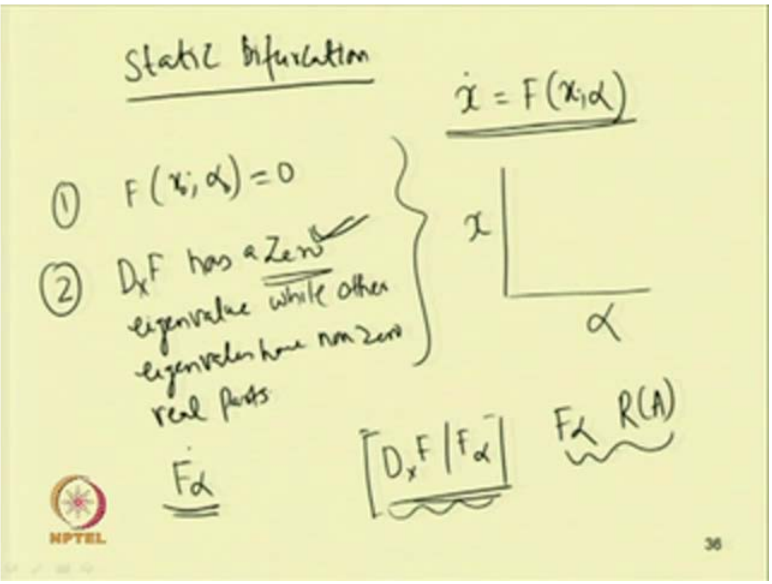

So, in case of saddle node bifurcation point, let us discuss about these static bifurcation. So, in case of static bifurcation, so we have a system. So, if we have the system x dot equal to F x alpha, so if this is our system, let us consider a first order system only, which we can extend easily for the higher order system. So, in this case, if our system is x dot equal to F x alpha, where alpha is the control parameter, then the bifurcation point, so to find the static bifurcation point first, so it should obey that this F, the first point will be F x alpha. So, x 0 alpha 0, so let alpha is the bifurcation point. So then, x 0 alpha 0 must be equal to 0. That means, substituting this x dot equal to 0, where x 0 is the equilibrium solution.

So, in this x alpha space, so if you plot this x alpha, so corresponding to this alpha 0, so if x 0 is the bifurcation point, then this F x 0 alpha 0 should be equal to 0. The second point should be this d x F, which is the Jacobian matrix. So, this d x F has a, so it should have a 0 Eigen value. So, it should have a 0 Eigen value, while at least 1 Eigen value should be 0. So, d x F has A 0 Eigen value and while all other Eigen values are non 0 real parts. So, at least one of the Eigen value should be or d x F has A 0 Eigen value corresponding to this alpha equal to alpha 0. So, these are the necessary conditions for the system to have a static bifurcation point. So, this to be the necessary condition, but these conditions may not be sufficient. So, to distinguish the saddle node bifurcation point from other bifurcation points, so we should find this F alpha. So, find F alpha. F alpha is the differentiation of F with respect to alpha. So, and then let us construct a matrix with d x F and F alpha. So, if we construct a matrix with d x F and F alpha and if at the saddle node bifurcation point one can show that, so this F, so we can find this F alpha. Now, so this, if this, so this will be, let us take this example, so then it will be clear.

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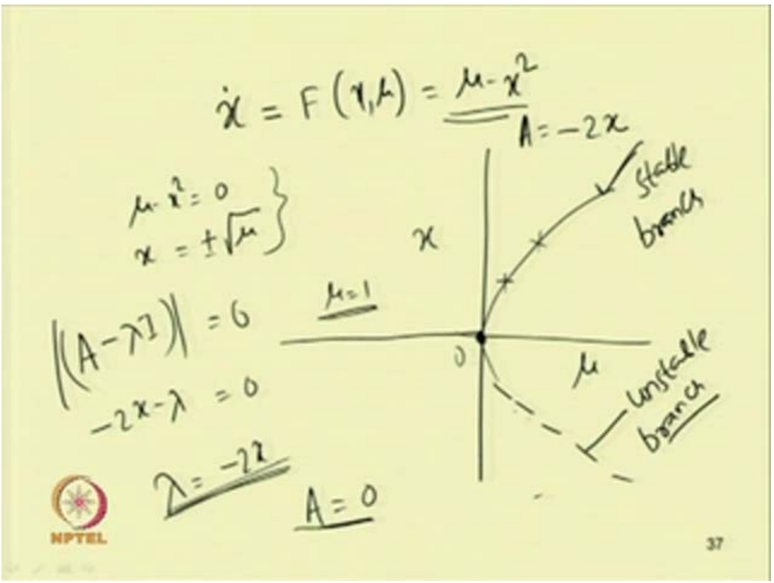

So, let us take this example x dot equal to F x mu, where this is equal to mu minus x square. So, in this case, we can write this. So, at equilibrium point, at equilibrium point x dot will be equal to 0. So, mu minus x square equal to 0 and in this case, we can find this x will be equal to plus minus root over mu. So, x will be equal to plus minus root over mu. So, we can plot this curve.

But, this already we have plotted several times this curve. So this is equal to, so x and mu if I will plot, so for mu value, negative value of mu, so x will be imaginary. So, it will have no solution for x equal to, for negative value of mu. For positive value of mu, so we have 2 solutions. So, in this case, we have 2 solutions and these 2 solutions, let us discuss about the static bifurcation point. Already we know the saddle node bifurcation point, pitch fork bifurcation point and trans critical bifurcation points or static bifurcation point. In static bifurcation point, it should satisfy 2 conditions. First condition, so for this first order system, x dot equal to f x alpha, where alpha is the control parameter. So, this f x 0 alpha 0 should be equal to 0. So here, x 0 is the solution corresponding to alpha equal to alpha 0 and x 0 is the equilibrium solution. So here, d x F has A 0 Eigen value, while other Eigen values has half non 0 real parts. So, to check whether this static bifurcation point, the saddle node bifurcation point or other different types of bifurcation point, we should construct a matrix with d x F and F alpha, where F alpha is the first derivative of F with respect to alpha and if this F alpha belongs to the range of a, so if this F alpha belongs to this range of this matrix, then, so if F alpha belongs to the range of matrix a, then we can tell that this is other type bifurcation. If F alpha does not belong to R A, then it will be a saddle node bifurcation point.

So for example, in this case, let us take the example of this x dot equal to F x mu equal to mu minus x square. Here, the solutions or the equilibrium points will be obtained by putting this x dot equal to 0. That means, mu minus x square equal to 0. So, x becomes plus minus root mu. So, at mu equal to 0, so this is corresponding to 0, so mu equal to 0. So, we have x equal to 0. Here a equal to, so by finding the first derivative, so we can find this a equal to minus 2 x. So, as a equal to minus 2 x, so a minus lambda I, I can write this equal to minus 2 x minus lambda equal to 0 or this. So, we can write this thing. So, lambda equal to minus 2 x. So, the Eigen value becomes minus 2 x. So, at mu equal to 0, so x equal to 0, so we will have a 0 Eigen value, as we have a 0 Eigen value, so it satisfy the second condition, that is d x F has a 0 Eigen value and also the first condition is satisfied at mu equal to 0. So, mu equal to 0 correspond to this static bifurcation points. So, at static bifurcation, so this point, that means this origin is a static bifurcation point. As it satisfy both the condition, that is F x 0 alpha 0 equal to 0 and d x F has a 0 Eigen value.

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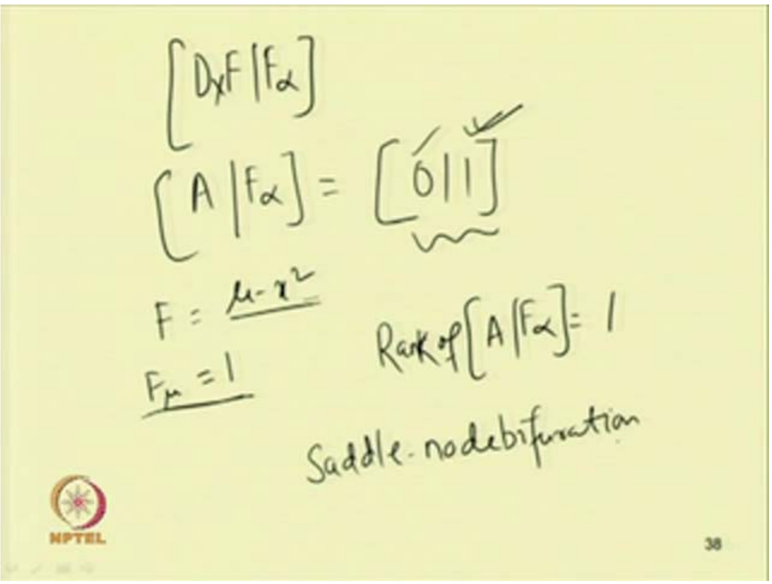

So, now as this is a, so let us find what is our, so now F alpha equal to, so our, so let us construct the matrix d x F. So, we have to construct the matrix d x F and F alpha. So, d x F is nothing but our a and F alpha, so here F alpha will be equal to our F equal to, so F equal to mu minus x square, so F alpha, so here alpha is nothing but our mu. So, F mu equal to, so we can find this F mu. So, F mu equal to 1. So, if you differentiate this with the respect to mu, so we have F mu equal to 1. But our a equal to, so already we have found this a. So, a equal to, a at this point at x equal to 0, our a becomes, so a equal to minus 2 x. So, a equal to, at x equal to 0, a becomes 0, so this matrix f d x alpha, so this matrix becomes, so 0 1. So, this matrix F a, F alpha matrix becomes 0 1, which has a rank. So, this has a rank of this matrix. So, this has the rank of 1. So, the origin x minus mu is a saddle point, saddle node point as F alpha, that is F alpha equal to 1 does not belongs to this a and a is only 0. So, as this does not belong to this, so this point is a saddle node bifurcation point. So, the saddle node bifurcation point also can be distinguished from other bifurcation point at this point.

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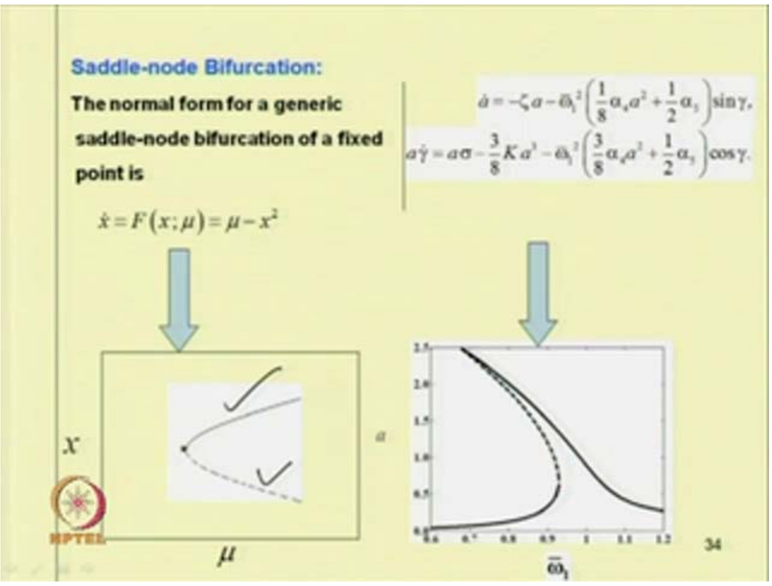

So the, at this bifurcation point, we can have same tangent from both the sides. So, by studying the stability, one can, we can see, so let us see about the stability of these two branches. So, for this branch, let us take a point on this branch, so that means this is positive. Let us take mu equal to positive. So, if mu is positive, then this lambda becomes minus as x equal to root over mu. So, let us take mu equal to 1. For example, mu equal to 1. So, then x becomes plus 1 and minus 1. So, for plus 1, x, when x is plus 1, so lambda becomes minus 2. So, lambda is corresponding to this point, lambda is minus 2 and corresponding to this point, lambda is minus. So, if it is 3, minus 3. So, Eigen values are negative.

So, as Eigen values are negative, so this branch is a stable branch and for this branch, so let us take a point here. So, let this correspond to minus 1. So, if mu equal to minus 1, so x becomes minus. So, if mu equal to, so let us take this. So, this corresponds to mu equal to 1, but x equal to minus 1. So, as x equal to minus 1, so our lambda becomes minus minus plus 2, so as lambda becomes positive, so on this branch, lambda value, so for corresponding to x equal to minus 3, this becomes plus 3 and this becomes plus 5. So, these points become negative. So, this point corresponds to negative Eigen value. This point corresponds to positive Eigen values. So, as this point correspond to the positive Eigen value, so I can plot dotted line here. Instead of solid line, I can plot a dotted line and this branch becomes unstable branch. So, in this case, we have seen, we have a stable branch and another unstable branch. So, they meet at the origin. So, there these 2 branches meet at the origin.

So, already we have plotted this curve here. So, this is the stable branch and this is the unstable branch and the stable and unstable branch meet at this point. So, this is the saddle node bifurcation point. So, the normal form or generic form of this saddle node bifurcation point for a first order equation is x dot equal to mu minus x square. Similarly, one can consider 2 equations also. So, we have a second order equation or we can deduce that thing to a set of 2 first order equations. So, if we have 2 first order equations, so by finding the Eigen values, so first one can find these Jacobian matrix a and then, one can find the Jacobian matrix corresponding to this point and one can observe that, corresponding to this point, it will, it should have a Eigen value equal to 0.

So, this point and this point are found to be, so by checking the Eigen value, one can see, so this branch is unstable and this is a stable branch. So, the stable and unstable branch meets at this point. So, one can see the slope at this point are same for both stable and the unstable branch. So, these two, so this point is a saddle node bifurcation point and in this case, by finding A, which is the Jacobian matrix, find the Eigen value and check at what point, by changing this control parameter, so this is the control parameter. So, by changing this control parameter, check at what point the Eigen value changes from positive to negative or negative to positive or it has a 0 Eigen value. So, by finding that point where it has A 0 Eigen value, we can tell that corresponds to a critical point or a bifurcation point. So, in this way, one can find the saddle node bifurcation point.

So, in case of saddle node bifurcation point, we have to construct the d x F and F alpha matrix and we have to check whether this F alpha belongs to the range of A. So, if does not belongs to this range of A, then we can tell this is a saddle node bifurcation. Otherwise, it may be a trans critical or either it may be a trans critical or it can be a pitch fork bifurcation point. So, let us see the generic form of the pitch fork bifurcation point.

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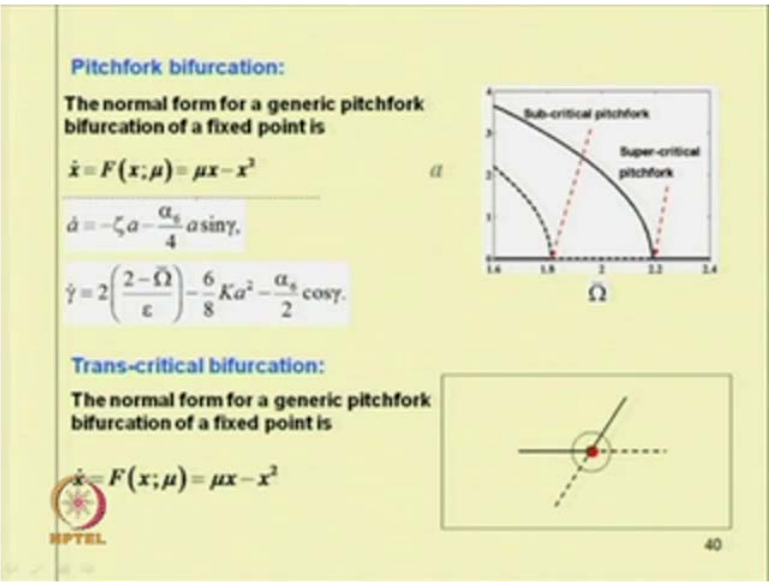

So, the generic form of the pitch fork bifurcation point can be written in this form. So, the generic form of pitch fork bifurcation point is mu x minus x square equal to x dot equal to this, so x dot equal to mu x minus x square. So, in this case, we have, so by putting this x dot equal to 0, so let us take this example, x dot equal to mu x minus x square.

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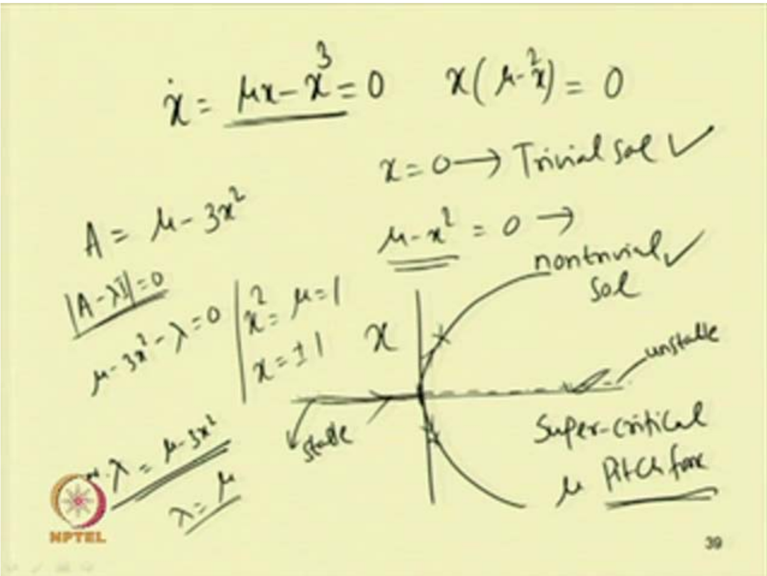

So, by putting this equal to 0, so we can write this x into mu minus x. For pitch fork bifurcation, so it will be mu x minus x cube. So, for pitch fork bifurcation, we have mu x minus x cube and trans critical bifurcation point, we have mu minus x square. So, in this case, so x into mu minus x square equal to 0. So here, x equal to 0 is the trivial solution. So, we have a trivial solution and mu minus x square equal to 0, so this is nontrivial solution. So, we have a set of trivial solution and a set of nontrivial solution. So, in this case, if one plot this, so one can say, so for example, corresponding to this, so we can have this and corresponding to this case, it will plot, so we have we have a 0 solution and then, we have a solution like this. So, this 0 solution, let us see this point. So, our a equal to, so in this case the Jacobian matrix, so we can find the Jacobian matrix A. So, A can be obtained by differentiating these with respect to x.

So, differentiating this with respect to x gives A equal to mu minus 3 x square. So, A minus lambda I, so we have to find A minus lambda I determinant of A minus lambda equal to 0. So, from this we can find, so mu minus 3 x square minus lambda equal to 0 or we have this lambda equal to mu minus 3 x square. So, corresponding to our x equal to 0, so I can plot this curve in x mu. So, let us plot this curve. So, this is x and this is mu. So, we have seen x equal to mu and we can, so this is 0 line. So, we can have the 0 line and we can have this other line also. So, in these cases, we can plot in this line. So, trivial and non trivial state, so this is the trivial state x equal to 0 and this curve becomes x equal to non 0, so for mu x minus x cube, now this is the solution.

So now, substituting x equal to 0, we can find lambda equal to mu, so x equal to 0 corresponding to lambda equal to mu, when mu is positive. So, if mu is positive, then lambda is positive, so the Eigen value becomes positive. So, this branch becomes unstable. So, this is, these 2, this branch becomes unstable. So, I can have this branch unstable and corresponding to mu negative, so lambda becomes negative, so this branch becomes stable. So, I have a stable trivial branch x corresponding to negative Eigen value. So, this is stable branch. So, at x equal to 0, so at x equal to or mu equal to 0, at mu equal to 0, lambda becomes 0. That means, one of the Eigen value becomes 0. So, this corresponds to a static bifurcation point. So, at this static bifurcation point, so we have this stable trivial branch and unstable trivial branch meet at this point.

So, along with this we have another two solutions, that is that correspond to x mu minus x square equal to 0. So, let us take mu equal to 1. So, if you take mu equal to 1, so we will have, so if mu equal to 1, so what will you have? So, we will have x square equal to mu or this is equal to 1. So, x will become plus minus 1. So, as x becomes plus minus 1, corresponding to mu equal to 1, so we have x equal to plus minus 1. So, our lambda, so let us consider the plus, so this point, so which correspond to x equal to 1.Then in this case, so lambda becomes, for mu we have 1, so minus 3 x square, so 3 into 1 square. So, this becomes, so 1 minus 3. So, 1 minus 3, this becomes negative. So, as it becomes negative, so we can have a, this is a stable branch. Similarly, this is also a stable branch.

So here, what we have observed, so from a stable trivial branch, we have other two stable non trivial branches and along with that, we have a unstable trivial branch. So, after this bifurcation point, the number of solution increases. So, before this bifurcation point, we have only a single solution. But after this bifurcation point, we have 3 solutions. So, 1 2, so this is 1, this is 2 and this is 3. So, out of these 3, so 2 branches are stable and this branch becomes unstable. So, if we increase this mu value, let we have increased this mu value from this side and we proceed. So then, we will have a solution along this line and then, either it may proceed this way or this way.

So, it will have a stable solution. So, as this branch is unstable or the trivial branch is unstable, actually the system will follow the stable branch. So, this is a super critical pitch fork bifurcation point. So, in case of super critical pitch fork bifurcation point, so from a stable branch, we have another stable branch and unstable trivial branch. So, this is known as this forward bifurcation point. So, we can have a backward bifurcation point also. So, we can have a backward bifurcation point corresponding to this.

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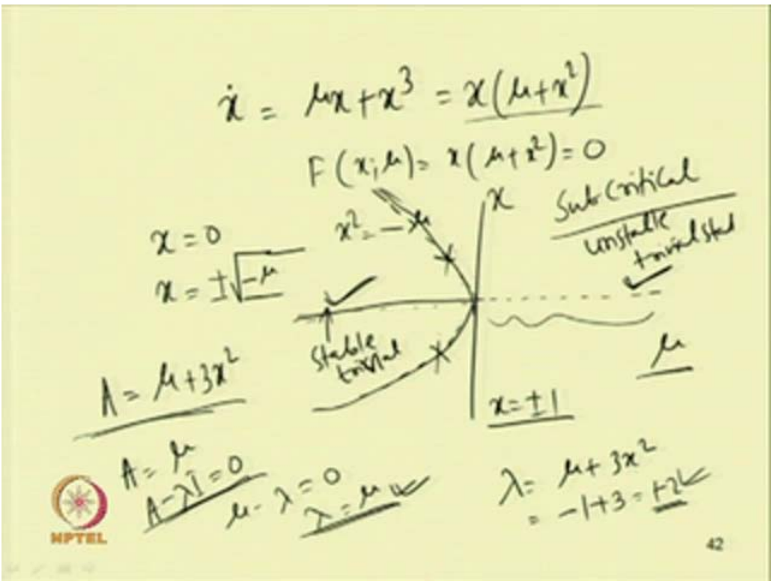

So for example, if we take, so if we take this x dot equal to mu x plus x cube, so let us take this example mu x plus x cube, so in this case, we will have this, taking this x common, so this is mu plus x square. So, our F x mu equal to x into mu plus x square, so this equal to 0. So x, the solution becomes x equal to 0 1 solution and the other solution becomes x square equal to minus mu or x equal to plus minus root over mu. No, so plus minus, so the solution becomes plus minus. So, we have x square equal to minus mu. So, this is minus mu. So, if we want plot the solution in this x mu, so this is x and this is mu. So, if we want to plot this curve, so in this case, we have, so for the negative mu, so negative value of mu, so this term becomes positive. So, we will have 2 positive solutions and also this is the trivial solution. So, we will have, so the solution becomes this.

So, we have a trivial branch and these two are nontrivial branch. So, in this case our A matrix, that is our Jacobian matrix becomes, so if you differentiate this thing, so x mu x plus x cube, so differentiating this thing, we have this is equal to mu plus 3 x square. So, for this trivial branch, we will have a equal to mu. So, as a equal to mu is for trivial branch, so when mu is negative, so this, so our, from a minus lambda I equal to 0, so we know this mu minus lambda equal to 0. So, our lambda equal to mu. So, as lambda equal to mu is the Eigen value, so for negative value of mu, that is from these two, this position, so we have a stable branch. So, this is stable and after the solution, after this branch, so that means, we can use this thing and up to this, we have a stable branch and we can see, so for positive value of mu, so we have a positive Eigen value. As positive Eigen value is unstable, so we have a unstable solution. So, this side is unstable solution, unstable trivial, so this is unstable trivial state and this is stable trivial state. So, this is stable trivial and this is unstable trivial.

Now, correspond to any point, let us take x equal to plus 1 or minus 1. So, corresponding to this, our a becomes, so for x equal, let us take x equal to plus minus 1, so these 2 points. So, in this case a or lambda will, so a minus lambda I equal to 0, so this lambda becomes, so lambda equal to mu plus 3 x square. So, as mu, so let us take this mu equal to minus 1. So, corresponding to mu equal to minus 1, we have x equal to plus minus 1. So, this 3 x square, this becomes 3 into 1. So, this becomes 3 and this becomes plus 2. Now, this becomes plus 2. So, 3 x square let us check it again. So, lambda equal to mu plus 3 x square, As mu I have taken minus 1, so corresponding to mu equal to minus 1, so x becomes root over plus minus 1. So, 3 x square becomes 3. So, then this becomes plus 2. So, as this is positive, so this branch is unstable. So, we can have a unstable branch. So, initially this is unstable. So, at this bifurcation point, what we have observed, we have a stable trivial state along with 2 unstable non trivial states. After the bifurcation, so this branch becomes unstable. So, this type of bifurcation is, as there is no solution after this, so there will be a catastrophic failure of the system and this type of solution are known as sub critical pitch fork bifurcation point. So, this becomes sub critical pitch fork bifurcation.

So, this type of bifurcation is what we have studied here. So, these are pitch fork bifurcation. So, this is one pitch fork bifurcation and this is one pitch fork bifurcation. So, we have sub critical pitch fork bifurcation and super critical pitch fork bifurcation. So, at this point, one see, so from a stable branch, we have a stable branch, so then this is super critical pitch fork and here from a stable branch, we have one stable branch. So, this is the sub critical pitch fork bifurcation. So, today class we have seen this. We have discussed more about this saddle node bifurcation point and pitch fork bifurcation point. So, next class we will study about the hopf bifurcation point, which is a periodic bifurcation point and is generated due to this dynamic bifurcation and along with that, we will study the solution of periodic response.

Thank you.