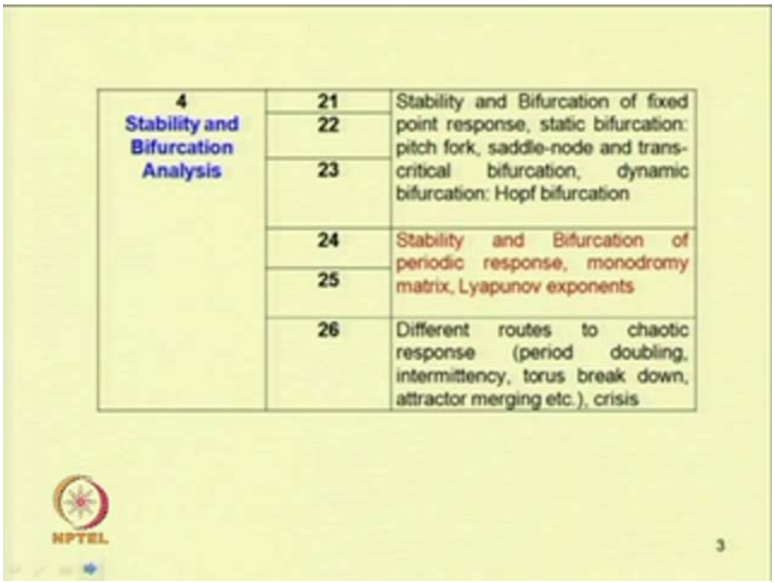
Non-Linear Vibration
Prof. S. K. Dwivedy
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 4
Stability and Bifurcation Analysis of Nonlinear Responses
Lecture - 1
Stability and Bifurcation of Fixed Point Response

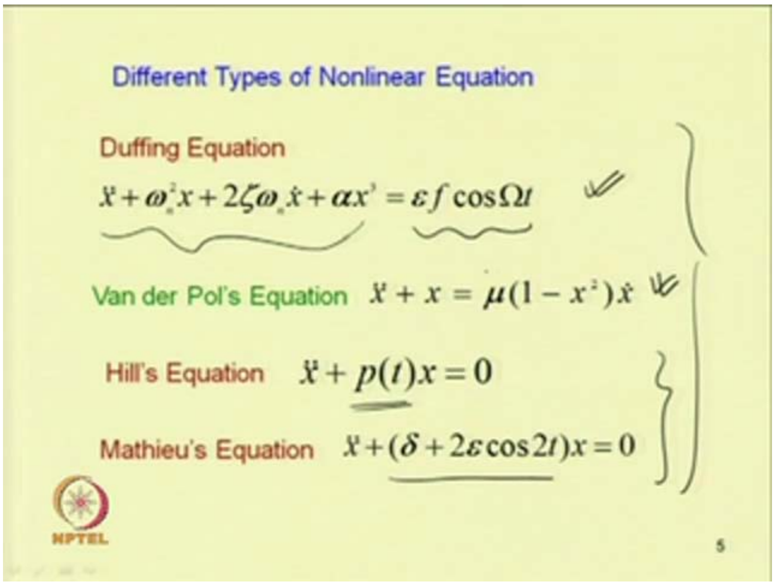
(Refer Slide Time: 00:32)



So, welcome to today's class of non-linear vibration. So, we will start new module today that is on stability and bifurcation analysis of non-linear responses. So, in this module we will have six classes. So, first three classes will report to the stability and bifurcation of fixed point response. So, there will study about the static bifurcations in which I will tell you about this pitch fork, saddle-node and trans-critical bifurcation then, about the dynamic bifurcation that is, Hopf bifurcation.

And next two classes, we will study stability and bifurcation analysis of these periodic responses where, we will study about this monodromy matrix will also study theory and Lyapunov exponents. So, then we will study about this quasi periodic and chaotic response and in this module, in the last class I will tell about different routes to chaotic responses. For example, this will be period doubling route to chaos, the torus break down route to chaos then, intermittency and also will discuss about this crisis. Also, some methods to control this chaotic response and the usefulness of this chaotic response also will discuss in that class. So, coming to the stability and bifurcation analysis, first we should know what we mean by stability of a system.

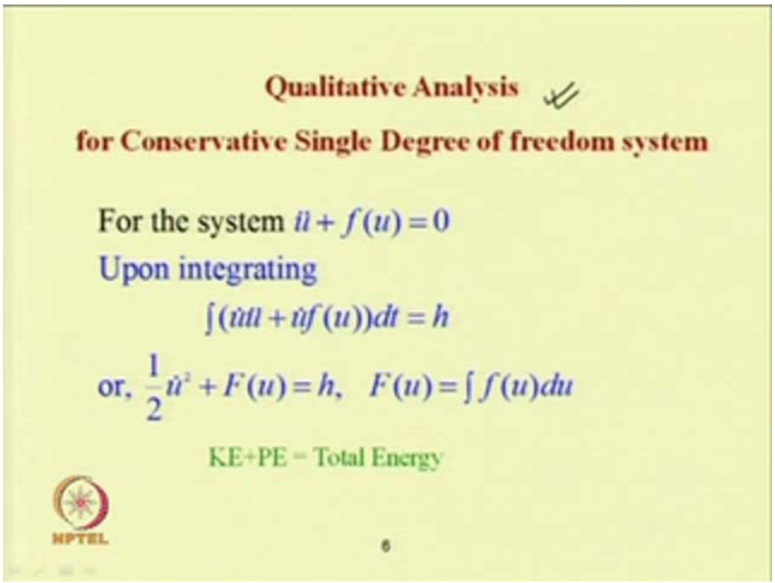
(Refer Slide Time: 01:55)



So, previous classes we have discussed or we know different types of non-linear systems. So, particularly we have studied about this duffing equation in which we have taken the system in this form that is, x double dot plus omega n square x plus 2 zeta omega n x dot plus alpha x cube. So, if you are taken cubic non-linearity otherwise we have this both quadratic and cubic. So, in case of quadratic we are adding a quadratic term that is alpha 1 x square we can add. So, either can be 0 so, we can put this side 0 for free vibration or we can put this forcing term that is epsilon f cos omega t to find the force response of the system. Actually, we are going to study in detail about this duffing equation or other type of equation in our applications. And in the previous module we have discussed about different solution methods how to solve this equations.

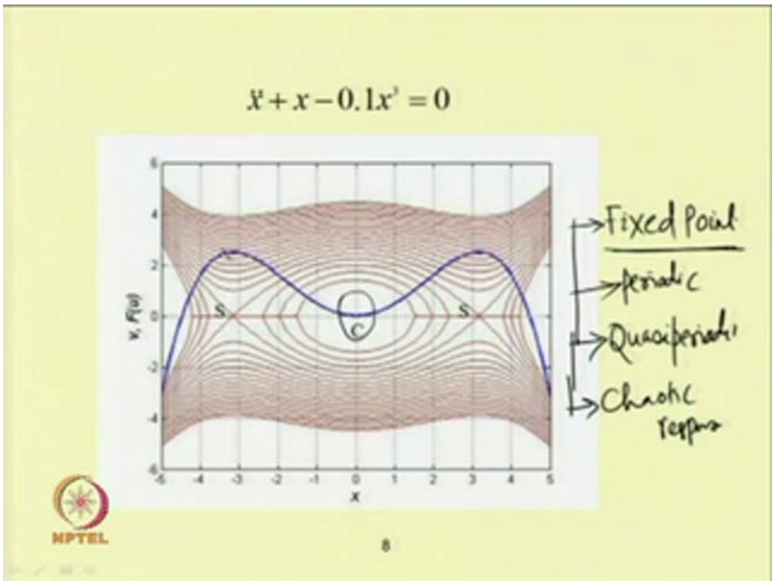
So, in most of the cases we reduce the second order differential equation to a set of first order differential equation and we obtain the steady state response by eliminating the terms which does not depend on time. So, we have studied this duffing or we know these different types of equations, that is duffing equation then, we also know about this van der pol’s equation. So, in case of this van der pol’s equation, the equation is in this type that is x double dot plus x equal to mu into 1 minus x square in to x double dot. Also we have we know this parametrically excited system equations of parametrically excited system in which a time varying term is the coefficient of the response. So, for example, in case of hill's equation we have the equation x double dot plus p t x equal to 0, when this p t that is periodic time takes the form delta plus 2 epsilon cos 2 t then, will tells this Mathieu's equation. So, in general now we are acquainted about these 5 different types of or 4 different types of equations also in addition to that we know this equation in their force vibration form or free vibration form. And in these equations we know different methods to solve them.

(Refer Slide Time: 04:30)



First, we use this qualitative analysis method in which we use both potential energy and kinetic energy terms and equating or by finding the potential well or finding the potential energy or variation of the potential energy so, we found different types of response.

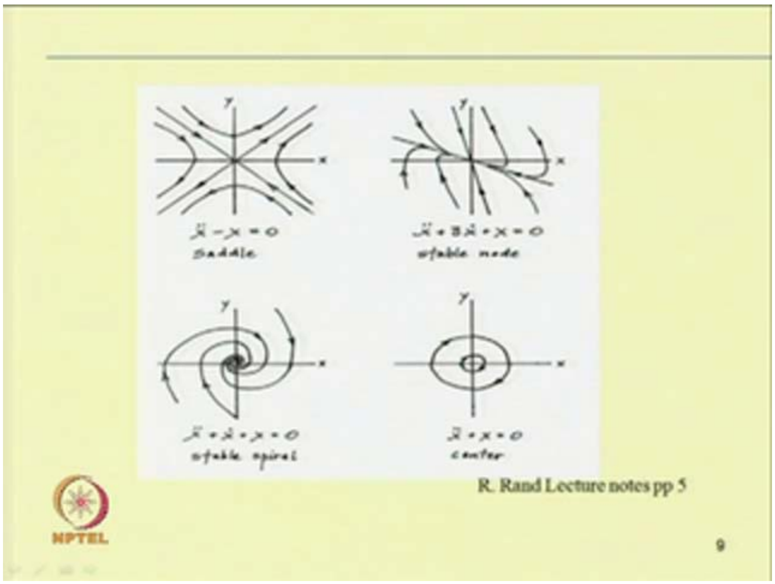
(Refer Slide Time: 04:47)



So, for example: corresponding to the maximum potential energy we have this saddle point and corresponding to this minimum potential energy we have the centre and here also corresponding to this maximum potential energy we have this saddle point. So, this is shown this is the variation of the potential energy and this other curves show the flow of the response. So, one can obtain or one can perform the qualitative analysis to find the different types of response of the system. So, this response that is what is written a c and s are the centre saddle point. So, these responses are the fixed point response of the system. So, 1 can obtain the fixed point response of the system by neglecting or by putting the time barring terms to be 0. Also in addition to the fixed point response we can have this periodic response.

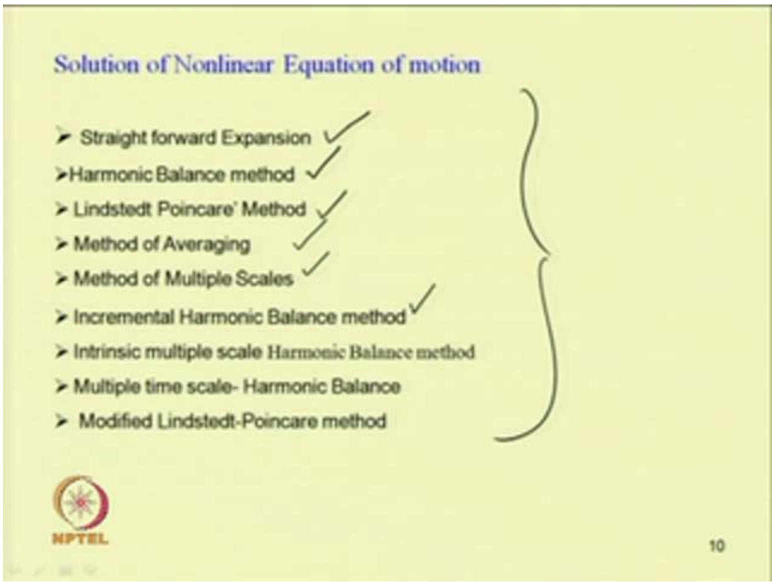
The response may be quasi periodic or the response may be chaotic. So, we have mainly four different type of response. So, first one is the fixed point response, second one we have a periodically varying system or periodic response, third one also we have this quasi periodic response and finally, we have this chaotic response. In case of fixed point response also, we can have a trivial response or non trivial response.

(Refer Slide Time: 06:43)



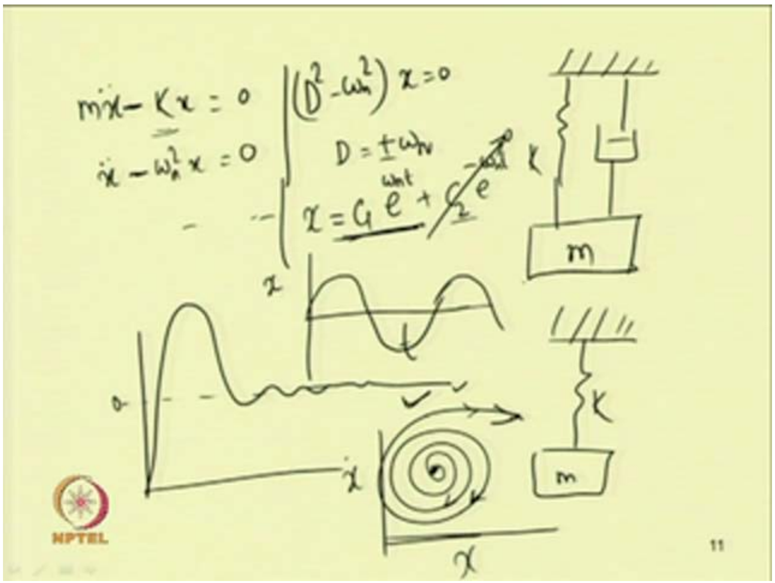
So, for example, in this equation x double dot minus x equal to 0. So, let us take this equation. So, in this case the solution so, if we put this x double dot equal to 0. So, what we can find is x equal to 0 is the equilibrium point. So, corresponding to this x equal to 0 if we find what will be the response of the system one can see that the system response may grow at this point or the system may be unstable.

(Refer Slide Time: 07:21)



So, first we should understand what we mean by the stability of a system, what is a stable system, what is unstable system. Then, we can perform the analysis or we can discuss their types of stability and types of bifurcation in this system. So, previously we have studied all this methods that is, straight forward expansion method, after qualitative analysis these are this quantitative analysis we perform. So, here we have used some perturbation technique. So, in these cases we have studied the straight forwarded expansion then, Lindstedt Poincare method then, harmonic balance method, method of averaging, method of multiple scale, incremental harmonic balance method, intrinsic multiple scale harmonic balance method, they are this multiple time scale harmonic balance method and modified Lindstedt Poincare method. So, in all these cases or in most of the cases except this harmonic balance method we reduced our second order differential equation to a set of first order differential equation and by solving those equations we obtain the steady state response of the system.

(Refer Slide Time: 08:35)



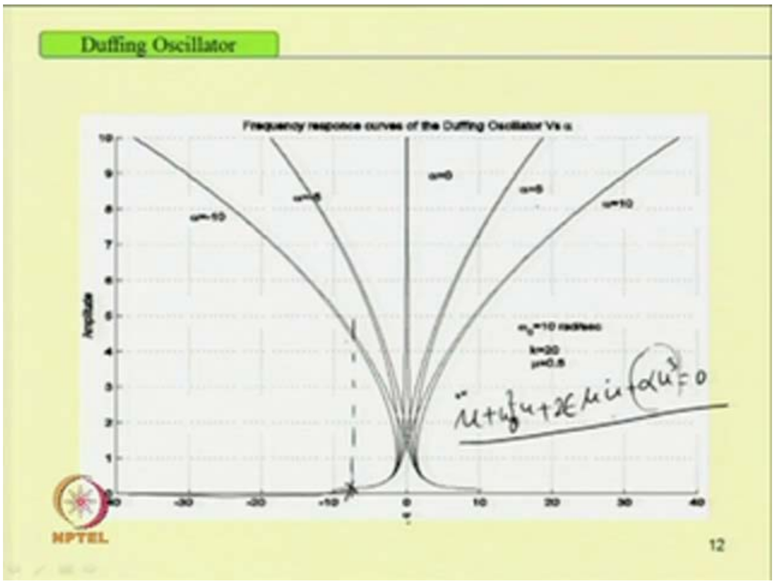
So, in those steady state responses now, we should know what we mean by a stable response or what we mean by an unstable response. For example, take the simple spring mass damper system, a linear system. So, in this linear system or simple spring and mass, if you take simple spring instead of taking damper, the system will be very simple. So, the equation already we know this is m x double dot plus k x equal to 0. So, in this case this x double dot plus k y m so if will write omega n square x so, this is equal to 0 and the solution we know so, the solution becomes a sin omega t plus phi. So, where we can obtain this a and phi from the initial condition.

So, if I will plot this x verses time so, we can observe that the response is bounded. So, we have a sin curve so the response is bounded. So, if there is damping in the system then, in case if there is damping and in case of the under damped system, we know the system response can be represented like this. So, it will be it will decay and finally, becomes 0. So, in this case the system response becomes bounded so, this is in this case as it comes to the 0 responses or trivial response. So, we can tell this is trivial response when the response becomes 0 and if it has certain magnitude or amplitude of vibration then, we can tell that response to be non trivial.

So, in this case what we observed that the response is stable that means it is bounded. So, we can have a bounded value of the response. But, instead of taking this m x double dot plus k x so, if I will have a value of minus k that means if the, if I will take the system in this way that is minus x if I will take. So, in this case in this case this x double dot minus omega n square x so, minus omega n square x will be equal to 0 and will have a. So, in this case the auxiliary equation becomes d square minus omega n square x equal to 0 or the roots becomes our d becomes omega n square or d will be d square in terms omega n square or d equals to plus minus omega n. So, the solution becomes x will be equal to C 1 e to the power omega n t plus C 1 e to the power of minus omega n t. So, in this case this part that is C 2 e to the power C 2 e to the power omega n t. So, with t tends to infinity this part tends to 0. But, this part that is C 1 e to the power of omega n t which increases in t so, the response expontionly this response will grow.

So, if one plot the phase portrait in this case. Phase portrait that means if can plot this x versus x dot term so, starting from this so, starting from a very small initial condition so, with time one can see that this response will grow. That means so, if you perturb from the initial condition that means if we start from initial condition this then, in this case a system with negative damping in negative stiffness so, in this case the system response will grow and with time it will be very large. So, there will large amplitude of vibration or the system may become unstable. So, by a stable system we mean that our response amplitude will be bounded or within certain limit. And in case of unstable system so, we can observe that the system response grows and it becomes it becomes more and more with time so that means it is not bounded. So, if we are not getting a bounded solution so the system becomes unstable and if where we bounded solution then the system is stable. So, this is the simplest way 1 can understand what we mean by stability of a system. That means, if we have a bounded system bounded response then, our system is stable and if we have unbounded solution then, the system becomes unstable. So, for studying the stability of the systems there are several methods.

(Refer Slide Time: 13:53)

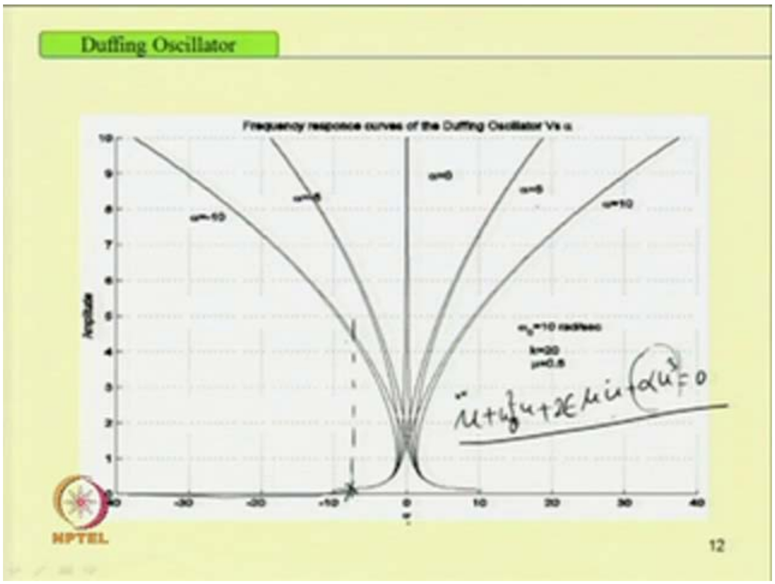


So, let us first see different responses of the system. For example, in case of the duffing equation with damping so, let us consider the duffing equation with damping that is it will take the equation in this form u double dot plus u double dot plus omega n square omega 0 square u plus 2 mu or 2 epsilon mu 2 epsilon mu u dot plus alpha u cube equal to 0. So, if we take this equation so, already we know the frequency amplitude relation in this case. So, unlike the linear system so, in case of the linear system that means if we put this alpha u cube so this non-linear term equal to 0. So, already we know in case of the linear system the response the steady state response of the free vibration will tends to 0. So, if this mu that is the damping term is very small that is if we have under damped system so, in case of under damped system will have a oscillatory motion and but, finally, the response will slow down to 0.

But, in case of this non-linear systems so, we already obtain 1 frequency amplitude relation so where sigma represent the determining parameter and this is the amplitude. So, the amplitude depends on the frequency of the system. So, with change in frequency of the systems we have observed that the amplitude of the system also changes. So, this figure shows the amplitude and determining parameter. So, for different value of alpha so, for alpha equal to plus and minus so, this is for plus 10 and this is for minus 10 so, alpha corresponding to plus value so this the system becomes or the system contains a hardening type of spring or hardening type of spring and in case of alpha equal to negative one can have a subtending type of spring. So, physically this alpha positive represents a hard spring and alpha negative represent a soft spring and in this case the response curve will become like this. So, this becomes alpha equal to 0 that is for a linear system and alpha equal to plus 5 so, this is for this non-linear system and with increase in non-linearity we have this different type of response.

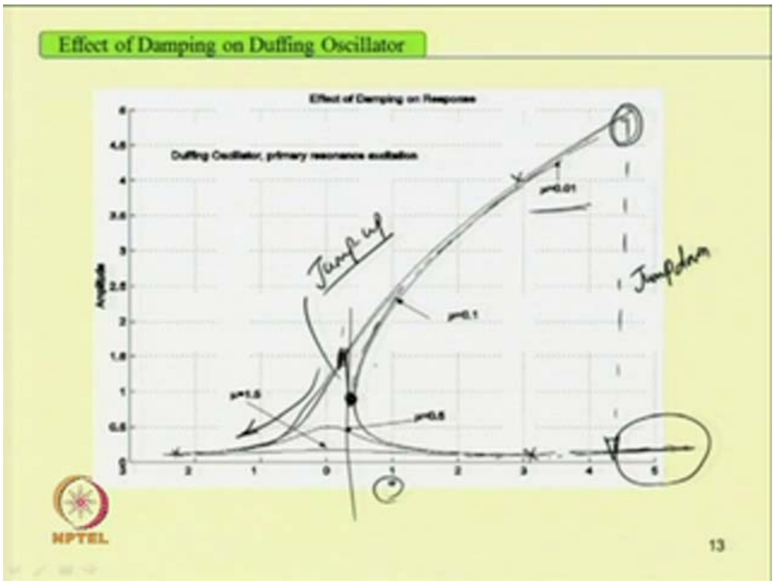
So, after getting this response curve one will be interested to know, what will happen to the system? So, if it is, if it is in the frequency if it is working or the working frequency let us take the working let the working frequency corresponding to determining parameter this. So, corresponding to determining parameter this is one we have observed that we have this is 1 solution, this is the second solution and this is the third solution. Unlike in case of the linear system so, for each and every frequency we have a single value of amplitude in this case we have multi solution. So, due to the presence of this multiple solution we should know actually at what will be the system response at that frequency. So, it depends actually on the initial conditions what we have taking. So, depending on the initial conditions so, either it will have this value or this value or this value. And whether this responses are stable or unstable so, one should study and then depending on the stable or unstable solution so, one can tells that in which state the system response will be there. So, to study this stability of the system we can take these points as the equilibrium points and perturb about these points to know whether the system response is growing or whether there is response becomes bounded.

(Refer Slide Time: 13:53)



So, if the system response grows then, in that case we have unstable system and if the system response becomes bounded then, we can have a stable system. So, if the response amplitude becomes 0 so, for example, we can have 0 amplitude so, if the system response becomes 0 then, this will be trivial state otherwise, this will be non trivial state. So, in case of the trivial state the response system response becomes 0 and in case of non trivial state the system response we have definite amplitude of system response. Both for the trivial state and for the non trivial state one should find this stability of the system. So, these curves show only the response. So, one can do further analysis to find whether the response is stable or unstable.

(Refer Slide Time: 19:17)



As we have changed this alpha value and we obtain different types of response similarly, one can change the value of damping or by changing the value of damping one can find different response. For example, in case of damping equal to damping parameter mu equal to 0.01 so this is the curve and with increase in damping one can see the response amplitude decreases. So, in this case let us assume that this part of the response is unstable so, which we can later will verify that whether it is unstable or stable but, for the time being let us assume that these part of the response is unstable.

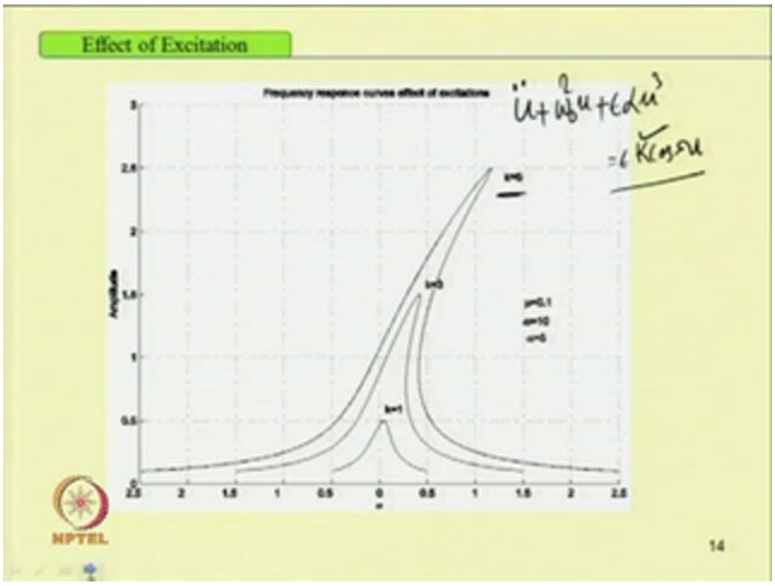
So, in this case with increase in this determining parameter or with increase in this frequency so, starting the system response at this point the system response will grow with increase in frequency and it will attain a maximum value after which further increase in this frequency will bring the system response down. That means will have a jumped down phenomena here. So, already in the module 1, we know different types of phenomena associated with this non-linear system. So, this becomes or the system response becomes so, system response from this point it will jump down to the system other type of response. Now, if we sweep down the frequency let us start at this frequency, if we sweep down then, it will follow this path this will follow this path up to this position and at this position as this branch at this position with further increase in frequency so, it will jump up to this position and with further increase in frequency it will follow this path.

So, it will follow this path with further increase in frequency. So, if we sweep up the frequency then it goes on increasing and finally, it jump down to this response. And it jump down to that response and again with decrease in or sweeping down the frequency, sweeping down the frequency at this point it becomes it moves up or jump up so, this phenomena is the jump up phenomena. So, it jump up to this response and further decrease in frequency it will move or follow this path. So, in this case these branch so, we have assumed these branch to be unstable that means the system will never attain this response so, either the response if this part and this part are stable so, either the system will have a stable response here and here and corresponding to this point it will have a unstable response. So, depending on the type of response so, here what we observe that further increase in this position so, it get jump down and after this we have only one solution and up to this so, up to this point so, we have a single solution and after this point after this point we have so, we have seen that we have three solutions but out of which two solutions are stable.

So, we can tell up to these the system is a single stable state, after these it has a bi stable region or the system has 2 stable region and 1 unstable region so, system has a bi stable region. In many cases also the system may have multiple stable regions and depending on the initial condition we can tell to which state the system can go. So, at this point so, this is a critical point after which this point is a critical point so, after which we have a multiple solution up to this we have only single solution and here we have change in this number of the solution. So, this point where the nature or the number of the response of the system changes so, those points are known as bifurcation point. So, this point at this point we have a bifurcation point this means that at this critical frequency or after this critical frequency the nature of the solution or nature of the response of the system changes or we have the number of solutions of the system changes.

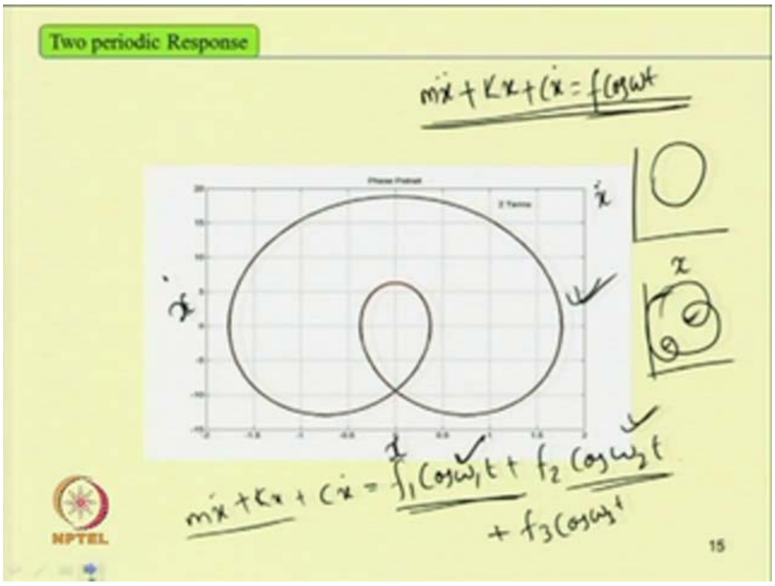
So, here also, this will be a critical point, after which we have only a single solution so, up to this we have multiple solutions and after this frequency we have single solution. So, in this way we know so, at bifurcation point either the stability of the system changes or the number of solutions or the number of responses changes. So, either qualitatively or quantitatively the stability of the response changes at bifurcation points.

(Refer Slide Time: 24:47)



So similarly, one can change different value of this k and one can find the response. So, those things will study in detail. So, this is a forced equation that is the forced duffing equation, equation in the form so, u double dot plus omega 0 square u plus epsilon alpha u cube equal to epsilon K cos omega t. So, in this case by changing the amplitude of the response also one can get different type of response. So, after getting the response one can study the stability of these responses. So, these responses what we obtain by finding this study state response so, these response either it has a fixed 0 value or it has a fixed non trivial value. So, that means the system will have certain definite amplitude of excitation. So, if the amplitude of excitation is fixed then, we can tell the response to be a fixed point response.

(Refer Slide Time: 26:05)

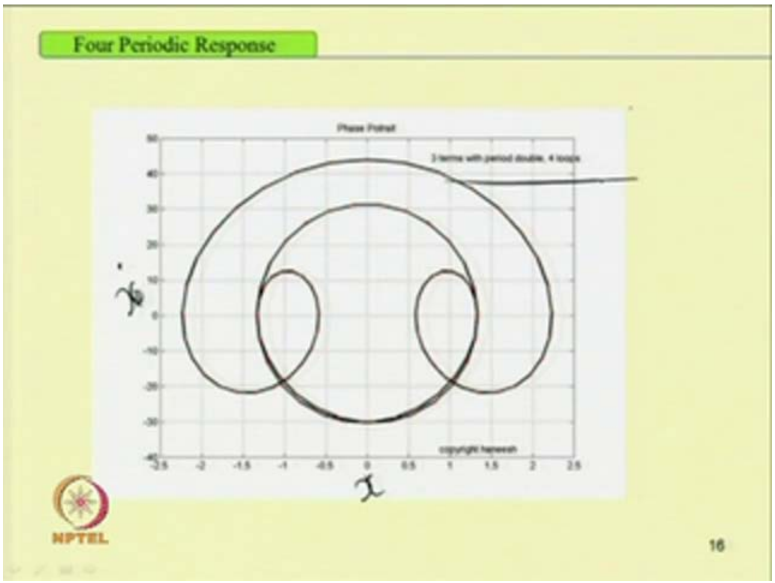


So, in addition to this fixed point response in most of the cases we have this periodic response. For example, in case of the spring mass system m x double dot plus k x plus c x dot equal to f cos omega t so, in this case we have a periodic response. So, if we plot phase portrait of the response that is x verses x dot then we have a close curve for this periodic response. So, now if will increase let us instead of taking this f cos omega t so, let us take 2 frequency so, m x double dot plus k x plus c x dot so, which is equal to f 1 cos omega 1 t plus f 2 cos omega 2 t.

Now, by applying the superposition theory so, we know that we can have 2 periodic responses. So, will have response with 2 periods so, one corresponding to this omega 1 and second one corresponding to this omega 2. So, if one plot this phase portrait that is x verses x dot so, in this case one can obtain 2 loops. So, by plotting this phase portraits so, one can know so, whether the response is 2 periodic or is having a single period. Similarly, by putting more number of terms forcing frequency that is, f 3 cos omega 3 t so, in this case we can obtain a response similar to this so, that will be so, it will contain 3 frequency. So, will have so, will have outside loop. So, this is the second loop and this is the third loop.

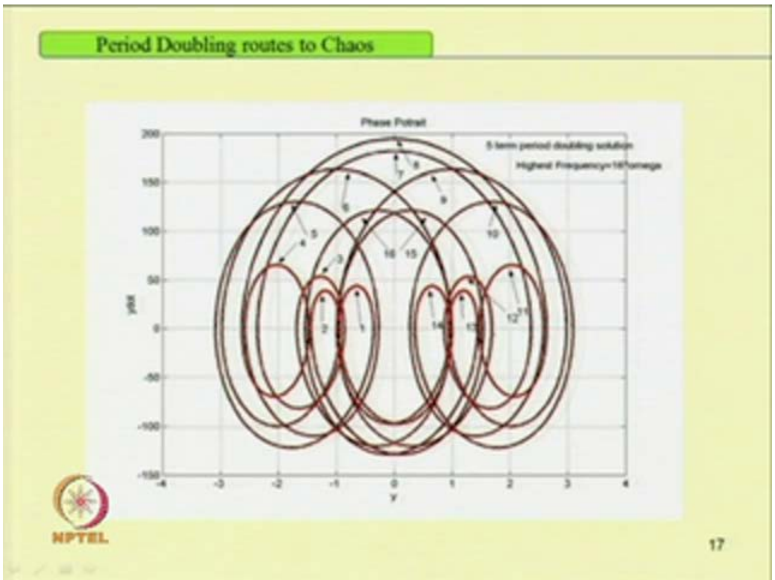
So, in case of these three frequencies if we have three period then, will have this 3. So, in this way 1 can go and increasing the number of term forcing term and in case of a linear system also one can obtain so, in case of the linear system also 1 can obtain a response with semi period with 2 periods or with multi period. Similarly, in case of the non-linear system also, for different resonance condition with a single so, in case of a single frequency excitation also, one can get periodic or 2 periodic or multiple periodic solution.

(Refer Slide Time: 28:46)

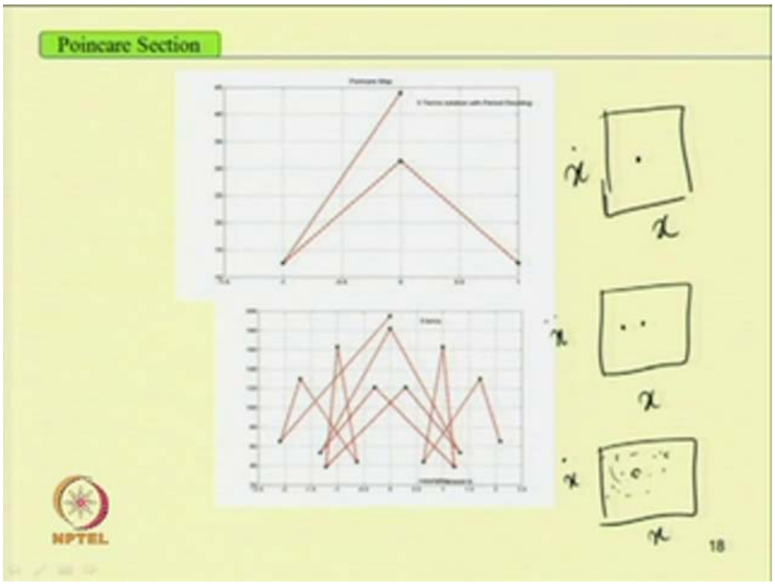


And so, this shows the number of periods increase that is, if we plot this x verses x dot so, in case of a number of periods increase. So, 3 terms with period doubling. So, we have shown this is 4 loops are there.

(Refer Slide Time: 29:05)



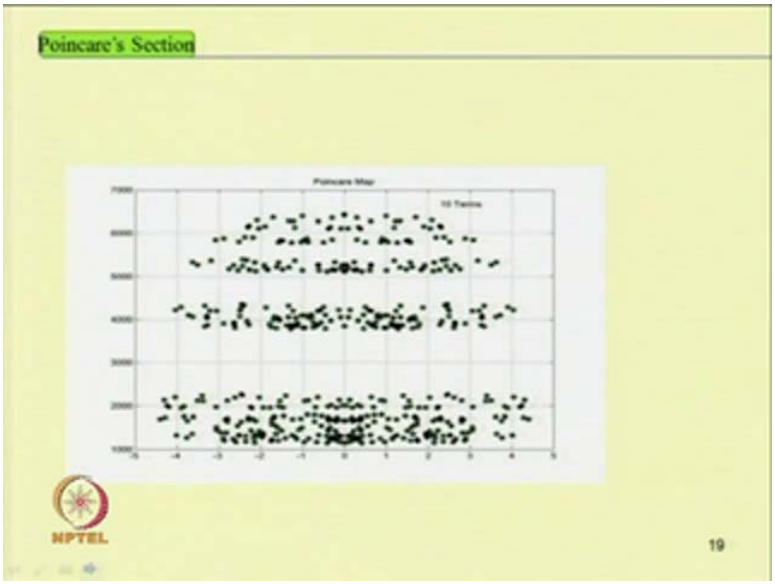
(Refer Slide Time: 29:21)



And this response shows a chaotic response. We have a number of periods available in this so, as we are taking a number of periods but the response becomes chaotic. So, in case of the chaotic response one can plot the Poincare section, which will study later by taking the Poincare section of the system.

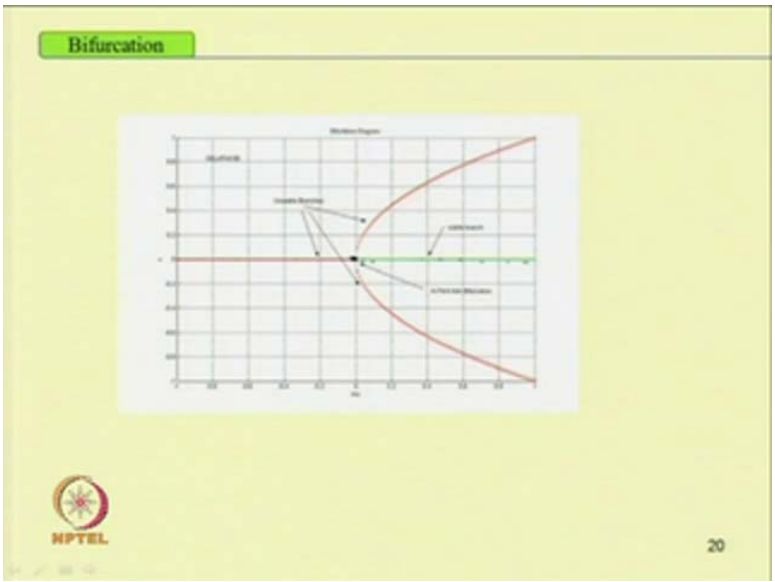
So, in this phase portrait one can know whether the system is periodic, chaos periodic or chaotic. So, in case of the periodic system, in phase portrait if one plot will have a single so, this is x dot verses x. So, will have a single point so, in corresponding to 2 periods so, will have 2 dots. And in case of the chaotic response, it will plot the Poincare's. If we plot the Poincare section then, it may fill up the space; the whole space should be filled up with these dots. So, these dot at present Poincare section of the system.

(Refer Slide Time: 30:17)



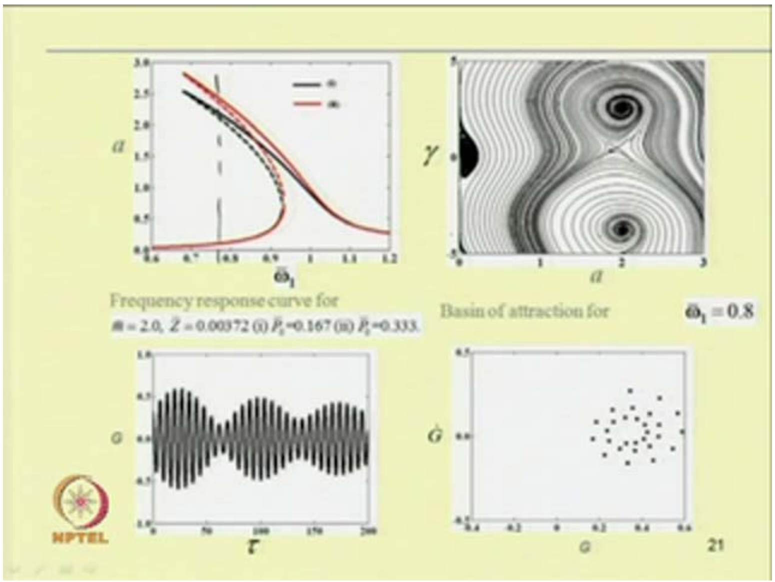
So, we will study about the Poincare section later in detail. So, this represents a chaotic solution.

(Refer Slide Time: 30:27)



Now, we know what we mean by a stable system, unstable system also we have seen some example of different responses and in these responses also we know what we mean by bifurcation. So, in case of bifurcation we mean either the stability of the system changes or the number of the response changes. That means, if we have a single branch before for example, in this case. So, initially we have a single branch and now we have two branch so, or we have three branches also. We may have 3 branch or 2 branch. So, at this point with increase in this frequency, we can see that number of branches changes. So, this point is critical or bifurcation point. So, will study what are the different type of bifurcation points for the fixed point response then, will study what are the bifurcation point for periodic response. So, first three classes will study about the bifurcation of the fixed point response.

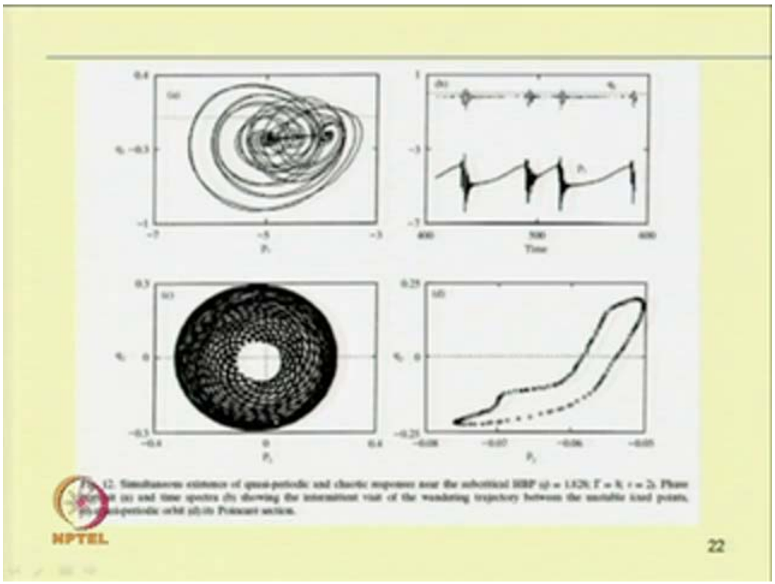
(Refer Slide Time: 31:41)



So, how to study those bifurcations? So, in today's class will briefly study and in case of the multiple solutions. For example, in this case we have multiple solution. So, in this multiple solutions already I told corresponding to a particular frequency we can know the solutions or we can find the response from the initial condition. So, different initial condition will lead to different response. So, by plotting those flows from the different initial condition the plot what we obtain is known as the basin of attraction. So, here this figure shows the basin of attraction of the system. So, in this case it may clearly be noted that this or these correspond to so, either one can have this negative value or positive value flow.

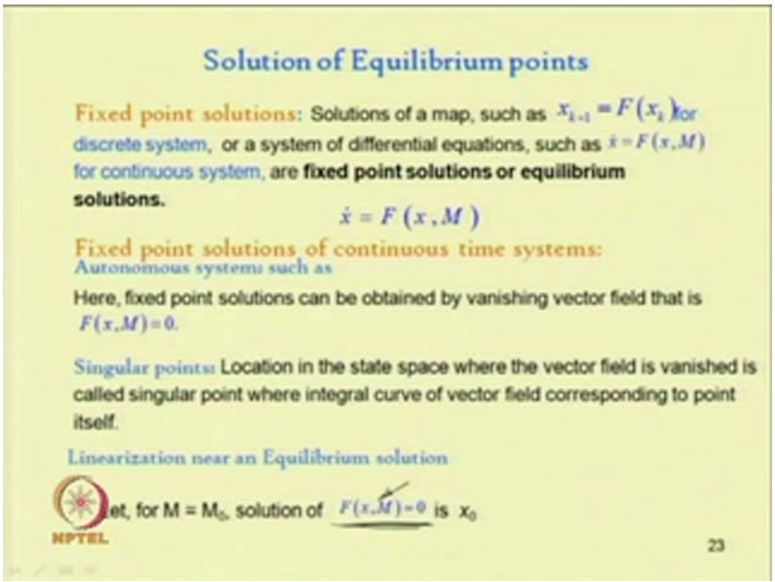
So, corresponding to a equal to 2 we have 1 equilibrium position, a equal to 2 and gamma equal to so, corresponding this value of gamma. Similarly, this is also equilibrium point which is a equal to 2 and also one can see that this point this is also 1 equilibrium point but this equilibrium point is unstable. This is 1 unstable equilibrium point that is that corresponding to the saddle node point and these are centre point. So, by plotting the basin of attraction so, we know what will be the response if one starts from 1 initial condition or initial point.

(Refer Slide Time: 33:21)



So, these are different type of quasi periodic response and this is cavity response. This is a quasi periodic response, because the Poincare shows Poincare response shows a close curve.

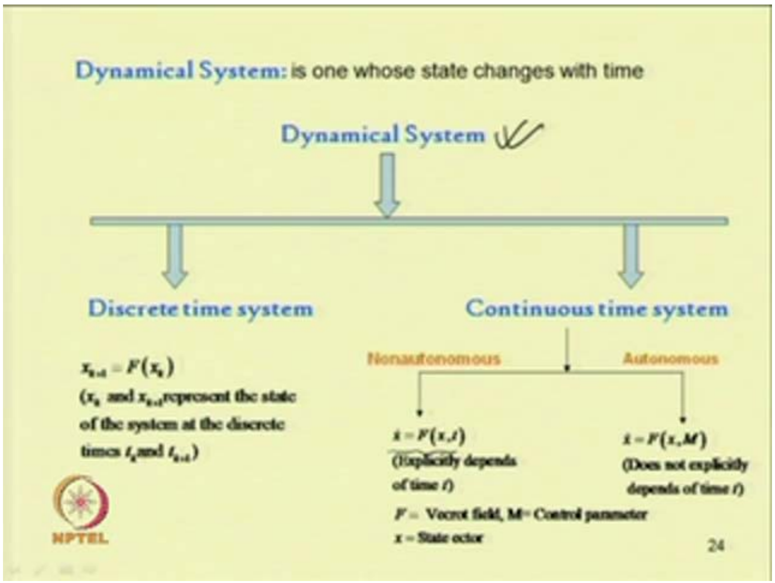
(Refer Slide Time: 33:36)



And now we see, what are the different types of solutions or equilibrium points, how we will find? Let us first take a first order equation so, will start with a first order equation and will find the response and it is stability and next class will study about the equilibrium of the higher order system or second order or higher order system.

So, today class will study about the first order system. So, let us take or in general the first order system can be written in this form that is, x dot equal to x dot will be a function of x and a control parameter m. So, either we can have a autonomous system. So, in case of the autonomous system we can write this f x m equal to 0. So, in this case the time terms will not be explicitly present in the system. So, this f x m equal to 0 so, for the fixed point response we can find the fixed point response by putting these x dot equal to 0 and by solving this equation that is, f x m equal to 0. So, to find the singular point location in the state space where the vector field vanished is called the singular point and in this case by changing that is f x m equal to 0 so, if you change this system parameter that is control parameter m so, we can obtain the response of the system.

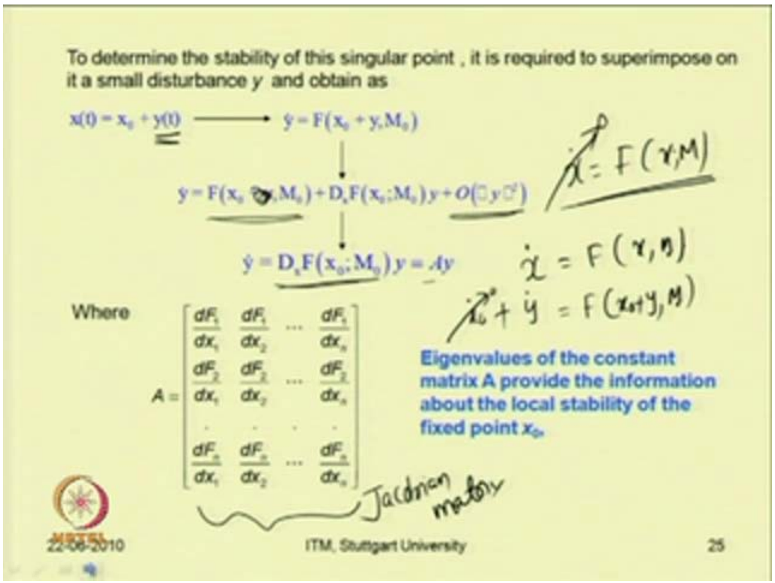
(Refer Slide Time: 35:11)



So, in case of the dynamical system so, either we can have a discrete time system in which will obtain the k plus finite state from the k h state. And we can have a continuous time system. So, in case of continuous time system so, we can have this non autonomous system or autonomous system.

So, in case of the non autonomous system so, the time curve is explicitly or this x term depends explicitly on time and in case of this autonomous system this time term will not come explicitly in this equation or we can have this x dot equal to f x m control parameter only and in case of non autonomous system we have the equation x dot equal to f x t. So, f is the vector field, m is the control parameter and x is the state vector. So, for a dynamic system so, we can divide it in to discreet time system or continuous time system or in discreet time system we can obtain the k plus finite state from the k eth state. In case of the continuous system, we obtain or we write the equation in this form that is, x dot equal to f x m, where m is the control parameter and this f does not explicitly depend on time.

(Refer Slide Time: 36:41)



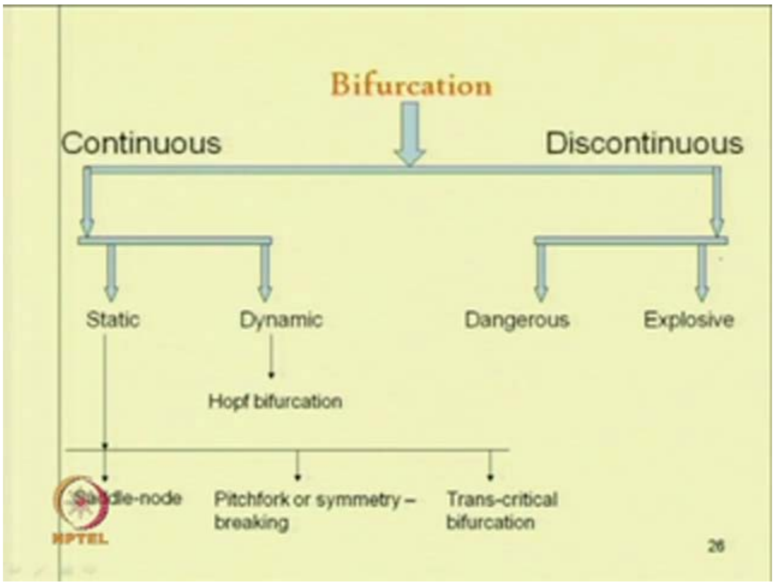
So, now to study the stability initially, I told you that we can perturb or we can change the state slightly and check whether it is the response are growing or the response is bounded. So, initially by taking so, let x 0 represent so, in our equation x dot equal to f x m. So, let us take that x 0 is the study state response or equilibrium response what we obtain by substituting this x dot equal to 0. So, by solving this f x m equal to 0 so, the solution what we got let it is x 0. So, if we obtain the solution to be x 0 then, by perturbing it that means slightly changing its value. So, let us assume that at time slightly different time that is x t so, the response becomes x 0 plus y t.

Now, this y dot will be equal to f x 0. So, it will be now we have to perturb this equation by perturbing this equation now we can obtain so, we can substitute these y dot equal to f so, in place of this let us our equation. So, we have taken a equation in this form x dot equal to f x t. Now, we are substituting this equation this y t is the perturb part. So, in our original equation that is x dot equal to f x m so, now you substitute this x t equal to x 0 plus y t.

So, now by differentiating this as x 0 is constant so this part becomes 0. So, will have x 0 dot which is constant which is 0 so, we can have this y dot so, y dot so, this x 0 dot is 0 so, plus y dot so, this will be equal to f. So, for x i will substitute it x 0 plus y and it will be let the control parameter becomes m. So, our m 0 I can put the control parameter so, in this way I can obtain this equation by perturbing this that means my equation becomes y dot equal to f x 0 plus y.

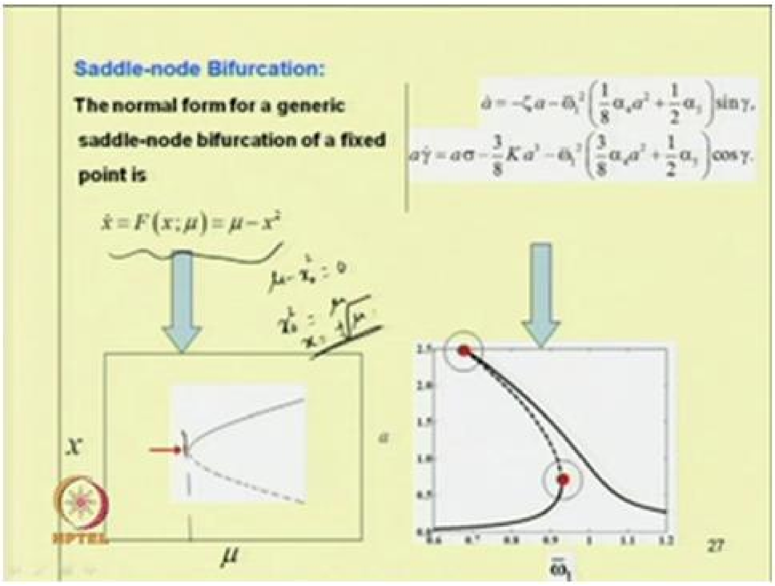
So, by applying the Taylor series or I can expand this equation applying Taylor series. So, I can write this y dot equal to f x 0 plus y m 0 plus f actually, I can write this is f x 0, m 0 plus d f x, x 0, m 0 into y plus higher order terms. So, as this part that is our f x 0, m equal to 0 for steady state so, we obtain this equation y dot that is the perturb part the rate of perturb part that is y dot. So, y dot equal to d x f x 0 m y or I can write this term in matrix form. So, I can write this y dot equal to a y where, A is this matrix which is known as the Jacobean matrix so, this part is written this is known as Jacobean matrix.

(Refer Slide Time: 40:11)



So, in this Jacobean matrix by finding the Eigen value of the Jacobean matrix we can study the stability of the system. So, here is a classification of the bifurcation of the system. So, in case of bifurcation either we can have this continuous bifurcation or a discontinuous bifurcation. So, in case of continuous bifurcation it can be static bifurcation or it can be dynamic bifurcation. In case of discontinuous bifurcation either we can have a dangerous bifurcation or explosive bifurcation. In case of static bifurcation we can have this saddle node bifurcation pitchfork or symmetry breaking bifurcation or trans-critical bifurcation. And in case of this discontinuous bifurcation we can have the sub critical bifurcation. So, the sub critical bifurcation may be either sub critical pitch fork or so, will study sub critical pitch fork, sub critical type of hopf bifurcations with those things will study later.

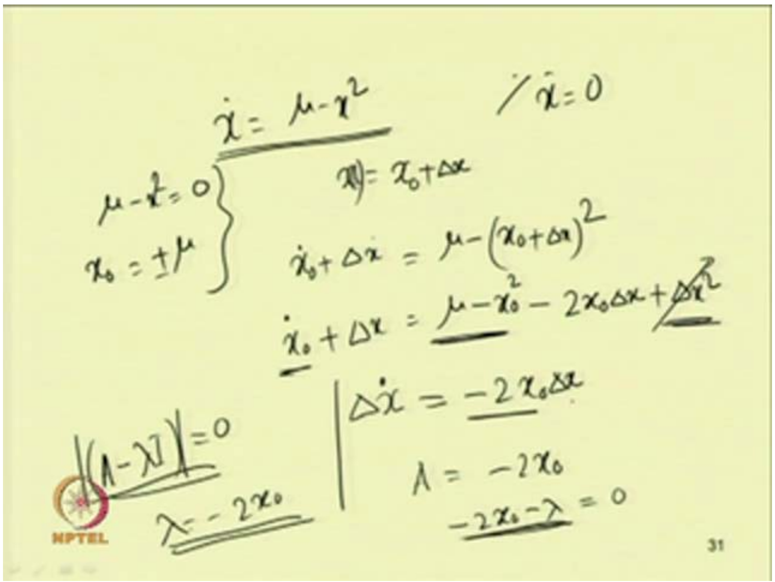
(Refer Slide Time: 41:18)



And so, let us see this equation for example, let us take this equation and study its bifurcation. So, let x dot equal to f mu x mu which is equal to mu minus x square so, in this case by substituting this x dot equal to 0 so, I have 1 equation that is mu minus x square equal to 0. So, if mu minus x square equal to 0 or mu minus x 0 square equal to 0 so, I can have this x 0 square equal to mu or x equal to plus minus mu root mu. So, the steady state solution becomes x equal to plus minus mu. So, by changing this control parameter mu for example, if I will take mu negative so, if I will take mu negative then, so, this will yield imaginary roots or imaginary value so, which is not possible. So, will have so, we do not have any response up to this and then at mu equal to 0 so, this x becomes 0. So, at mu equal to 0, x becomes 0. But, for mu greater than 0 so, for mu greater than 0 we have 2 solutions.

So, corresponding to mu equal to 0, we have the response equal to 0. But, with slight increase in mu so, one can see that we have 2 responses. So, as the number of the response changes so, this point is a bifurcation point. So, we can have this plus root mu or we can have minus root mu. So, this is the value of x corresponding to different value of mu. Now, we have to see whether this part is stable or unstable.

(Refer Slide Time: 43:20)

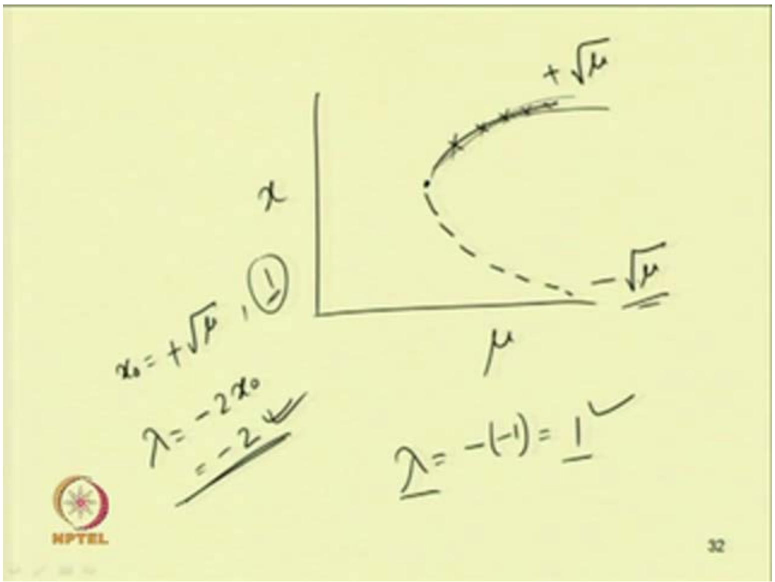


So, to find the stability of the system so, let us take. So, for example, in this case our equation is x dot equal to mu minus x square. So, by substituting this x dot equal to x dot equal to 0 so, we have written so, corresponding to x dot equal to 0 so, our equation becomes mu minus x square equal to 0 or we have this x 0 equal to plus minus mu. Now, we have to let, let us take this x so, we are perturbing this thing so, let us put this x equal to x t with time. So, we have started with x 0 so, x 0 plus delta x.

So, we have this x t equal to x 0 plus delta x. So, in this equation if I will substitute this then I can have this. So, x 0 dot plus delta x dot will be equal to mu minus x 0 plus delta x whole square or I can write this equation x 0 dot already I know that this x 0 dot for steady state, let me write first and then I will tell you this mu minus so, this by expanding this thing so, this becomes mu minus x 0 square minus 2 x 0 into delta x plus delta x square but, this x 0 dot corresponding to the equilibrium position x 0 dot. So, x 0 dot minus so, equal to mu minus x 0 square so, this becomes 0. So, our remaining terms becomes delta x equal to minus 2 x 0 delta x so, in this case we have neglected the terms as delta x is very small so, delta x square will be further small so, we can neglect this thing. So, we obtain this delta x equal to minus 2 x 0 delta x. So, this way one can find this delta x that is delta x dot equal to minus 2 x 0 delta x. So, our Jacobean matrix becomes so, our Jacobean matrix so, in this case this Jacobean matrix becomes this only. So, that is our A becomes minus 2 x 0.

So, we have to find the Eigen value of the Jacobean matrix. So, to find that thing so, we should write this a minus lambda I or determinant of a minus lambda I equal to 0. But, in this first order equation so, we can write as this A equal to this so, minus 2 x 0 so, minus lambda so, this becomes equal to 0. So, from this we obtain our lambda equal to minus 2 x 0 as lambda becomes minus 2 x 0 now, we have seen that we have 2 solutions.

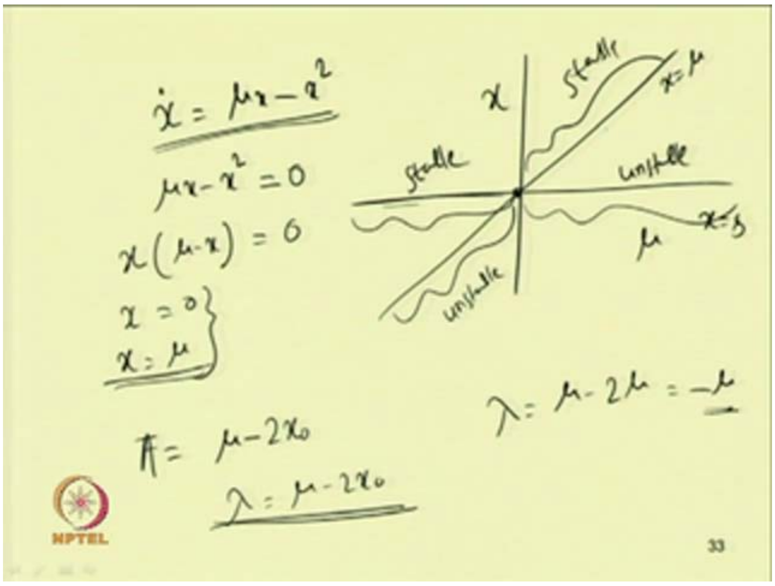
(Refer Slide Time: 46:20)



So, 1 solution so, this is mu, this is x. So, we have this two solution. So, corresponding to this point this is x equal to 0. So, now let us take this is root over plus mu. So, this is plus root mu and this becomes root over minus root mu. So, in this case as our Eigen value so, corresponding to x 0 equal to so, corresponding to x 0 equal to plus root mu. So, let us take mu equal to so, let us take this plus 1 value so, let this is plus 1. So, our Eigen value lambda becomes so, already lambda we have written so, lambda equal to minus 2 x 0, minus 2 x 0 so, this so, corresponding so, if this 1 that is positive then, this becomes minus 2. So, as lambda becomes minus 2 or negative or the real part of the Eigen value in this case we are only real. So, as lambda becomes negative so, this response becomes stable that means will have a stable solution here. So, this is a stable solution stable.

And this part so, corresponding to our corresponding to our x equal to minus mu. So, if will substitute this thing. So, in that case our lambda value will be equal to minus. Let me take this value equal to minus 1 so, if this is minus 1 so, minus of minus 1 this becomes 1. So, our lambda value becomes 1 that is positive.

(Refer Slide Time: 48:31)



So, this branch so, instead of a stable branch we should have unstable branch. So, this is unstable branch. So, this is stable and this branch becomes unstable. Similarly, we can take some other examples also. Let us take one more example so, in this case will take let us take one more example, in which will take this let us take x dot equal to mu x minus x square so, x dot equal to mu x minus x square. So, in this case for equilibrium point so, x dot becomes 0 so, our mu x minus x square equal to 0. So, for equilibrium position we can find this way so, if I will take common x so, this becomes mu minus x equal to 0.

So, our either x equal to 0 or x equal to mu. So, we have 2 solutions. So, one is x equal to 0 and other one is x equal to mu. So, in this case I can plot the response. So, if I will this x verses mu so, I will have two solution so, x equal to 0 is 1 solution. So, this line x equal to 0 so, this line represent x equal to 0 and then, we have x equal to mu so, x equal to mu so, we have another line so, this is x equal to mu and this is x equal to 0. So, we have 2 solutions. So, in this 2 solution we should know which part is stable and which part is unstable.

So, to find the stability of this thing so, we can so, we can perturb this equation and write so, either we can perturb this equation or from our previous analysis we have seen this Jacobean matrix can be obtained by finding the first derivative of this part. So, by finding the first derivate so, our derivative becomes so, we obtain this equal to mu minus. So, our Jacobean matrix that is A I can write so, A becomes mu minus 2 x so, this becomes mu minus 2 x. So, corresponding to the equilibrium position so, I can write this is equal to mu minus 2 x 0. So, our lambda will becomes mu minus 2 x 0.

So, corresponding to different value of mu so, for example, if I will take the trivial state that is x equal to 0 so, in this case lambda equal to so, x equal to 0 let me take so, if x equal to 0 so, this line so, if will take this line then lambda becomes mu. So, depending on the positive and negative value of mu for example, this to this. So, mu for negative mu so, this when mu becomes negative so, this lambda becomes negative that means this branch so, this to this branch we have a stable solution and corresponding to this branch so, this to this branch will have unstable solution. And coming back to this x equal to mu so, if I will substitute this equation in this so, lambda becomes so, mu minus 2 mu so, this becomes minus mu so, that means lambda equal to minus mu. So, for this part that is when mu equal to negative. So, when for this part so, we have in the previous case this is stable, this is unstable.

And now this is for negative value so, this becomes positive that is the real part becomes positive so that means this is unstable. So, this part is unstable and corresponding to this as this becomes positive negative so, this becomes stable. So, what we have seen? So, previously we have 2 solution 1 stable and 1 unstable so, after this point so, this is a critical point after which though the number of the solutions are not changing but, the type of solutions are changing. Here, a stable branch becomes unstable branch and this unstable branch becomes stable. So, this point is a bifurcation point and this type of bifurcation or this bifurcation is known as trans-critical bifurcation. So, in next class we will study more in more detail about the bifurcation of the fixed point response and we will solve some more examples on this.

Thank you.