Non-Linear Vibration
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Module - 4
Stability and Bifurcation Analysis
of Nonlinear Responses
Lecture - 2
Stability and Bifurcation Analysis of
Nonlinear Fixed Point Responses

Welcome to today's class of linear vibration. So, in today's class we are going to discuss about the stability and bifurcation analysis of non-linear fixed point responses.

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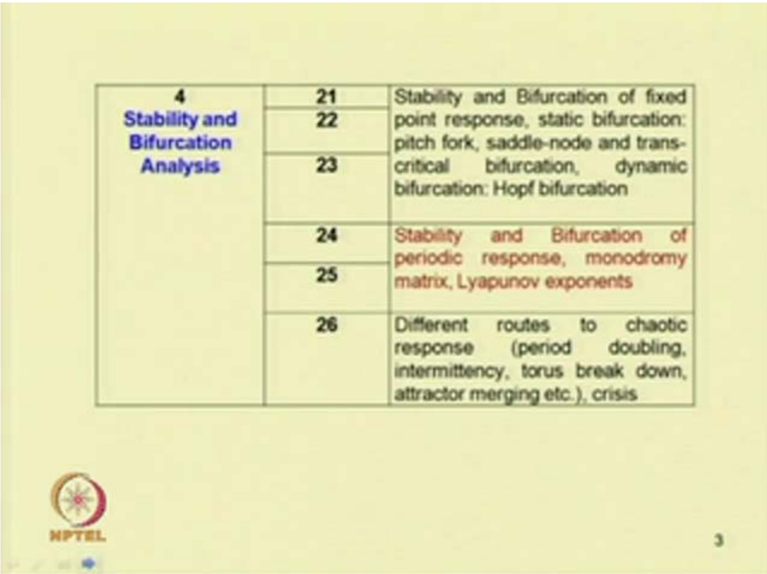

Particularly, we will discuss about the equilibrium points and their stability and about the static bifurcation points. So, in static bifurcation points we will discuss about these saddle-node bifurcation, pitch fork and trans-critical bifurcation. Next class will study about these Hopf bifurcation points.

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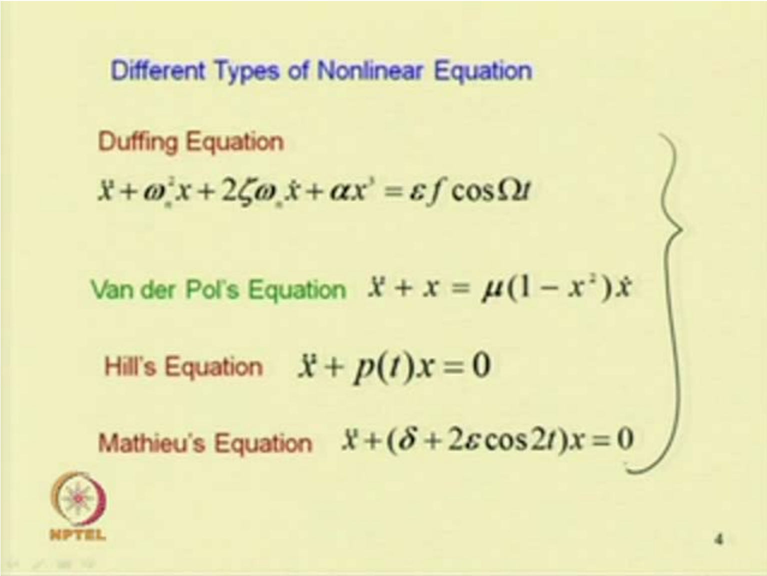

So, in the previous class we have discussed about several non-linear equations and in those equations how to find the equilibrium points. So, equilibrium points are points for which there is no or these will be these steady state solutions of the systems which will not vary with time. So, to obtain these equilibrium points we will not consider the terms containing the time so, in different equations we can find the equilibrium points and we have to study their stability. So, in case of to find the equilibrium points in different non-linear equations so, previously we have studied different types of methods, different perturbation methods or directly we can use this harmonic balance method or other different types of method solution methods to find the equilibrium points.

So, after finding the equilibrium points we are interested to find so, what should be its stability whether, the obtained solutions is stable unstable or what will happen to the system as time progresses.

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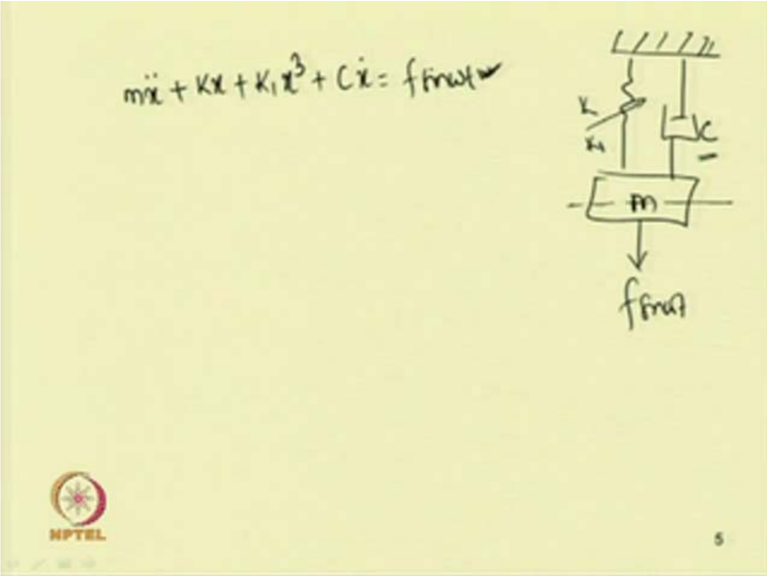

So, in case of a stable system for example, if you consider the simple spring mass system which can be represent by a Duffing equation also so, if we are considering the non-linear spring so, it can be represented by this Duffing equation so, this spring we have considered is non linear. So, this equation can be written in this form that is m x double dot plus k x plus let me take a non-linear spring so, in that case it will be k 1 x cube plus c x dot equal to if a force f sin omega t is acting on the system then, it will be f sin omega t. So, this is k, this is c damping and this is linear stiffness k and non-linear stiffness k 1. So, in this equation so, first we have to solve the equation to obtain the equilibrium position.

So, for example, the static equilibrium position so, we for which x equal to 0 so, will be the equilibrium position. So, by using several methods we can find what will be the response of the system one may also, solve this equation using numerical methods to find the solutions. So, after getting different types of solution so, we will find the stability of these solutions.

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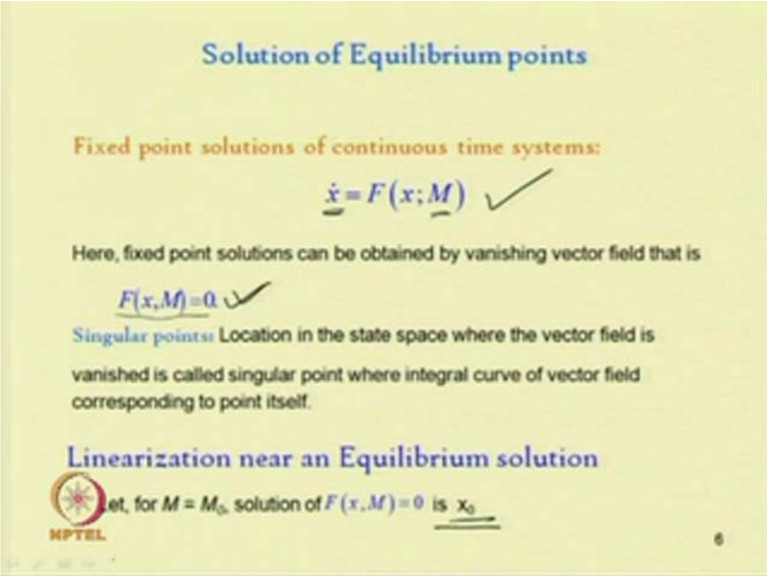

So, to obtain the stability for example, let us take a one dimensional equation so, in this one dimensional equation can be written in this form that is x dot equal to F x M where, M is the control parameter and F is the so, to find the equilibrium position the time derivative term that is x dot we have to put this x dot equal to 0 so, it will reduce to F x M equal to 0. So, it will solve this equation for different value of x so, that will give us the equilibrium position. So, singular points or equilibrium positions are the location in the state space where, the vector field is vanished and we can find that thing by equating this F x M equal to 0. So, we may have to solve a set of algebraic equation or trans-critical equation or different types of equations to find this equilibrium position.

So, to study the stability near the equilibrium position one may due to this linearization. So, let for the control parameter M equal to M 0 the solution becomes x equal to x 0.

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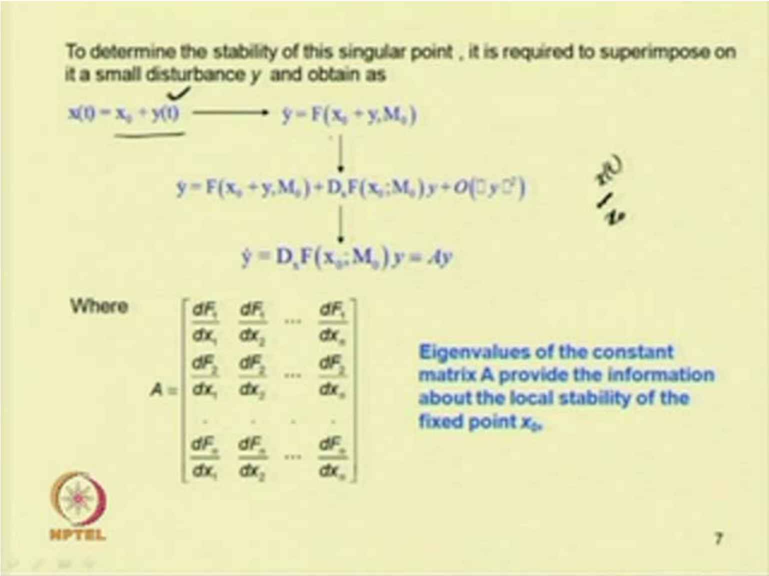

So, if x equal to x 0 is the solution for this then, one can write by whatever being the solution that means so, let this is the equilibrium position 0. So, now what will happen next to x 0 so, if we perturb this thing then, we can write this at x t that means a time which is slightly aware from this x 0. So, we can write this x t equal to x 0 plus y t. So, y t is the small perturbation you could take.

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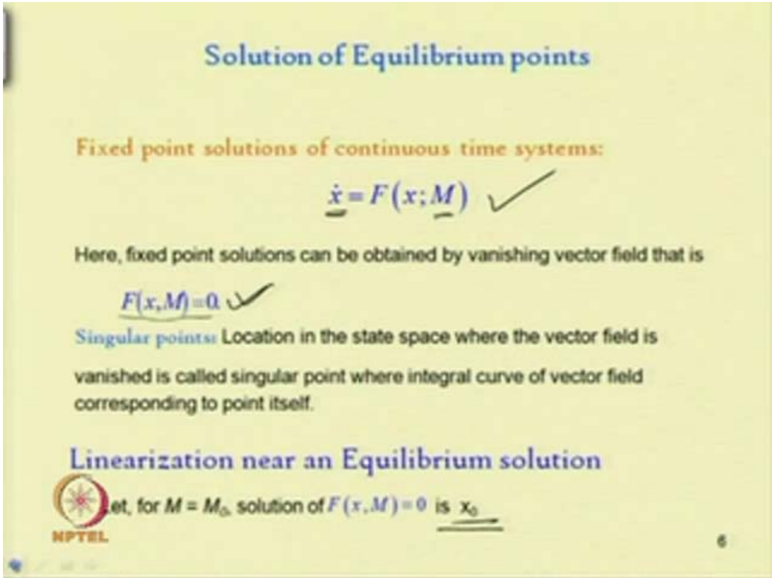

Then, we can write this equation previous equation this x dot equal to F x M by this equation so, y dot equal to F x 0 plus y M.

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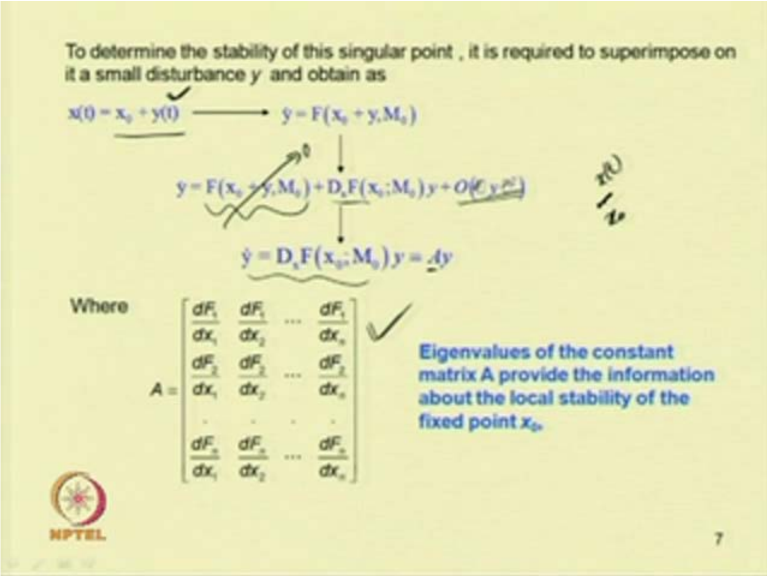

Now, expanding this thing using Taylor series one can write this y dot will be equal to F x 0 plus y M 0 plus d x F. So, that is first derivative of this F at x 0 M 0 y and one can neglect these higher order terms so, higher order neglecting the higher order terms one can write this equation i dot equal to as already this F x 0 y M 0 equal to 0. So, according to our assumptions here, F x 0 M equal to 0 which where we are getting x equal to x 0 so, this part is 0 so, one can obtain this y dot equal to y dot equal to d x F, x 0 M. So, this matrix that is d x F or this a is known as the Jacobian matrix. So, by finding the Eigen value of this Jacobian matrix so, we can study the stability of the system. So, we after finding these Eigen values so, if all the real parts of this Eigen value are negative then, the system is stable otherwise the system is unstable.

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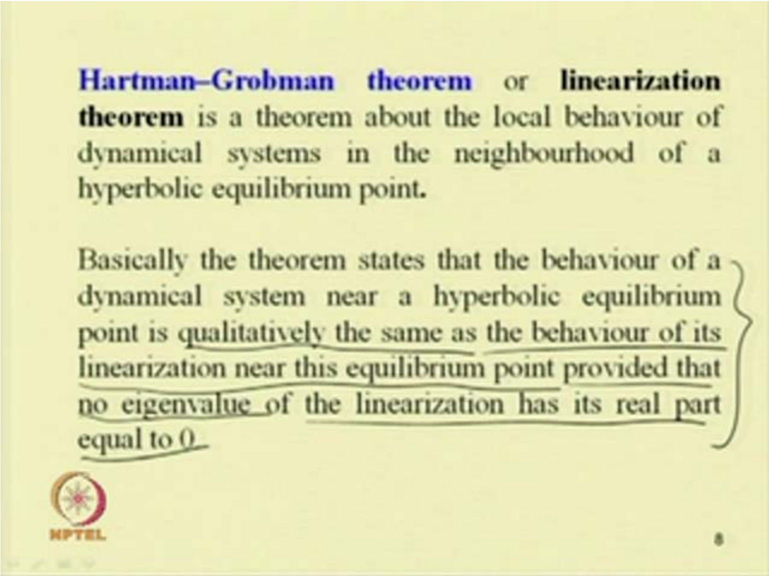

So, let us take one example so, in case of the so, before that let us see whether this linearization technique so, what we are finding so, whether this is valid for all the cases. And so, there is one theorem that is by a Hartman and Grobman theorem or linearization theorem so, this tells about this theorem is about the local behavior of the dynamical system in the neighborhood of a hyperbolic equilibrium point.

Basically, the theorem states that the behavior of a dynamical system near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization near its equilibrium point provided, that no Eigen value of the linearization has its real part equal to 0. So, if we have a hyperbolic fixed point then, according to this theorem so, this linearization linearized system behavior and the original system behavior will be similar. So, if we can linearize near the equilibrium position and study its stability so, this will give the local stability of the system.

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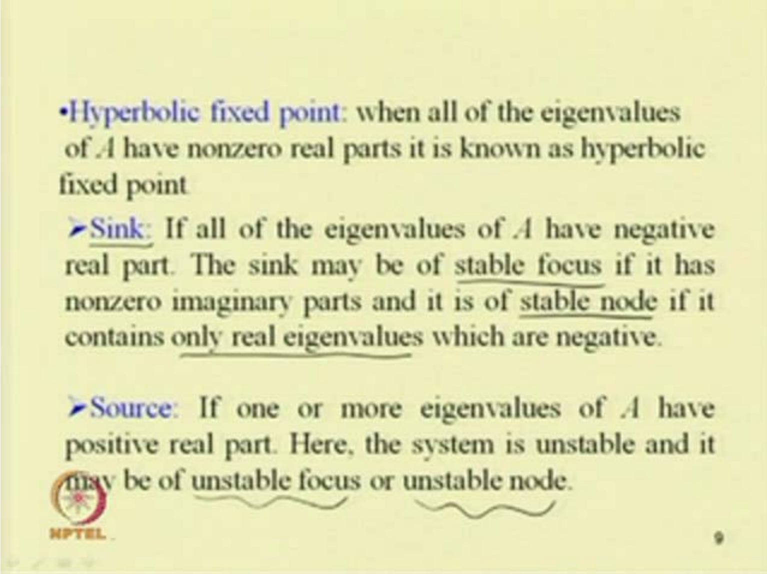

So, for finding the global stability of a system one may use this Lyapunov method. So, here some definitions also we one can see. What is hyperbolic fixed point? When all the Eigen values of the Jacobian matrix are non 0 real parts it is known as hyperbolic fixed point. So, if all the Eigen values of a have negative real part then it is known as sink. The sink may be stable focus if it has non 0 imaginary parts and it is a stable node so, either it can be stable focus or it can be stable node.

So, in case of stable focus so, it has non 0 imaginary parts and in case stable node it contains only real Eigen it contains only real Eigen values that means the imaginary parts are 0. So, in the Eigen value so, in case of a sink all the Eigen values have negative real parts and if it contains non 0 imaginary part then, it is stable focus and if it contains only real Eigen values that means the imaginary part is 0 then, it is known as stable node. So, the sink is stable so and source if one or more Eigen values of a have positive real part then, it is known as source. So, here the system is unstable and it may be unstable focus or unstable node. So, in case of unstable focus so, it will have non 0 imaginary parts and in case of unstable node it will it will have 0 imaginary parts. That means, all the Eigen values will be real and some of the Eigen values will contain positive real parts. So, in that case it will be a source or the system will be unstable.

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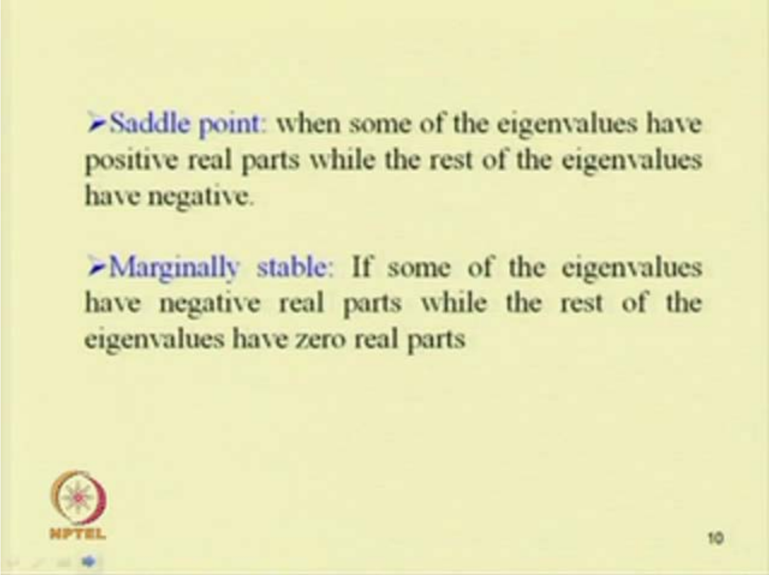

Similarly, saddle point when some of the Eigen values have positive real parts while the rest of the Eigen values have negative parts then, it will be a saddle point.

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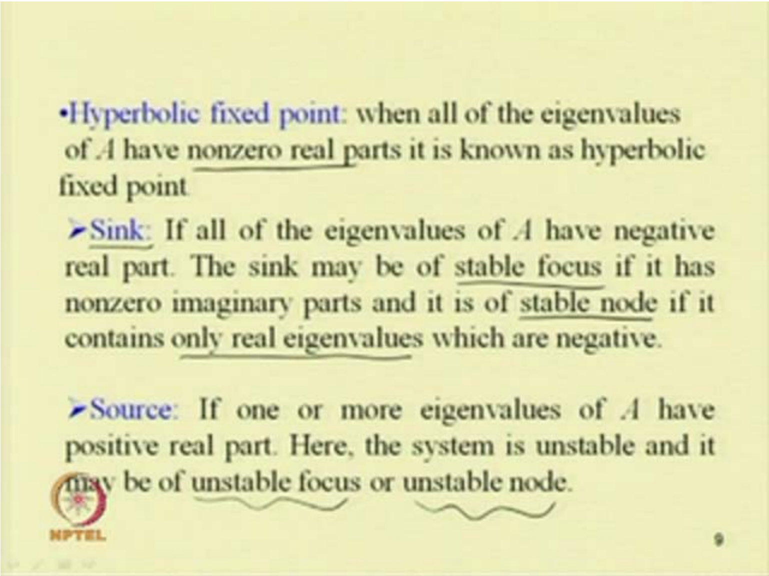

So, in case of the source if one or more Eigen values have positive real part this is source and in case of saddle point that is some of the Eigen values have positive real parts while the rest of the Eigen values have negative real parts. So, the systems can be called marginally stable if some of the Eigen values have negative real parts while the rest of the Eigen values have 0 real parts. So, these are different terminology for the equilibrium points so, we have this hyperbolic fixed point when all the Eigen values of a has non 0 real parts then, it is hyperbolic so, if it has 0 real part if some of the Eigen values have 0 real part then, it is called non hyperbolic fixed point. So, in case of the hyperbolic fixed point we have seen source, saddle point and based on these things we can study the equilibrium system.

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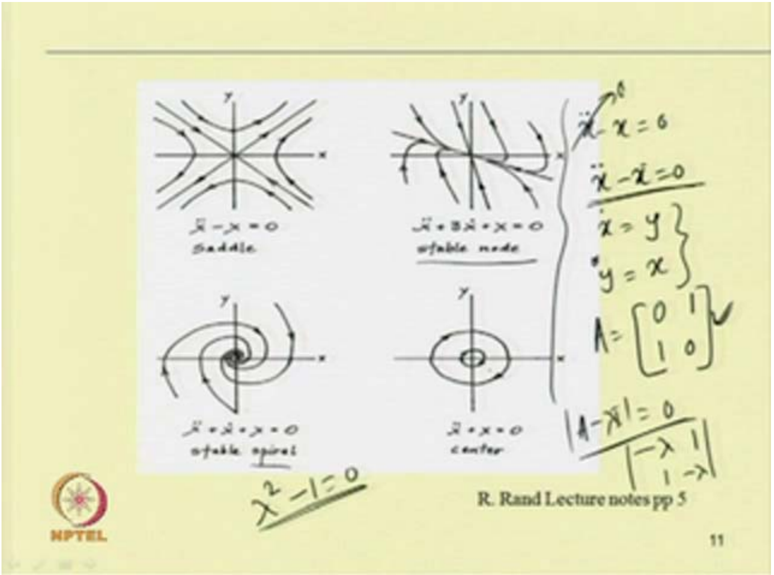

For example, already we have seen in this example x double dot minus x equal to 0 so, if we have x double dot minus x equal to 0. So, by putting this part equal to 0 which varies with time so, the equilibrium position is x equal to 0 that is, the origin so x equal to 0 is the equilibrium position.

So, in this equilibrium position one can see that this point is a saddle point. So, later we will see how we have checked its Eigen value to find whether it is a saddle point or sink or source. For example, this equation so, this x double dot minus x equal to 0 can be written by using 2 equations so, this is second order differential equation that thing can be written as a first order differential equation. So, let me put the first equation x dot equal to y so, the second equation will be this second order differential equation can be written by using these 2 so, x dot equal to y and I can write this x double dot that is equal to y dot so, y dot will be equal to so y dot equal to this x so, x double dot equal to x so, y dot equal to x. So, these are the 2 equations these are the 2 first order equation one can solve to check whether the point is a saddle node or whether the point is a saddle node point or not. So, in this case x dot equal to y and y dot equal to x so, one can write this A matrix as so, A matrix can be written so, for the first equation x dot equal to y so, this is x equal to 0 and x there is no x term that is 0 so, this is 1 and in case of the second equation one can write this is equal to 1 0.

So, this is the equation. As this is the linear equation, one can find its delta x dot delta x dot will be equal to delta y and delta y dot equal to delta x. So, A can be written in this way or a can be written by del f by del x del f by del x equal to 0 and del f by del y equal to 1. Similarly, in the second equation del f by del x equal to 1 and as there is no y term so, this is equal to this. So, from this a minus lambda i so, a minus lambda i determine of A minus lambda i equal to 0 where lambda i is the Eigen value so, from this one can obtain this lambda so, this is lambda A minus lambda so, minus lambda so, this equation so, from this equation one can write so, this is minus lambda 1 so, then this is 1 minus lambda. So, A minus lambda is 0 determinant of this thing so, from this one can get this lambda square so, lambda square minus 1 equal to 0 so, if lambda square minus 1 equal to 0 so, 1 so, from this one can find the roots so, from these roots so lambda square minus 1 equal to 0.

So, by finding the roots one can study whether the point is stable, marginally stable or it is a saddle point or node point. So, in this case also one can similarly, proceeding in this way write writing the set of first order equation so, one can check whether the system is a stable node whether, the system is a so, this spiral means this so, this spiral out or in so, this is a stable spiral as all the points are spiral in inside this is a stable point similarly, this is a center point.

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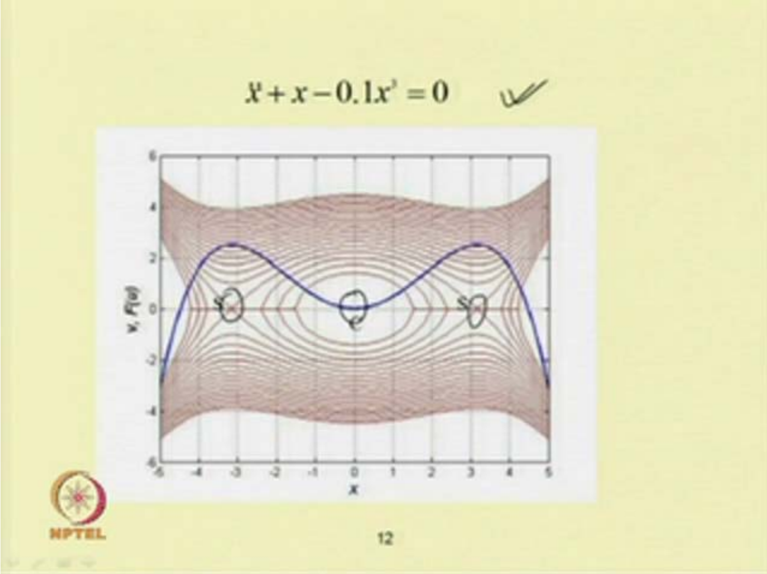

So, in this case also in this example x double dot plus x minus point 1 x cube so, which is similar to that of a duffing equation equal to 0. So, one can see this point is a saddle point this point correspond to the center point and this point correspond to the saddle point corresponding to this maximum the saddle point corresponding to the maximum potential energy and center correspond to the minimum potential energy.

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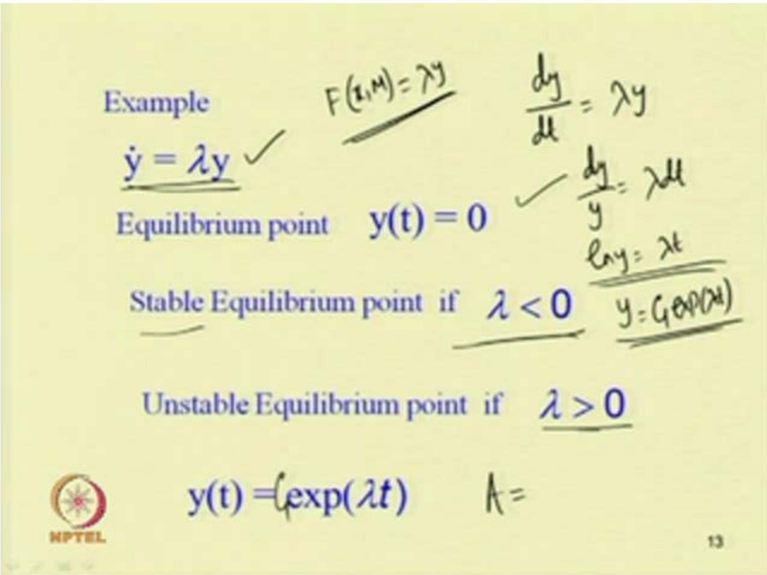

So, as another further one example one can take this y dot simple example y dot equal to lambda y so, the equilibrium position y t will be equal to so, in this case y dot we will put this y dot equal to 0. So, y t can be written equal to 0 the stable equilibrium position. So, for stable equilibrium position we can write so, our equation reduces to so, as y dot equal to lambda y so, we can find this A minus lambda i so, we can write A so, if lambda less than minus lambda less than 0 so, the point will be stable and if lambda greater than 0 so, it will show the system unstable. For example, in this case the solution is y t equal to as d y by d t equal to lambda y so, y dot equal to this or d y by d t equal to lambda y or one can find this d y by y equal to lambda d t. So, by integrating one can find this l n y equal to lambda t so, from this one can write y equal to some constant some constant let C 1 or C so, e to the power lambda t. So, this is the solution. So, in this case if lambda is positive then, exponentially it will increase the responsible will exponentially increase and it will becomes unstable and if lambda is negative exponentially it will decrease to make the system stable. So, depending on the value of lambda one can find this thing.

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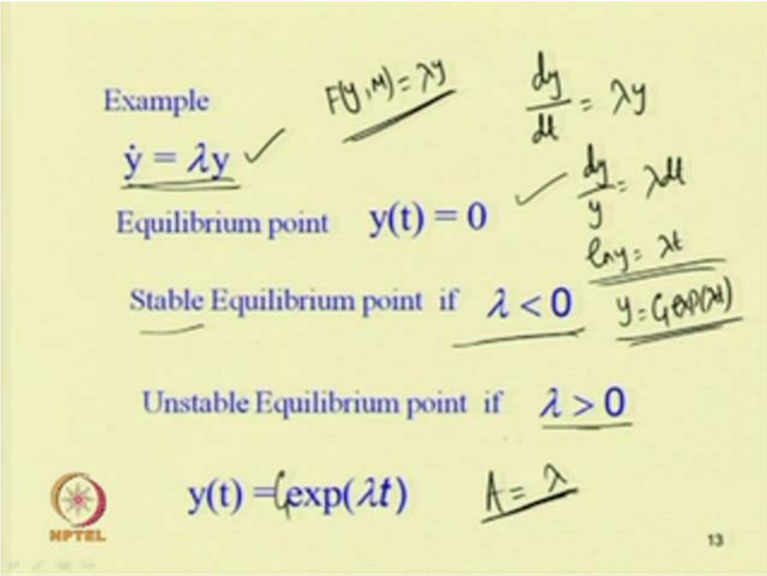

So, in this case if we use our original method that is y dot equal to in this case so, our A matrix to obtain this a matrix one can find this del f y so, this is so our F x M. So, in this case our F x M equal to lambda y so, del f by del x or you can write so, for x instead of writing x as our variable is y so, we can write this is y m and so, in this case we can write del f by so, del f by del y so, A equal to lambda A so, A equal to lambda or i can so, from this one can study or one can find the Eigen value and from this Eigen value we have seen that if lambda less than 0 it is stable and if lambda greater than 0 it is unstable.

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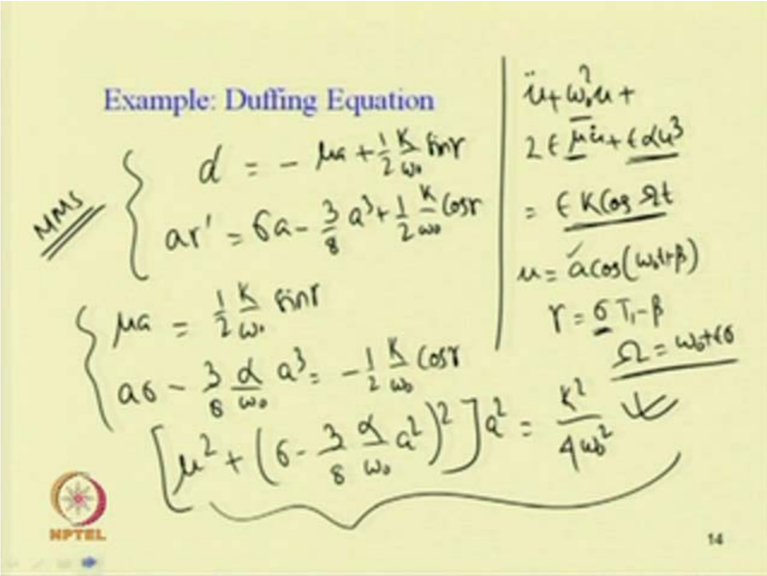

So, let us take this another example that is duffing equation. In case of the duffing equation either we can directly solve this equation to find its stability or we can use the perturb equation using method of multiple scale. For example, if one use method of multiple scales then, one can obtain a set of reduce equation which can be written in this form. So, a dash equal to minus mu a. So, let us take the dufffing equation in this form that is u double dot plus omega 0 square u plus 2 epsilon mu u dot plus epsilon alfa u cube equal to epsilon k cos omega t.

So, here this is the epsilon mu u dot that is the damping and this is the non-linear term and this is the forcing term this part is the forcing term so, here we have assumed the force to be very small that is why have used this book keeping parameter epsilon omega square e t square of the frequency and mu is the damping factor. So, in this equation the solution can be written in this form u will be equal to a cos omega 0 t plus beta where, this beta or gamma can be written as sigma t 1 minus beta where, sigma is the detuning parameter sigma is the detuning parameter to express the nearness of this parameter with respect to the excitation frequency omega. So, in this case this equation the reduce equation can be written a dash equal to minus mu a plus half k by omega 0 sin gamma and this a gamma dash equal to sigma a minus 3 by 8 a cube plus half k by omega 0 cos gamma. So, these 2 are the reduced equation obtained by using method of multiple scale so, using method of multiple scale one can find these 2 reduced equation. Now, for steady state motion so, as they are not function of time so, this a dash and gamma dash will be equal to 0. So, one can write the equation in this form mu a will be equal to half k by omega 0 sin gamma and a sigma minus 3 by 8 alpha by omega 0 a cube so, this will be equal to minus half k by omega 0 cos gamma.

So, from these 2 equation right hand side, squaring the right hand side and add adding these 2 so, one can obtain the frequency equation so, the frequency equation will be in this form so, mu square plus sigma minus 3 by 8 alpha by omega 0 a square whole square into a square equal to k square by 4 omega 0 square. So, this is the frequency this is the frequency response equation. So, from this one can find the amplitude so, we have seen the solution of this equation equal to a cos omega 0 t plus beta so, this amplitude a and frequency they are related by using this equation. So, in this equation we have this sigma which is detuning parameter so, that can be that shows the nearness of this omega so, omega will be equal to omega 0 plus epsilon sigma.

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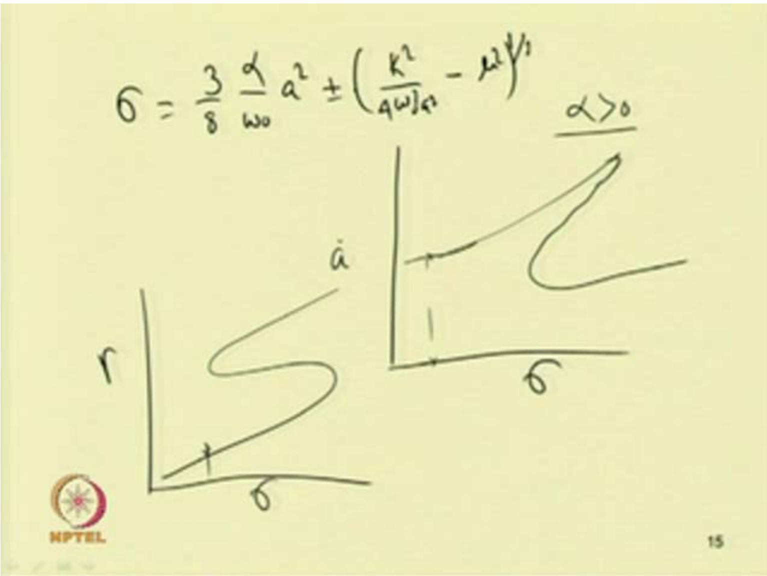

So, from this equation from this frequency equation so, one can plot the response so, this equation can further be written using this sigma. So, sigma will be equal to 3 by 8 alpha by omega 0 a 0 a square plus minus k square by 4 omega 0 a square minus mu square to the power half. So, mu square to the power so, using this sigma and a equation this equation is a quadratic equation in terms of a, in terms of sigma so, that is why solving this quadratic equation one can obtain the sigma. But, one can numerically also solve this equation this will be a sixth order equation in terms of a square as we have a square here square term is there another a square is multiplied outside so, this is a sixth order equation in terms of a or a quadratic equation in terms of sigma as it is easier to solve this quadratic equation so, one can write or one can solve this quadratic equation to write sigma otherwise, one can use the numeric method some numerical method to solve this equation for a. Now, solving this equation one can obtain the response so, the response a and sigma one can obtain for example, for alpha greater than 0 the curve will be curve will looks like this. So, in this case each point on this curve whether these points or this solution what we obtained so, this is for a and one can plot the gamma equation also gamma also, this is a versus sigma is the detuning parameter a is the amplitude.

Similarly, gamma is the phase and with respect to sigma one can plot and here one can see that the plot will look like this. So, in this case for corresponding to a particular value of a for example, sigma equal to for this sigma value so, this is a value and this is the sigma value so, correspond this is the gamma value. So, corresponding to this a and gamma value so, one can check whether this point is stable or not.

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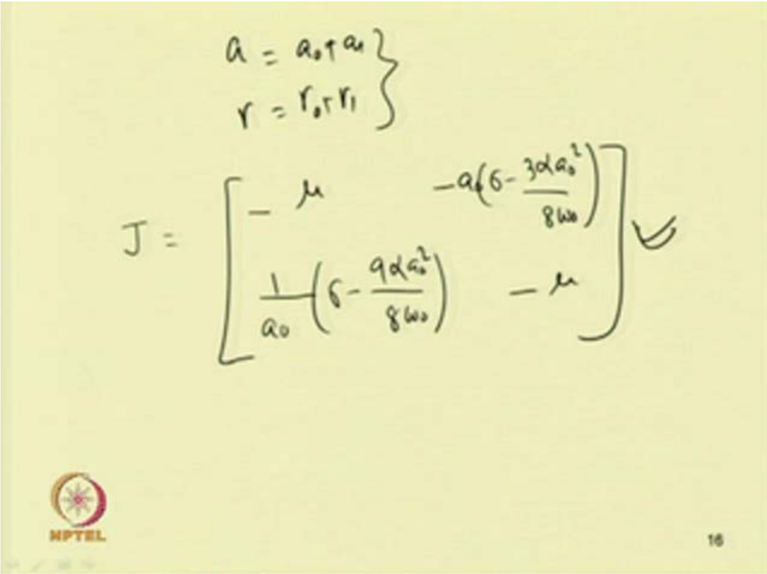

So, to check the stability of this point so, one can perturb the solution that means taking this a equal to a 0 plus let me take this is equal to a 1, a 0 plus a 1 and gamma equal to gamma 0 plus gamma 1.

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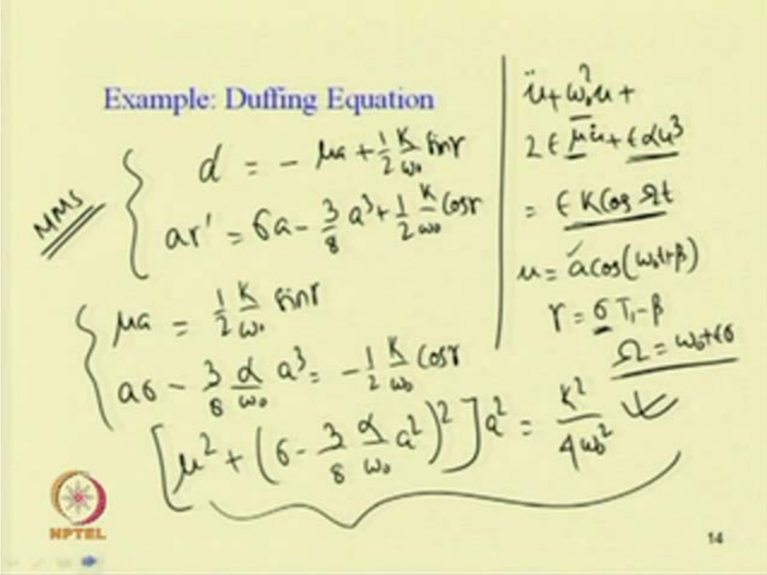

So, one can perturb this equation that means this equation a dash gamma dash equation can be perturbed.

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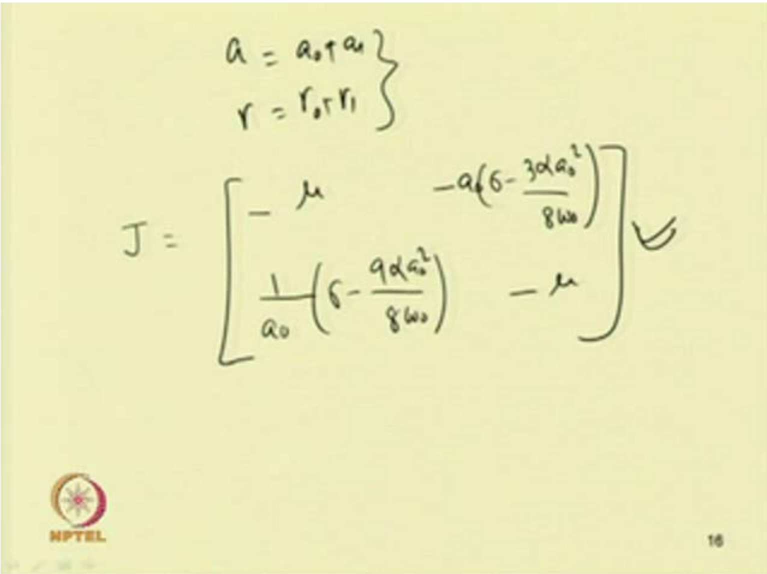

And by substituting this a and gamma in that equation one can write this one can obtain the Jacobian matrix. So, Jacobian matrix can be written in this form that is minus mu minus a 0 into sigma minus 3 alpha a 0 square by 8 omega 0. So, this is 1 by a 0 into sigma minus 9 alpha a 0 square by 8 omega 0 so, this is minus mu.

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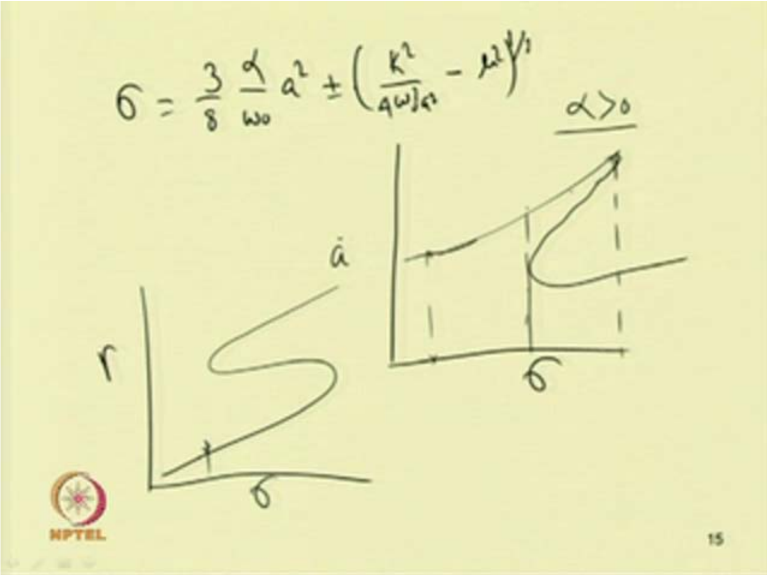

Now, one can find the Eigen value of this Jacobian matrix corresponding to this value of a and gamma and study the stability whether, this branch is stable or unstable and after studying the stability of each branch one can know whether the corresponding response is stability or unstable. So, one can note in this case so, up to the sigma so, the system has a single branch but, after this value of sigma so, one can have 3 solutions so, out of these 3 solutions some solutions maybe stable some maybe unstable and one can study this stability by finding the Eigen value of this Jacobian matrix.

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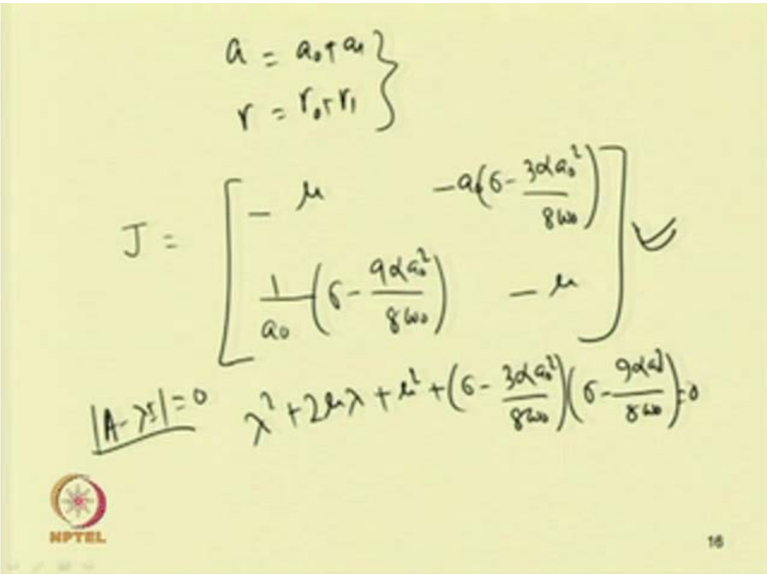

So, these Eigen value so, for example, in this case so, by putting this a minus lambda i equal to 0 we will have the equation lambda square plus 2 mu lambda plus mu square plus sigma minus 3 alpha a 0 square by 8 omega 0 into sigma minus 9 alpha a 0 square by 8 omega 0 equal to 0 for steady state motion. So, the steady state motion will be unstable.

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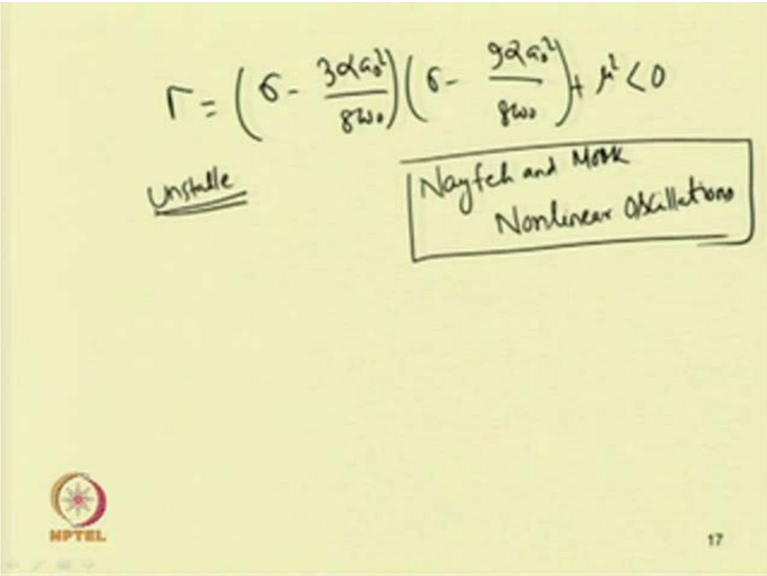

So, if we have this term gamma equal to sigma minus 3 alpha a 0 square by 8 omega 0 into sigma minus 9 alpha a 0 square by 8 omega 0 plus mu square less than 0. So, one can see the book by Nayfeh and Mook non-linear oscillation non-linear oscillation to know more about the stability of this type of system non-linear oscillations.

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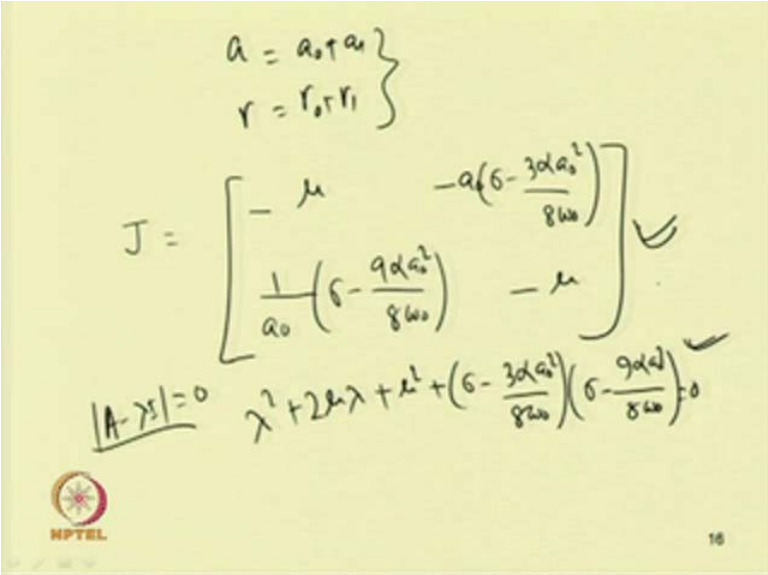

So, one can obtain the stability from this equation when lambda is less than 0 the real part of lambda is less than 0 then, the system will be stable so, from this one can solve and see this equation or find that if this term less than 0 then, the system will be unstable. So, in this way one can study the stability of the system. Sometimes, this Jacobian matrix may have distinct Eigen values or sometimes they may not be distinct so, depending on these Eigen values one can study the stability and one can obtain the bifurcation diagrams.

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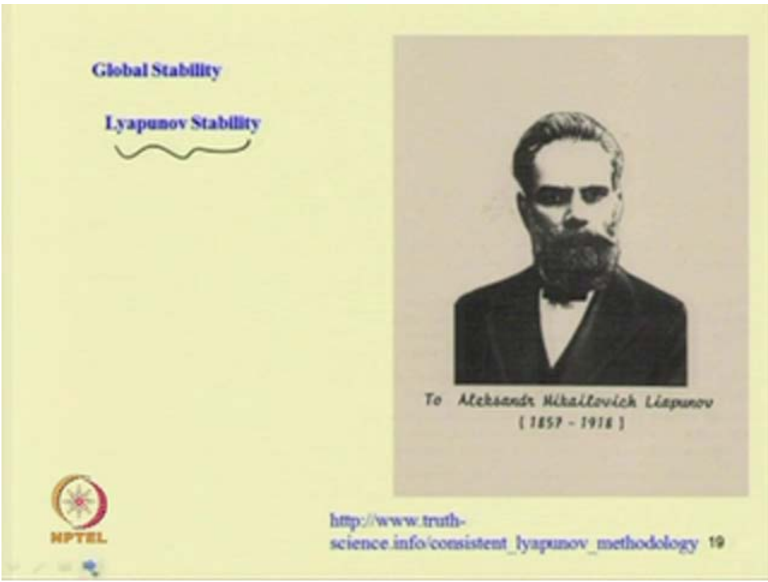

So, let us to study the global stability one can use this Lyapunov stability method so, instead of going for the linearization technique so, one can use Lyapunov stability. So, in case of the Lyapunov stability one has to develop the difficulty of this Lyapunov stability method. One has to develop a function Lyapunov function to find to study the stability of the system.

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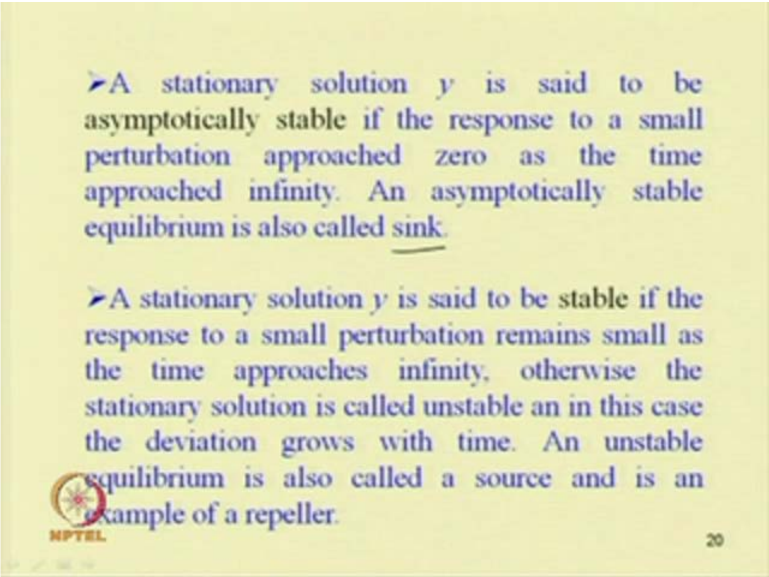

So, in case of this Lyapunov stability according to this a stationary solution y is said to be asymptotically stable if the response to a small perturbation approach 0 as the time approach infinity. An asymptotically stable equilibrium is called a sink. Stationary solution y is said to be stable if the response to a small perturbation remains small as the time approaches infinity. Otherwise, the stationary solution is called unstable as in this case the deviation grows with time an unstable equilibrium is also called a source and is an example of repeller. So, in this case one has to find one Lyapunov function and one has to check whether the system is asymptotically stable or stable or unstable.

So, in this case so, let this is the equilibrium solution so, near this equilibrium solution if we or if we perturb this equilibrium solution so, if it remains within a bounded region then it becomes stable and if with time it approaches this equilibrium position then, it is asymptotically stable. So, let us take one example so, let us take the van der pol equation so, according to this Lyapunov method we have to take one Lyapunov function which is a positive definite function so, if its derivative becomes negative definite then, the system is stable otherwise, the system is unstable.

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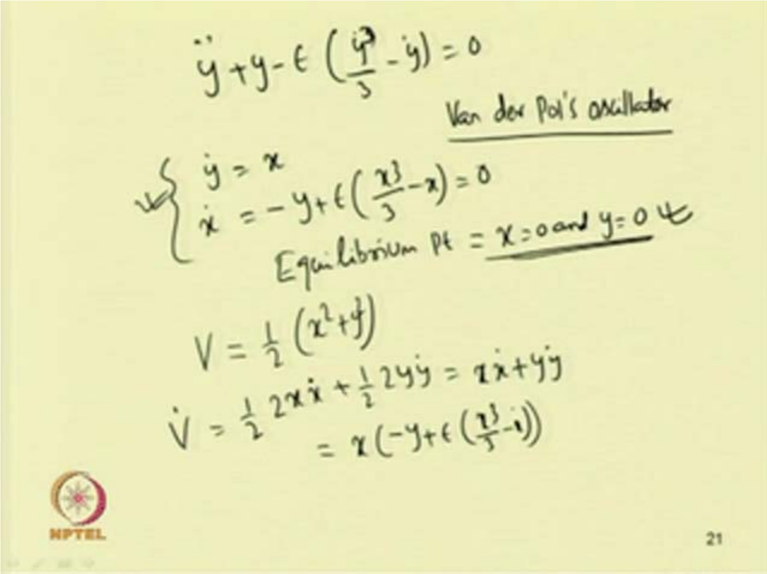

So, for example, taking this van der pol equation let us take the van der pol equation that is, x y double dot plus y minus epsilon y dot cube by 3 minus y dot so, this is equal to 0. So, in this case so, this is the well known van der pol equation so, one can write this equation by using a set of first order equation.

So, in this case first order equation will be in let me put this y dot equal to x so, the second equation becomes x dot that means y double dot x dot equal to minus y plus epsilon into x cube by 3 minus x so, this is equal to 0. So, in this case the equilibrium points will be x dot equal to 0 and by putting this x dot equal to 0 and y dot equal to 0 so, we will find the equilibrium position by putting this y dot equal to 0 so, x equal to 0 is x we obtain x equal to 0. Now, by putting x dot equal to 0 and x equal to 0 so, we obtain the equilibrium so, the equilibrium point so the equilibrium point is x equal to 0 and y equal to 0.

So, we have to check whether this equilibrium point is stable or not so, we can take a Lyapunov function let us take a Lyapunov function V equal to half x square plus y square so, this function is positive so, this is a positive definite function so, this is always positive for all the values of x and y. Now, we can form this equation so, we can write this V dot so, if you differentiate this thing so, this V dot equal to half into 2 into x into x dot plus half into 2 into y into y dot so, this becomes x x dot plus y into y dot. So, this thing can be written so, from this we know, what is our x dot and y dot value from this equation.

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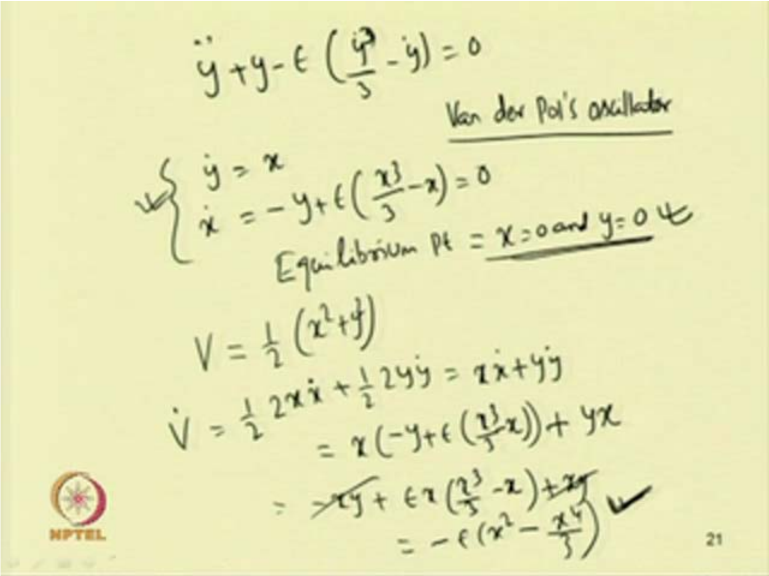

Now, substituting that thing so, we can write x into so v dot will be x into so for x dot one can substitute this is equal to minus y plus epsilon into x cube by 3 minus x so, into minus x plus y into y dot so, y dot equal to x so, this becomes so, minus x y plus epsilon x into x cube by 3 minus x plus so, this is x y so, minus x y plus x y cancel. So, this term can be written as minus epsilon x into x becomes x square minus x fourth by 3. So, this becomes minus epsilon x square minus x fourth by 3. So, this term is as x square and x fourth so, they are positive so, this term can be so, this term for so, this term will be always negative. So, from this we have seen that this point so, will have a stable point at x equal to 0 and y equal to 0.

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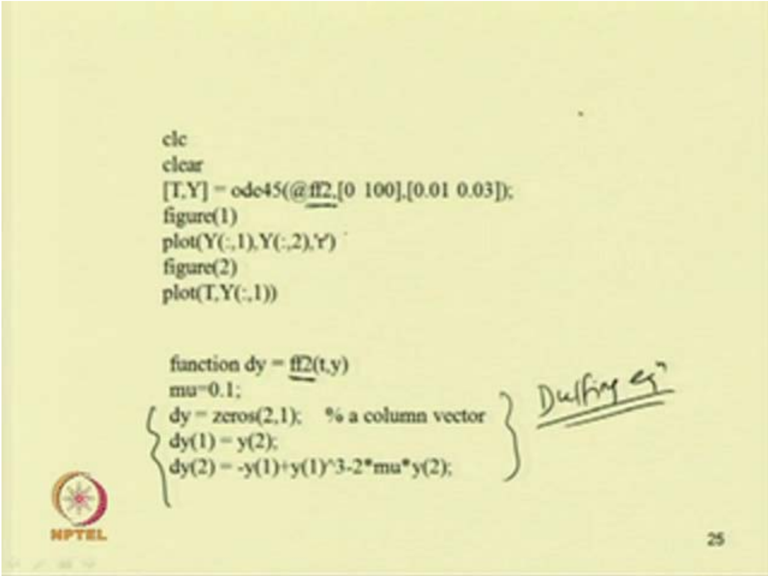

So, let us see to obtain the solution so one can use this numerical technique using mat lab also one can find the response of the system. For example, in case of this duffing equation so, one can write this program using this ode45 so, for ode45 one can use the simple comments so, t y equal to ode 45 so, this is the function and using ff 2.

So, this is the function so, which where this first order differential equations 2 first order differential equations are written that is d y 1 equal to y 2 and this d y 2 equal to minus y 1 plus y 1 cube minus 2 mu y square. So, this is for the duffing equation. Similarly, one can write so, similarly, one can write for the van der pol equation and by using this command one can find the response of the system. So, here the response are plotted that is y 1 y 2 y 1 and y 2 are these displacement and velocity so, one can obtain the phase portrait so, also one can obtain by plotting this t and y 1 so, one can obtain the time response so, from the time response and frequency response one can study whether the system is stable or unstable.

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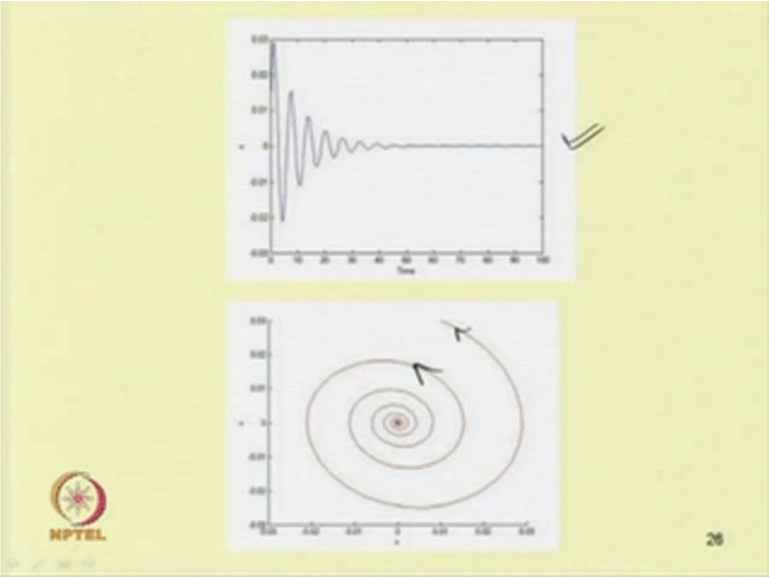

So, one can find the response so, from this response also one can study whether it is decreasing exponentially decreasing or this is decreasing and finally, it is stable. In this case or from the phase portrait so this is the time response so, one can plot the phase portrait also. So, in case of the phase portrait so, starting from this initial position if the response grows that means with time it increases and increases so, than it will be unstable.

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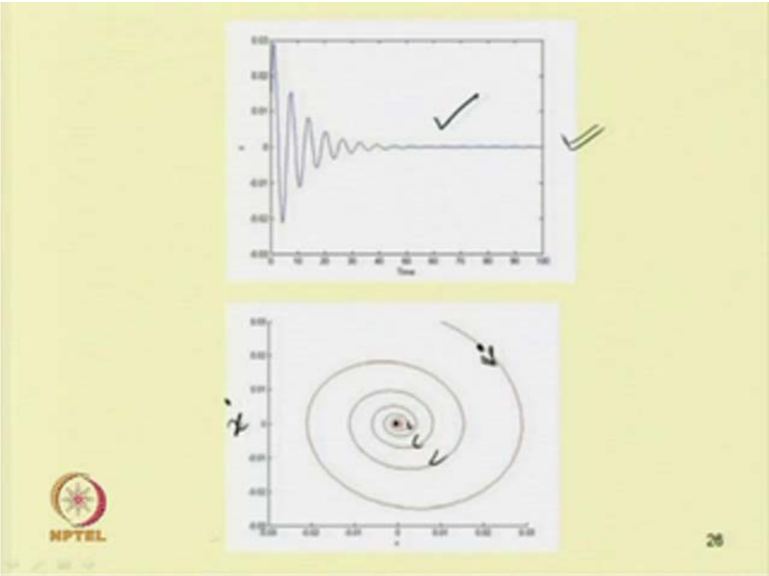

Or if it is starting from this position let us let me start from this position. So, in this case if the response decreases and finally, comes to this position so, this position is a stable position. So, if one plot so, for this cases the phase portrait is this so, this shows a stable point. So, by using these numerical methods also one can find the time response phase Portrait and from this time response and phase portrait one can visualize whether actually this system is stable at the equilibrium position. So, this is the equilibrium position corresponding to x equal to 0 and x dot equal to so, you are so, this is x equal to 0 and x dot equal to 0 in the equilibrium position. So, it shows that this equilibrium position is stable.

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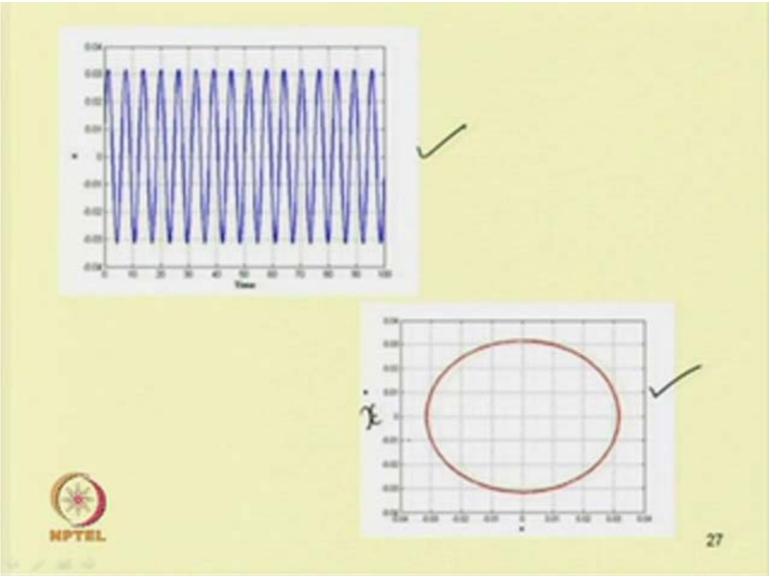

So, from this time response one can understand whether the response is stable or not. So, this also a periodical response in case of a periodical response the phase portrait so, the phase portrait is periodic. So, one can use these methods this time response and phase portrait to visualize actually whether the system is stable or not and by finding the Eigen values from the Jacobian matrix one can theoretically or analytically predict whether the systems will be stable or unstable at that point.

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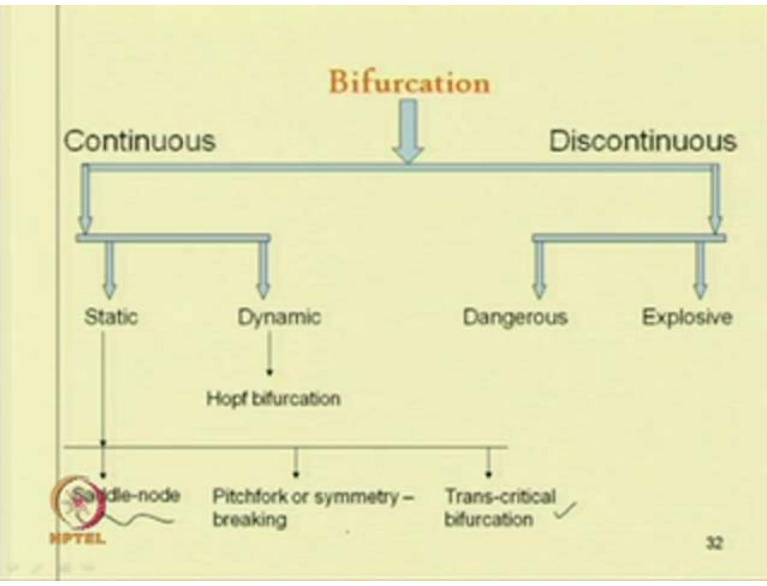

So, already we have seen the bifurcation so, bifurcation points and in case of the bifurcation point we know we have 2 different bifurcations one is continuous bifurcation and other one is discontinuous bifurcation. In case of continuous bifurcation we have static bifurcation and dynamic bifurcation. In case of static bifurcation, we have this saddle node bifurcation pitchfork or symmetry breaking bifurcation and trans-critical bifurcation and in case of the dynamic bifurcation we have hopf bifurcation. So, today class we will basically study about the saddle node and pitchfork symmetry breaking bifurcation.

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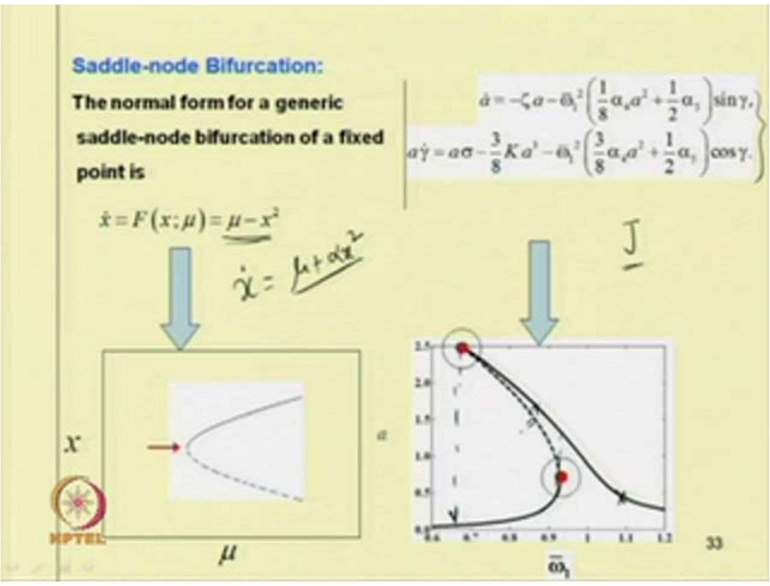

And so, in case of the saddle node bifurcation the normal form for a generic saddle node bifurcation fixed point is so, x dot equal to or x dot equal to F x mean or this thing can be written so, this is the generic form that is mu minus x square or one can write also this generic form x dot equal to so, x dot equal to mu minus alpha or mu plus alpha x square. So, in this form also one can write the equation so, in this case alpha equal to minus 1 so, it becomes mu minus alpha x square.

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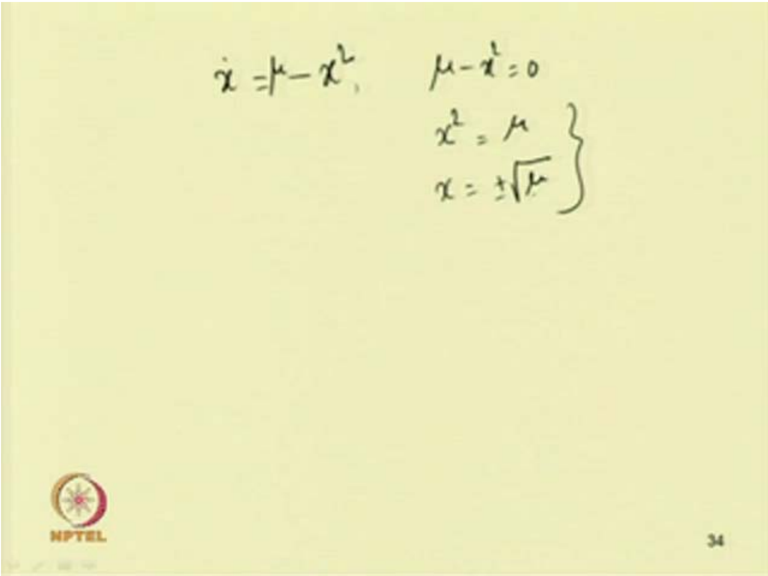

So, for the saddle node bifurcation point now, one can find the Jacobian matrix as x dot equal to minus mu x square so, as we have this x dot equal to minus mu minus x square. So, in this case this equilibrium point becomes putting this x dot equal to 0 so, equilibrium point mu minus x square equal to 0. So, from this one obtain this x square equal to mu or x equal to plus minus root mu. So, corresponding to positive and negative value of mu one can plot the responds plot. So, this is the x mu plot.

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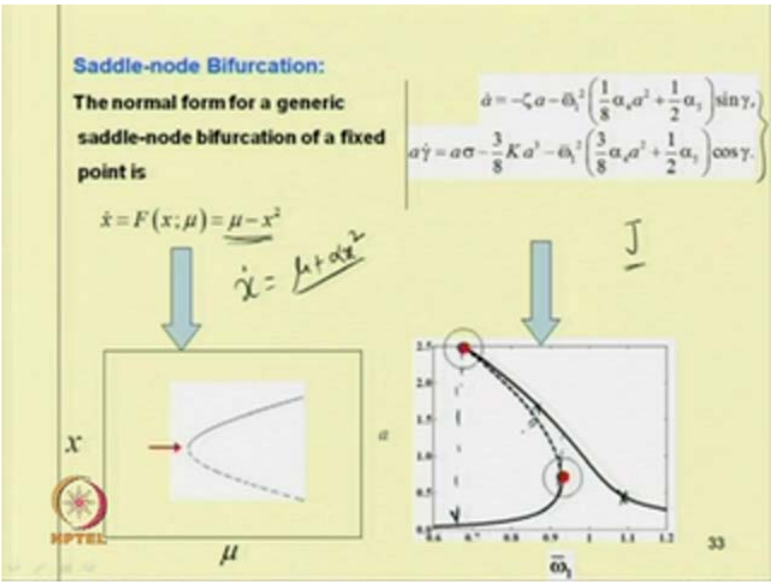

So, one can see that corresponding to different value positive value of mu so, this is positive value of mu and negative for negative value of mu one can see so from this equation so, if mu is negative so, this is imaginary so, the root will be or x will be imaginary so, that is why it is not plotted and for x after x equal to mu equal to 0 only we will have the solution. So, here one may note that before mu equal to 0 so, there no solution exists in this case and after this mu equal to 0 so, we have 2 branches of solution. So, this point mu corresponding to 0 is a bifurcation point as here there is change in the number of the solutions and also we can see that the quality of the solutions that is the stability type of the solutions also changes. So, this is for a one dimensional one dimensional equation.

So, for two dimensional equations also, this is example of a two dimensional equation so, in this case one can find the Jacobian matrix so, find after finding this Jacobian matrix so, one can obtain the Eigen values of this Jacobian matrix and one can observe that this part of the solution is unstable and this is stable so, as it is unstable at this position. So, when one sweep off the frequency so, at this position the system will show a jump of phenomena and further increase in this frequency it will follow this part. Similarly, during sweeping down the frequency that means if you decrease the frequency so, one can follow this path so, one can follow this path during sweeping down and at this point so, it will jump down. So, one can observe this jump up and jump down phenomena at the bifurcation points.

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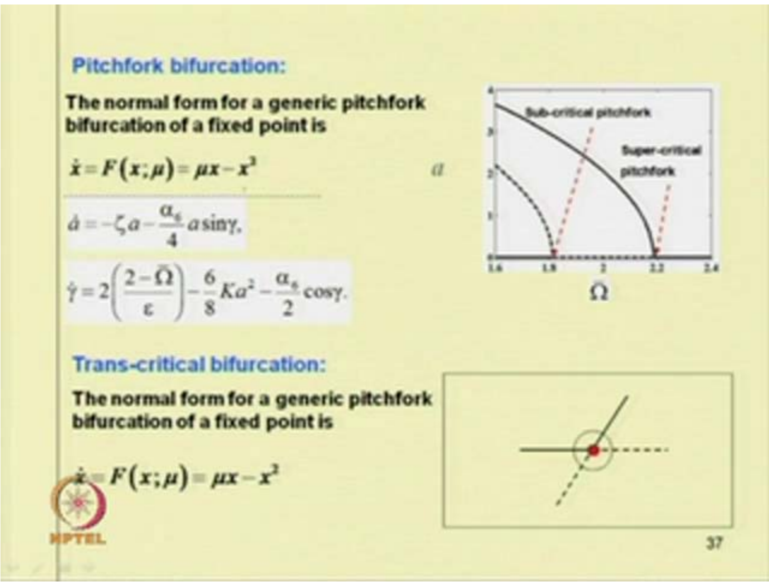

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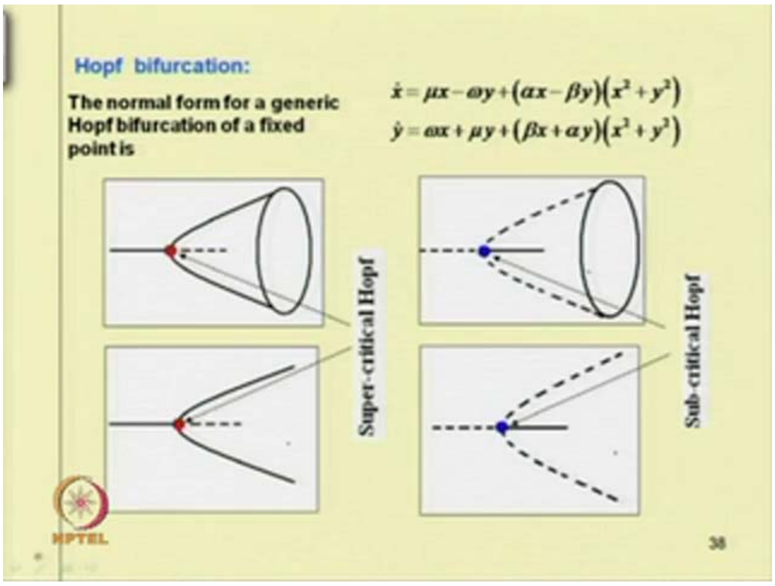

So, similarly, one can take this pitchfork bifurcation so, in case of pitchfork bifurcation the normal form of the pitchfork bifurcation is mu x plus alpha x square equal to 0.

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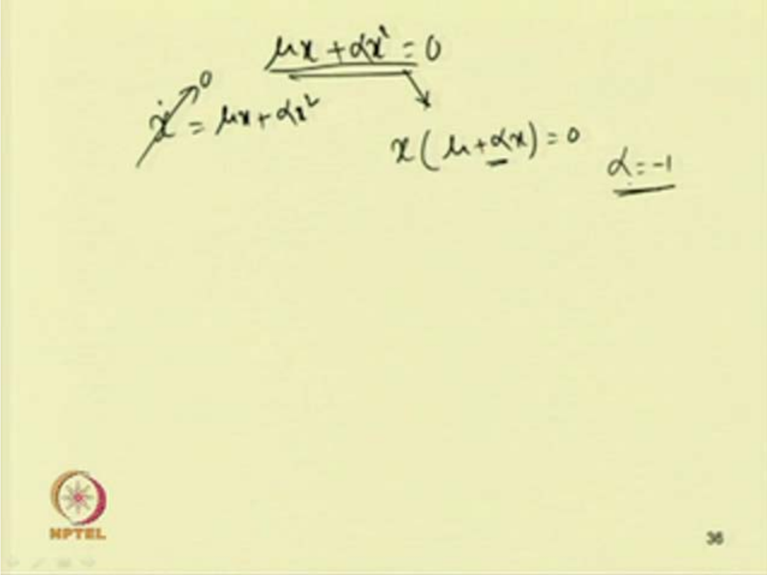

So, if we write this mu x plus alpha x square equal to 0 so, this is the alpha x square equal to 0. So, in this case our equation one dimensional equation becomes x dot equal to mu x plus alpha x square. Now, putting this x dot equal to 0 so, we have this equation so, from this equation we can find the equilibrium position so, in this case to find the equilibrium position this will give mu plus alpha x equal to 0 so, if I will take this alpha so, depending on different value of alpha we can have a set of curves. So, let us take alpha equal to minus 1 so, this curve is shown in this figure.

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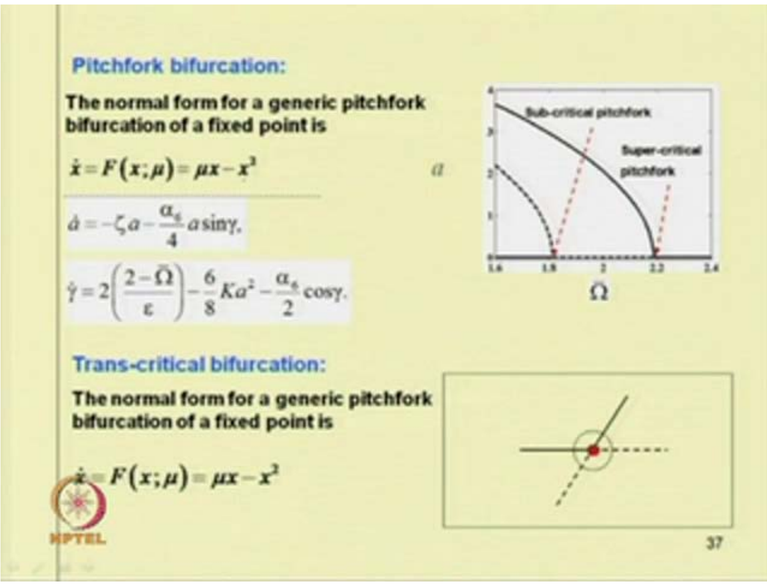

So, corresponding to alpha equal to minus 1 so, we have this equation x dot equal to mu x minus x square.

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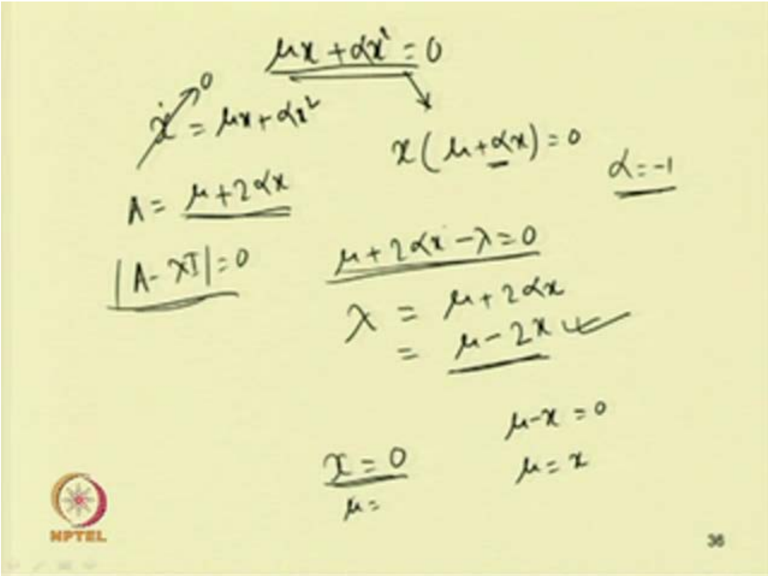

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So, in this case the so, our Jacobian matrix so, to obtain this Jacobian matrix we will find this del f by del x the matrix with del f by del x at is as it is one dimensional so, we will have only 1 so, this becomes mu plus so, this becomes mu plus 2 alpha x. So, as our A minus lambda I so, we have to make A minus lambda I equal to A minus lambda I determinant of A minus lambda I equal to 0. So, in this case as only we have one term so, we can write this mu plus 2 alpha x minus lambda equal to 0 so, our lambda becomes the Eigen value becomes mu plus 2 alpha x. So, as alpha equal to minus 1 so, we can write this lambda equal to mu minus 2 x so, as lambda equal to mu minus 2 x so, depending on the value of mu we can obtain different value of x and we can study whether the branch is stable or unstable.

So, in this case we have seen this trivial solution so, one solution is so, x into mu plus alpha x equal to 0. So, the equilibrium position becomes x equal to equilibrium position becomes x equal to 0 so, that is the trivial solution and also we have a non trivial solution that is mu plus alpha x by putting this mu plus alpha x equal to 0 so, we have or this putting this alpha equal to minus 1 so, mu minus x equal to 0 or mu equal to x so, we have 2 points one is x equal to 0 and another one is mu equal to x equal to mu.

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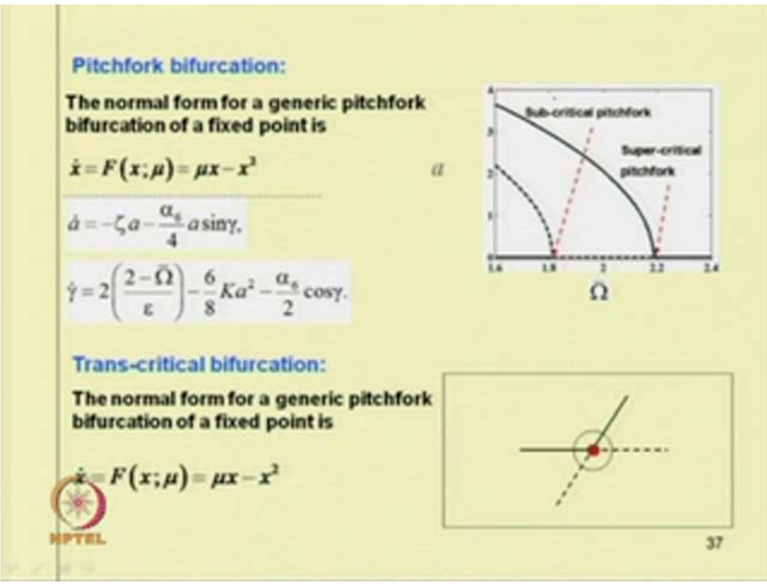

So, we have these 2 x equal to mu. Now, we can plot so, from this plot we can see so, we have this trivial branch. So, if this trivial branch becomes stable so, if the Jacobian matrix has negative real parts similarly, so here one can observe that at this point at this so, from this branch so, if one solve this equation so, one can find the solution. So, in this case for pitchfork bifurcation it will be x dot will be equal to mu x minus x cube.

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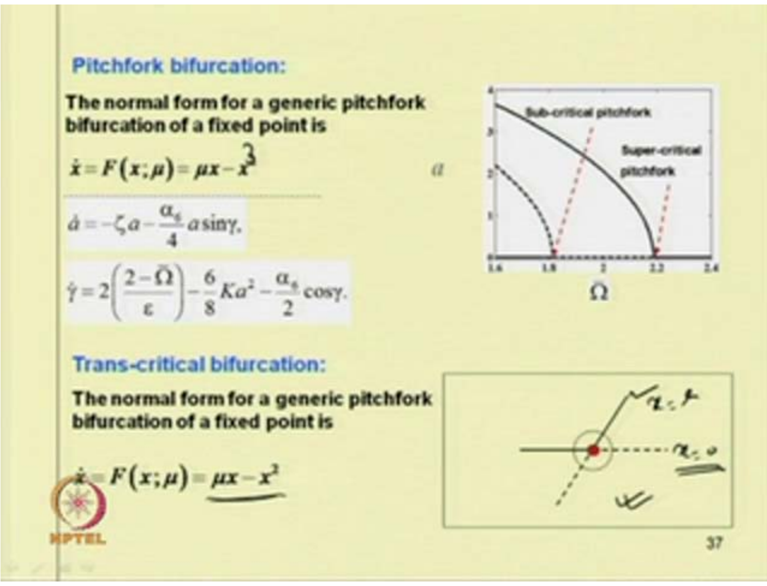

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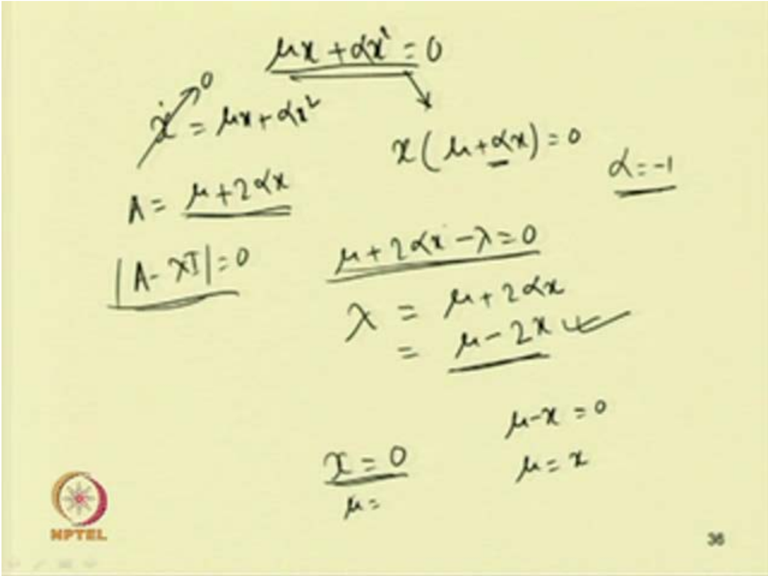

So, the previous things what we have taken this example mu x plus alpha x q equal to 0 so, we have 2 solution that is x equal to 0 and x equal to mu so, this is the example of this trans-critical bifurcation so, in this case this is mu x minus x square. So, this is the x equal to 0 line x equal to 0 line and x equal to the second one is x equal to mu so, this is x equal to mu line so, x equal to mu this is x equal to 0 so, this is pitch fork bifurcation and in case of this is trans-critical bifurcation.

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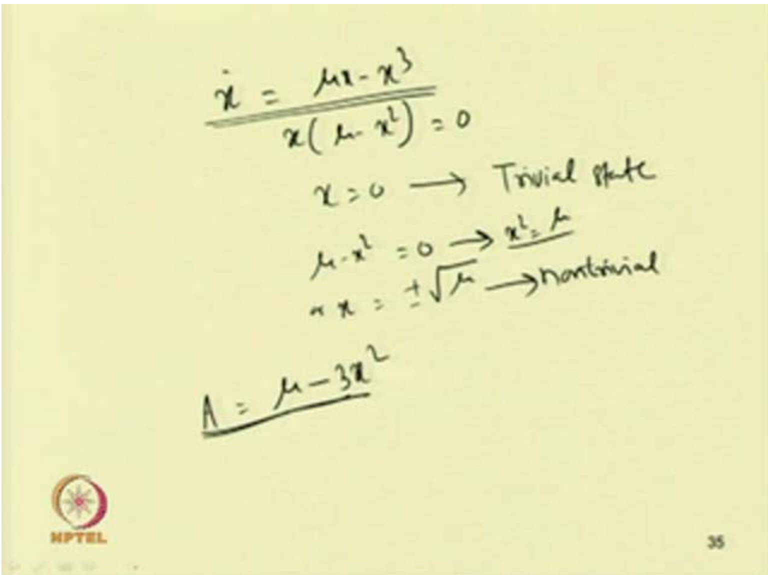

So, in case of the pitch fork bifurcation so, mu x minus x cube will be equal to 0 so, x dot equal to mu x minus x cube so, taking this x common we have mu minus x square equal to 0 or x equal to 0 is the trivial state response trivial state response and mu minus x square equal to 0 or x square equal to plus minus root mu so, x equal to plus minus root mu so, from this x square equal to mu or x equal to plus minus root mu so, this is non-trivial response. So, in this case one can find the Jacobian matrix from this. So, the Jacobian matrix A will be equal to so, this becomes mu minus 3 x square.

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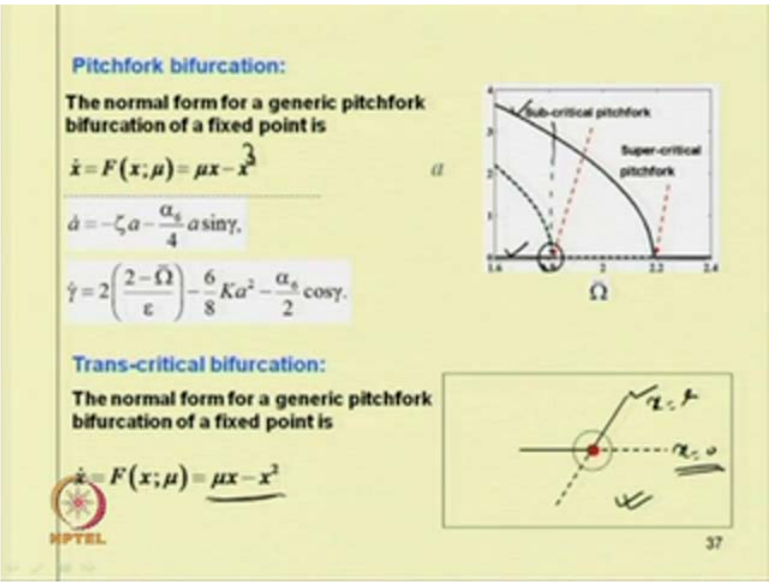

Now, by substituting this value for example, taking this x equal to 0 that is, for the trivial state so, one can observe this branch to be stable this branch to be stable and this is unstable and this is stable. So, at this point so at this at this point one can observe that the response before that one has 3 solutions so, out of the 3 solution the upper branch and lower branch were stable and the middle branch is unstable and at this point so, this becomes the trivial state become unstable and one has a one has 2 solutions. So, out of in before this thing out of these 3 solutions, these 2 are stable and this is unstable and after this one has a unstable solution. So, these points where it tends to jump from this position to this upper one is known as sub critical pitch fork bifurcation point. Similarly, while sweeping down this thing so, we have this super critical pitch fork bifurcation point.

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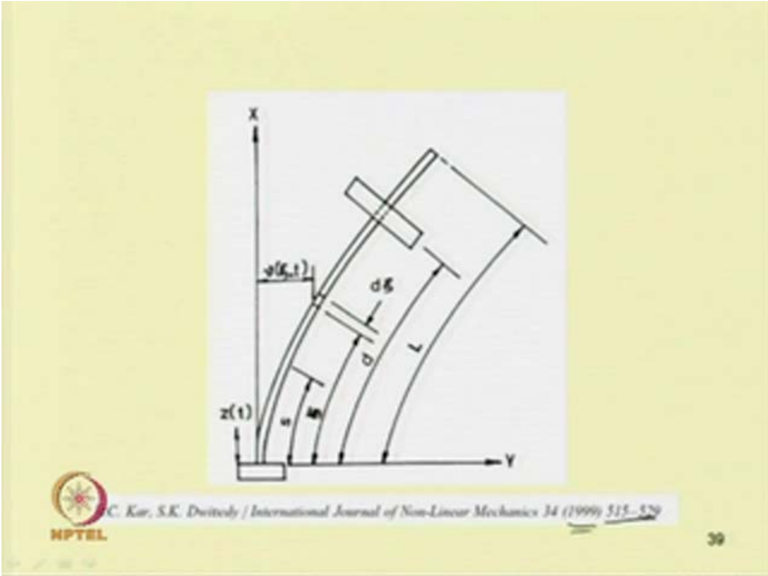

So, next class we will study about the hopf bifurcation point. One more example one can see so, were this one can apply the stability analysis. For example, one can refer this paper by Kar and Dwivedy international journal of non-linear mechanics 1999 page number 315 to 515 to 529. So, this is where the non-linear dynamics of a slender beam is studied.

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$$\ddot{u}_n + 2\varepsilon\zeta_n\dot{u}_n + \omega_n^2 u_n - \varepsilon \sum_{m=1}^{\infty} f_{nm} u_m \cos\phi\tau + \varepsilon \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \{\alpha_{klm}^n u_k u_l u_m + \beta_{klm}^n u_k \dot{u}_l \dot{u}_m + \gamma_{klm}^n u_k u_l \ddot{u}_m\} = 0, \quad n = 1, 2, \ldots, \infty$$

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So, in this case equation can be written in this form and here 2 mode interactions has been taken.

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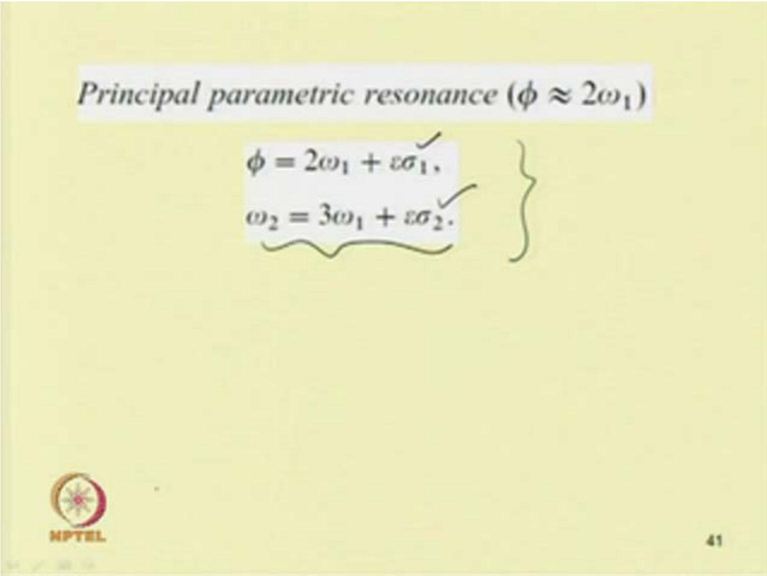

So, to study the principal parametric resonance case by taking this phi external frequency nearly equal to twice the first mode frequency and using this detuning parameter sigma 1 and sigma 2 as the internal resonance is taken so, this second mode frequency was taken thrice near trice the first mode frequency so, this is for internal resonance and this is for external resonance. So, in this case one can get by using this method of multiple scales.

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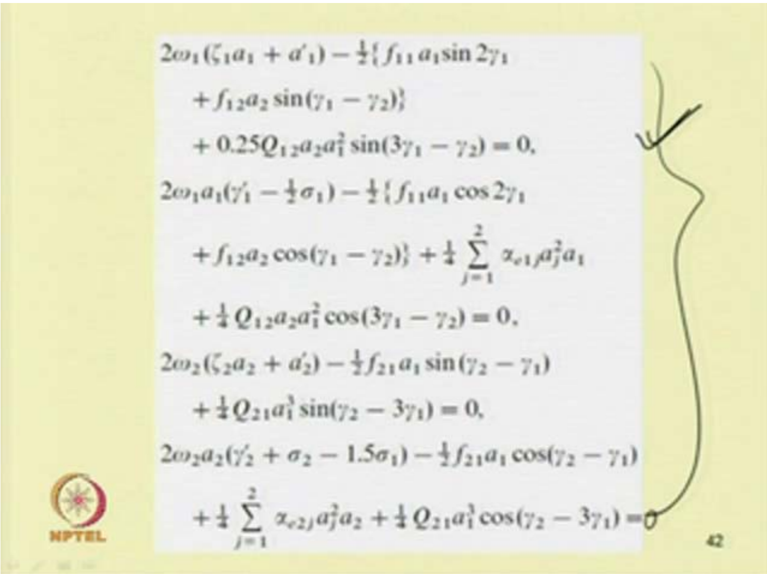

So, one can get a set of 4 first order equation. So, previously we have seen 2 equations now, one can take a set of 4 equations.

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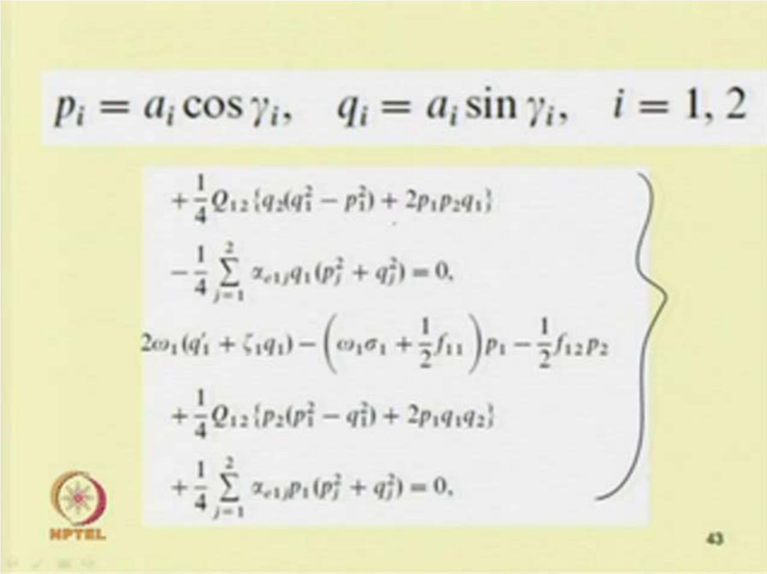

By perturbing this thing so, one can obtain the Jacobian matrix and one can study the stability. But, in this case one can see that while perturbing some of these perturbation will not will for the trivial state whose, overcome that thing so, one can use this transformation. So, here p i equal to a i cos gamma i and q i equal to a i sin gamma i has been taken. So, then one can write this equation by writing this transform form in terms of p and q so, one obtain these 4 equations so, using these 4 equations.

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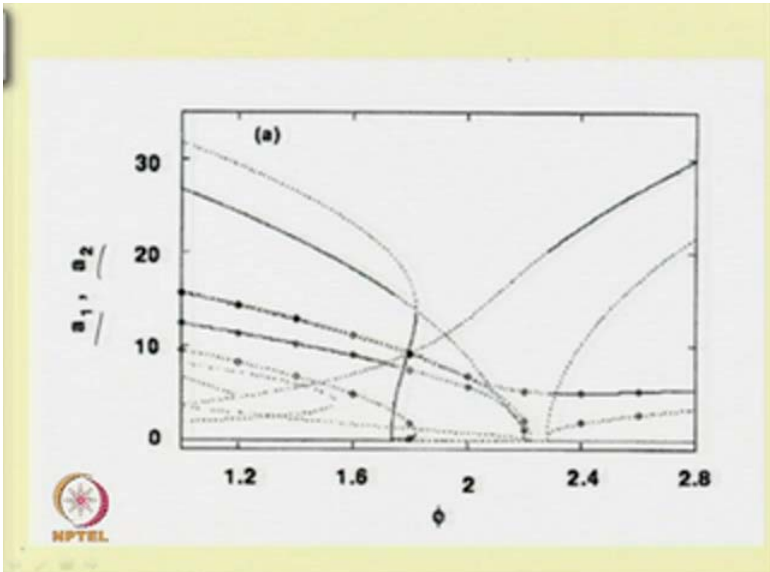

Now, perturbing these 4 equations one can find the Jacobian matrix and slowly the stability. So, in this case one can plot this a 1 and a 2 so, this is your a 1 and a 2 have been plotted so, a 1 a 2 have been plotted.

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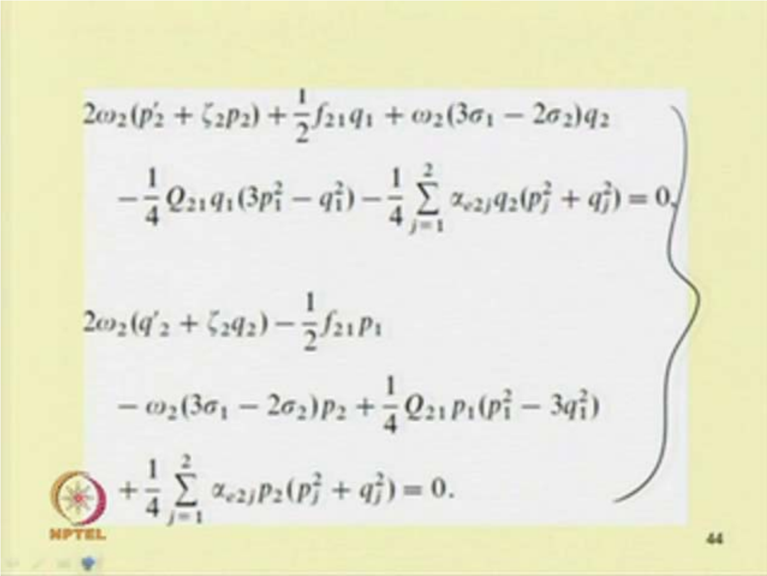

So, using by solving the set of equations by substituting first by substituting this p dash q dash p 1 dash q 1 dash p 2 dash q 2 dash equal to 0.

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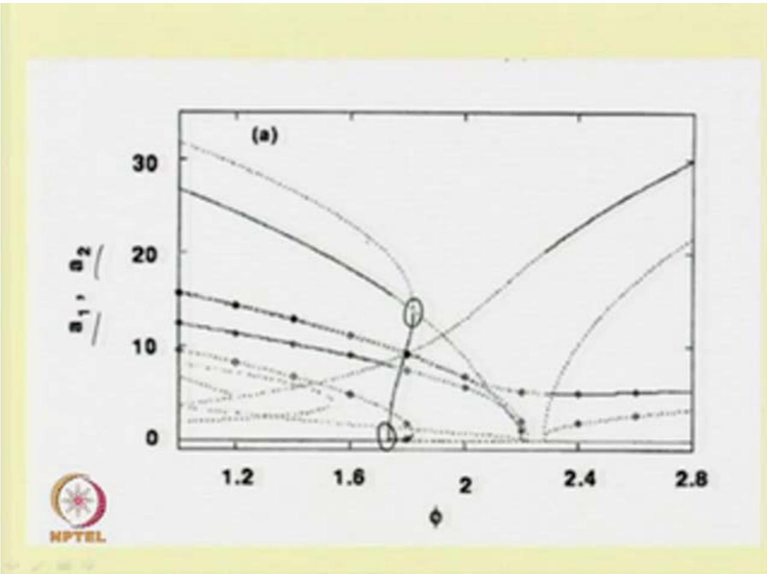

So, one can plot this response. So, after plotting this response, one can study whether the branch is stable or unstable so by using the Jacobian matrix. So, in this way one can study the stability of a system. So, here one can observe so, this point of the pitch fork bifurcation point so, this point is a saddle node bifurcation point. So, later we will see that in this curve, there are also some points which bifurcation points those things we will study in the next class.

Thank you.