**Non-Linear Vibration**
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**Module - 3**
**Solution of Nonlinear Equation of Motion**
**Lecture - 10**
# Modified and Extended Lindstedt Poincare Method

Welcome to today class of non-linear vibration. Today class will study about the modified Lindstedt Poincare method and also will summarize, whatever the methods we have learnt in this module.

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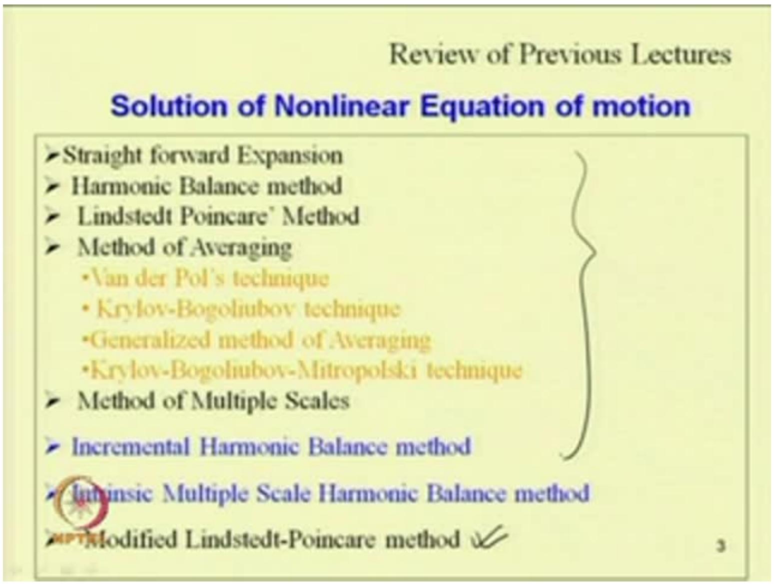


So, in this module already we have studied about the straight forward expansion method, harmonic balance method, Lindstedt Poincare method and method of averaging. So in case of method of averaging we know about this Van der Pol's technique, Krylov-Bogoliubov technique, then generalized method of averaging and Krylov-Bogoliubov-Mitropolski technique. And also we know about this method of multiple scales in which we have taken different time scales to solve the differential, non-linear differential equation of motion.

Also last class we have studied about this incremental harmonic balance method and intrinsic multiple scale harmonic balance method. So, today class, we are going to study about this modified Lindstedt Poincare method and though there are several other methods available, we will limit our study up to this modified Lindstedt Poincare method and then we will study about the stability of the obtain response. So, in case of the or in all these methods, what we have studied mostly these methods are used for weekly non-linear system. So, for systems with large non-linearity, so this Lindstedt Poincare technique has been modified and one can use this modified Lindstedt Poincare technique to solve those equations. So, in case of classical Lindstedt Poincare method.

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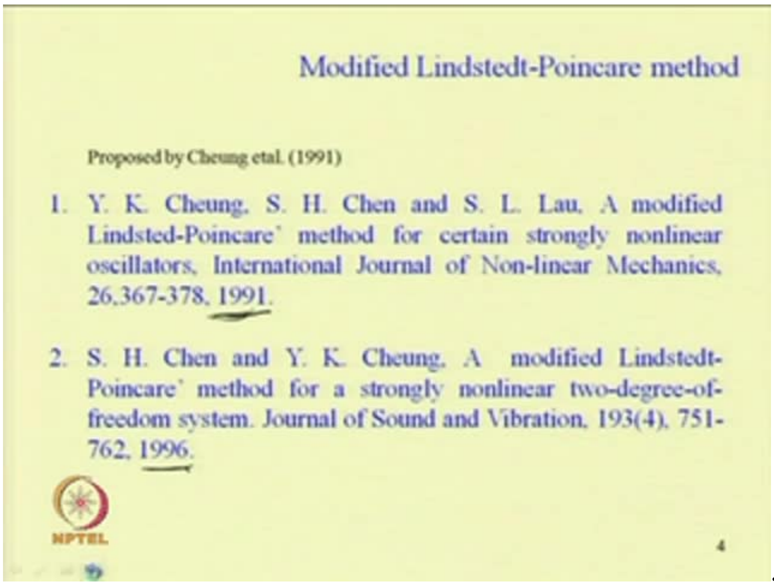


So, this modified Lindstedt Poincare technique was developed by this Cheung, Chen and Lau so in 1991. So, this method they have developed for a strongly non-linear oscillator also they have use this method for in another paper by Chen and Cheung, a modified Lindstedt Poincare method for a strongly non-linear two degree of freedom system. So, they have modified or they have used this modified Lindstedt Poincare method for a two degree of freedom system in this paper in 1996 and original paper. So, they have use this modified Lindstedt Poincare method in 1991, so first briefly we will see what is the Lindstedt Poincare method, and how one has to modify this method to solve this non-linear equations.

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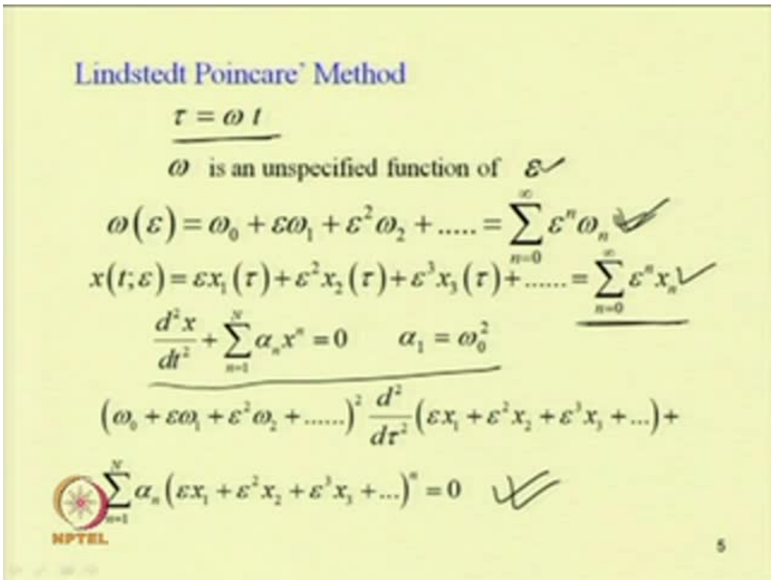


So, in case of classical Lindstedt Poincare method, so to solve a differential equation let us take this differential our differential equation motion this is the differential equation motion. That is d square x by d t square plus n equal to 1 to n alpha n x to the power n equal to 0, or if I will take up to let n equal to 2 then this equation can be written as d square x by d t square plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube equal to 0. So, here this first term that is alpha 1, so we can take this alpha 1 equal to omega square. So, alpha 1 is the coefficient of the first or the linear term, so that is d square x by d t square plus alpha 1 x.

So, this alpha 1 will be nothing but this is omega 0 square, so this is the omega square of the natural frequency of the system, or this part is the linear frequency linear so for the linear part if you take, so this is the frequency obtained in this case. So, when we are adding the non-linear terms to the system that is alpha 1 alpha 2 x square or alpha 3 x cube, so no longer this solution of the linear solution will be applicable so in that case one has to modify the equation.

So, in case of Lindstedt Poincare method so first what we did so we have taken a different time scale that is tau equal to omega t, where omega is an unspecified function of epsilon. So in this case we have taken this omega as an unspecified function of epsilon that means, this omega which is a function of epsilon is written as omega 0 plus epsilon omega 1 plus epsilon square omega 2, or in other word one can write this in this form that is n equal to 0 to infinity epsilon to the power n omega n, and again by taking this x t which is a function of or parameter epsilon, in this form. That is epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3, so it can be written in summation form in this way n equal to 0 to infinity epsilon to the power n x n.

So, now substituting this omega equation and this x equation in the original equation that is this d square x by d t square plus alpha 1 x plus alpha 2 x square equal to 0. So, one can get this equation, so in this equation now separating the terms with epsilon to the coefficient epsilon to the power 0 with epsilon to the power 1 and epsilon to the power 2, or higher order. So, one can get a set of equations and from those equations by.

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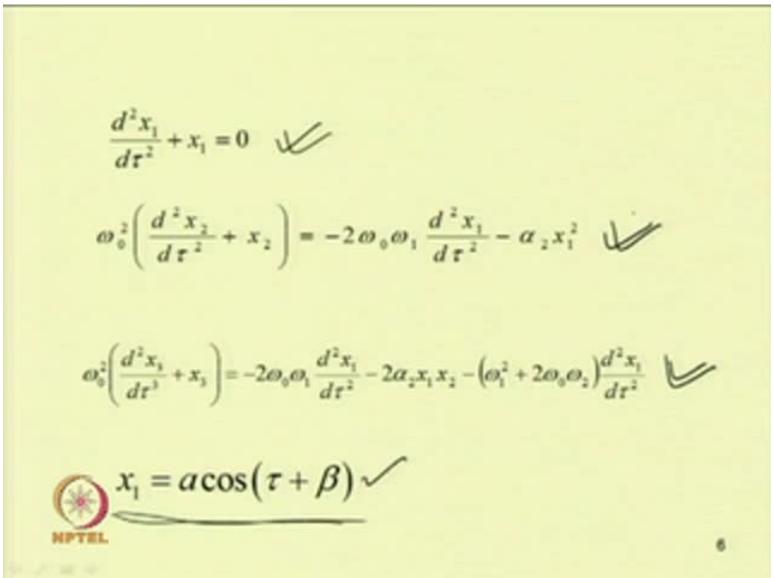


Separating or by killing the secular terms one can get the frequency equation. For example, in this case so already we have solved this problem. So, in this case the first equation can be written in this form that is d square x 1 by d tau square plus x 1 equal to 0 and second equation in this form, and third equation in this form. And then the solution of the first equation is x 1 equal to a cos tau plus beta by substituting this equation in the second equation.

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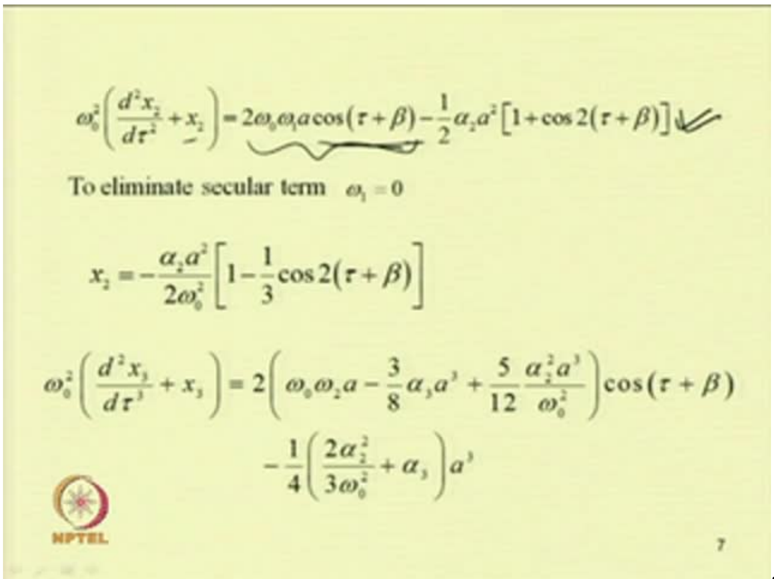


One can get this expression in which we can see that coefficient of this x 2 equal to 1, so the terms that is this cos tau plus beta the coefficient of this one, should be eliminated otherwise this will leads to a secular term. So, in this case so we have to put this omega 1 equal to 0 so that this term can be eliminated otherwise, this term will not be equal to 0. So, by eliminating the secular term we get.

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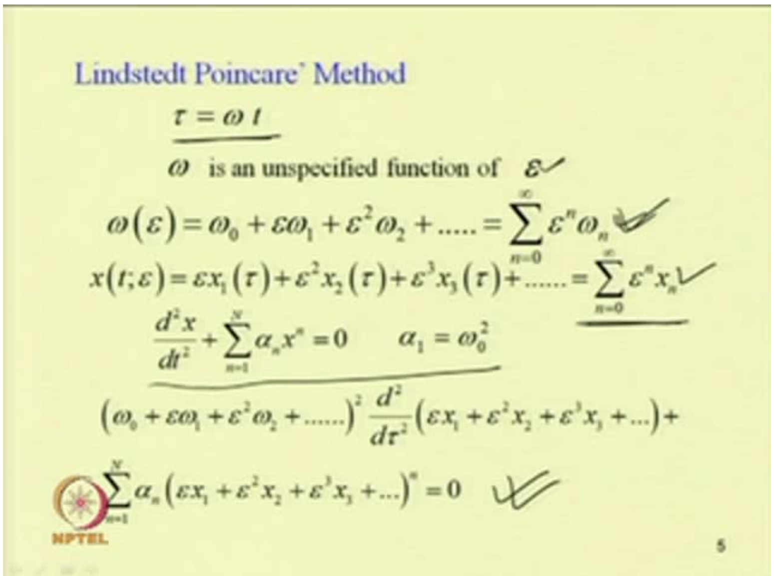

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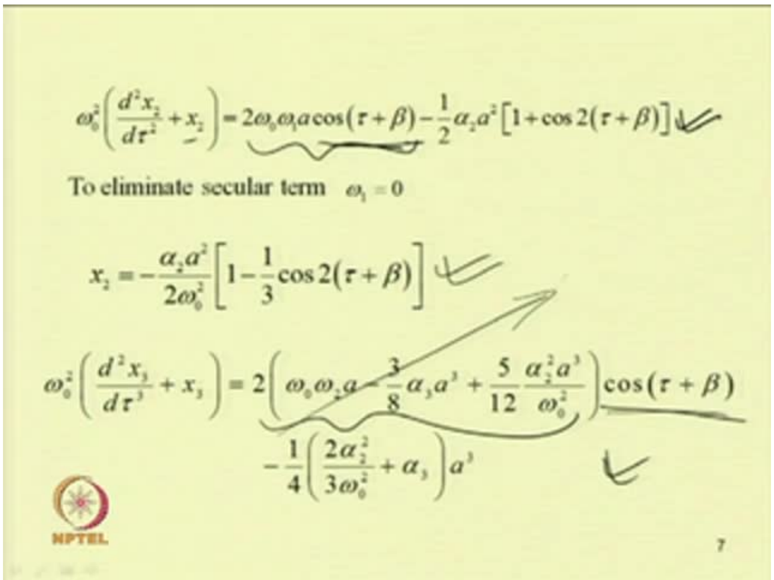


This functions that is this omega 1 omega 2 and higher order omega by eliminating the secular terms. So, in this case we got this omega 1 equal to 0 similarly, so by putting this omega 1 equal to 0, so we got the expression for x 2 by putting this expression for x 1 and x 2 in the third equation, we can get this expression. So, in this expression the coefficient of cos tau plus beta again we can take this one and this term has to be eliminated so that we can have, so we can have a solution which is bounded. Otherwise, this will leads to a unbounded solution. So, this term has to be 0 so by putting term equal to 0.

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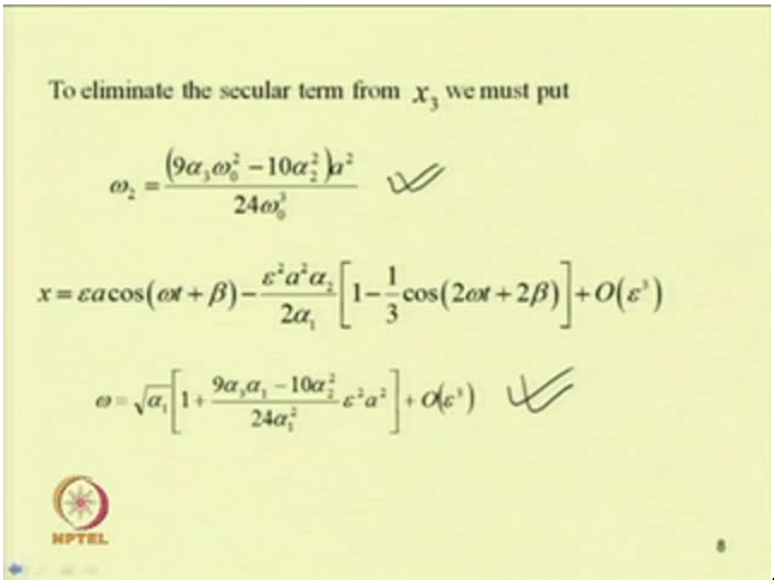

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So, you can get another frequency relation that is this omega 2 relation with respect to a. Now, we got omega 1 and omega 2 in terms of a and omega 0, so we can get this expression for x and also this frequency equation, but. So, this is the classical Lindstedt Poincare method which is in which, so we have modified this straight forward expansion method by substituting this or by taking into account.

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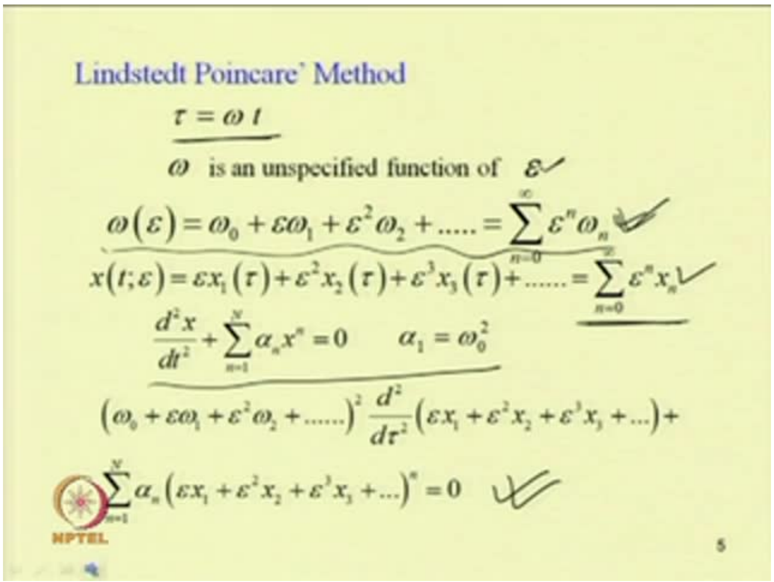


Taking into account taking, into account this omega term that is omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2, but in case of this Lindstedt modified Lindstedt Poincare technique. So, this modified Lindstedt Poincare techniques differs from this classical Poincare technique in four steps. So, in the four procedural step so in the first, so instead of expanding this in terms of omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2 or higher terms, so here the expansion is taking for this omega square. So, instead of expanding omega so here the expansion is carried for omega square.

This is for the simple design that when we substitute this tau equal to omega t, so the second order differential will yield or will give raise to a term omega square. So, instead of expanding this omega, so if one expand this omega square then the result it will give better result. So, in modified Lindstedt Poincare technique so one a expand omega square instead of expanding this omega. So, this is the first difference between this two method and the second difference method, second difference is that instead of using these so a new parameter alpha is also introduced in this method.

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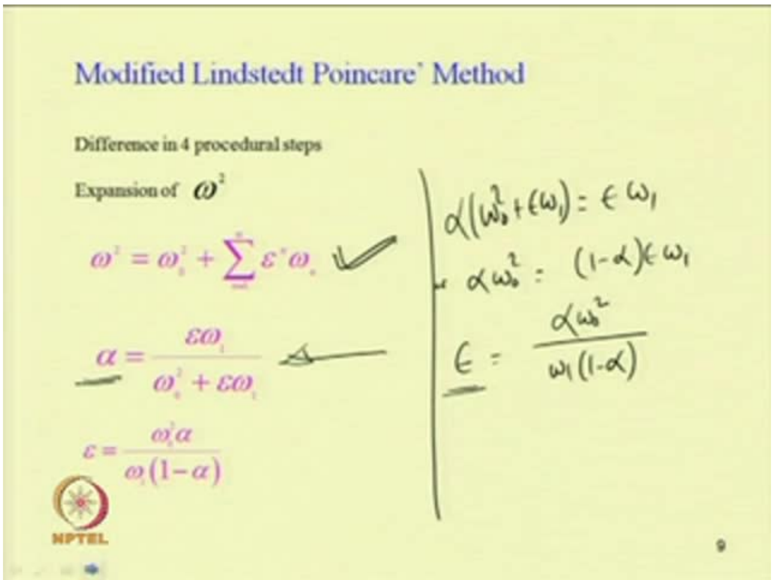


So, a new parameter alpha is introduced in this way so, this is the first difference; in the second step is that so a new parameter alpha is introduced here. So, this alpha equal to epsilon omega 1 plus omega 0 square plus epsilon omega 1. So, or one can so from this so introducing this alpha equal to epsilon omega 1 by omega 0 square plus epsilon omega 1, so one can find the expression for epsilon, so in this way, so it will be alpha into omega 0 square plus epsilon omega 1, so this is equal to epsilon omega 1 or this alpha omega 0 square equal to, so I can write this equal to by taking this 1 minus alpha into epsilon omega 1.

So taking this epsilon alpha omega 1 to write inside and taking epsilon omega 1 common, so one can write this way or one can write this epsilon equal to so one can write epsilon equal to alpha omega 0 square by alpha omega 0 square, so by omega 1 into 1 minus alpha. So, in a given differential equation so as we know this epsilon; so one can find this alpha so alpha can be obtained so from this expression for epsilon. So, it will be equal to or epsilon can be written in terms of alpha equal to so epsilon equal to alpha omega 0 square by omega 1 into 1 minus alpha.

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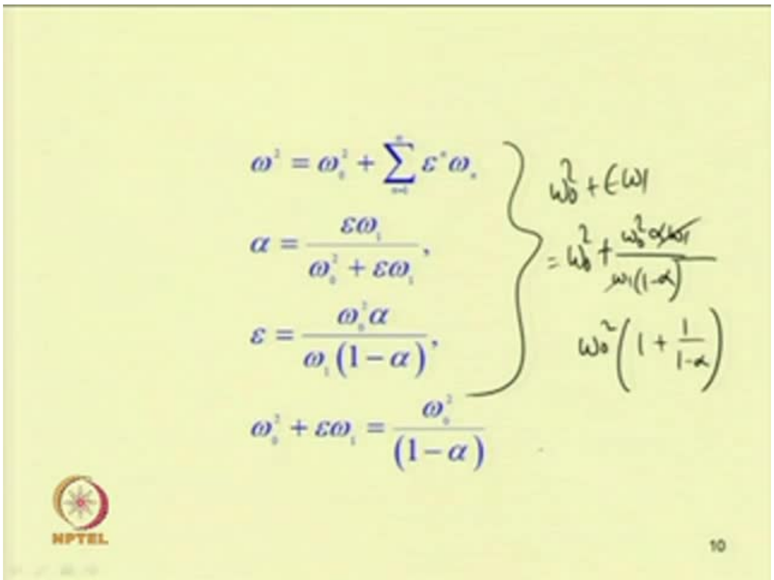


So, from this expression one can write this omega square now expanding this already, we have expanded this thing omega square equal to omega 0 square plus epsilon omega 1 plus epsilon square omega 2 plus epsilon square omega 3 in this way. So, in this case this omega 1 omega 2 omega 3 those terms, which will be function of omega, so we have to find this. So, now this omega square term we can write this way so omega square equal to omega 0 square already we have written.

This three terms epsilon equal to omega 0 square alpha by omega 1 into 1 minus alpha or this equation we can write, this omega 0 square plus epsilon omega 1 omega 0 square plus epsilon omega 1 that is the first term, so that thing can be written as omega 0 square. Already we have written omega 1 epsilon equal to alpha 0 omega 0 square alpha by omega 1 into 1 minus alpha.

So, epsilon omega 1 will be so omega 0 square plus epsilon omega 1 so can be written as omega 0 square plus for epsilon I will substitute this thing omega 0 square alpha by omega 1 into 1 minus alpha. Now, with multiplication of omega 1 omega 1 omega 1 cancel, so by taking this omega 0 square common, so this becomes omega 0 square into so this is 1 plus 1 by 1 minus alpha or this is equal to omega 0 square into. So, this is 1 minus alpha, so by substituting this thing we can find this expression.

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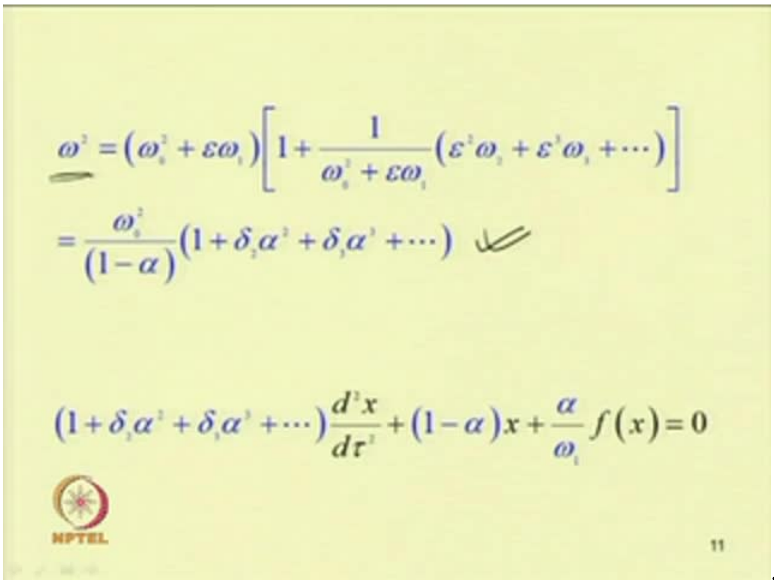


Then this omega square again can be written in this form, so omega square equal to omega 0 square plus epsilon omega 1 into 1 plus 1 by omega 0 square plus epsilon omega 1 into omega square epsilon square into omega 2 plus epsilon cube into omega 3 plus the higher order term, or this thing can be written by omega 0 square by 1 minus alpha into 1 plus delta 2 alpha square plus delta 3 alpha cube plus the higher order terms.

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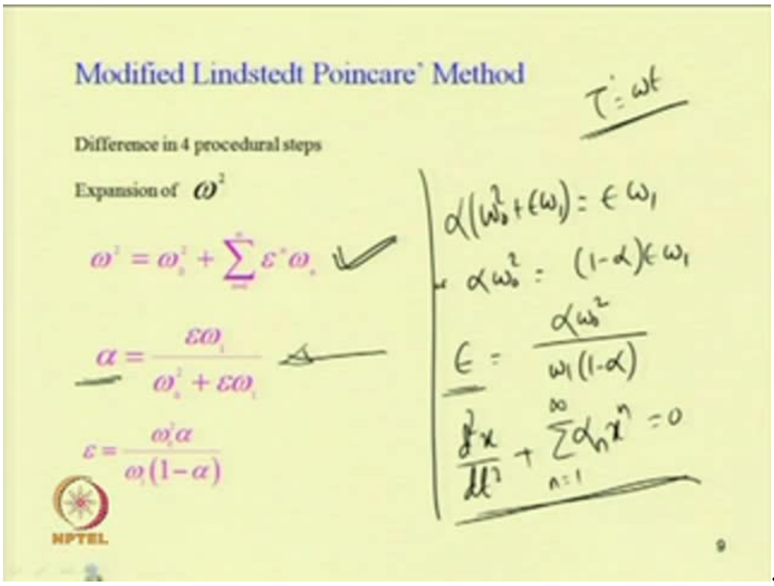


Now, substituting this expression for omega square, so in the original equation that is that is the governing equation. So in this governing equation that is your our d square x d square x by d t square plus summation alpha, n when x n, n equal to 1 to infinity equal to 0. So, in this equation cause substituting t equal to or tau equal to omega t, so by substituting this thing in this equation. So, it becomes omega square into d square x by d tau square plus alpha n x to the power n equal to 0, so again substituting this expression for omega square.

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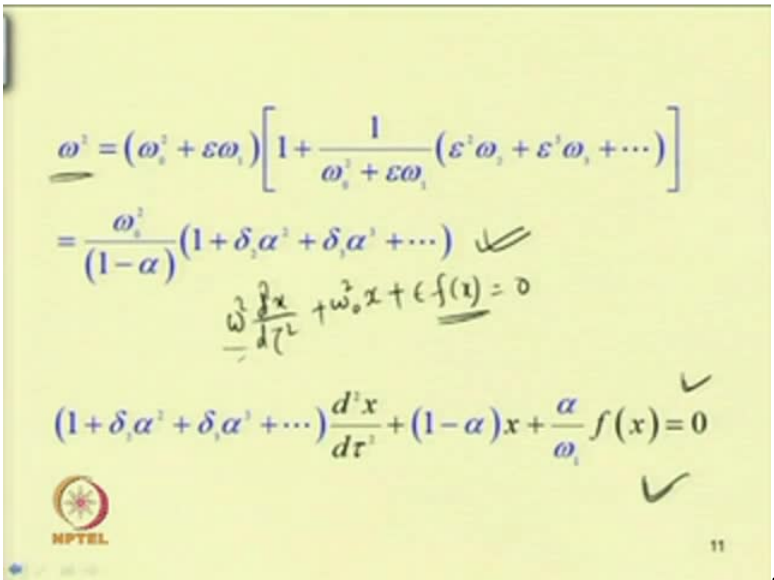


So, one can write this equation in this form, so or this equation so one can write that governing equation in this form that is omega square d square x by d tau square plus omega 0 square x. That is alpha 1 x plus a f u equal to so plus epsilon f x, the governing equation we can take in this way so plus epsilon f x equal to 0. So, this f x will contain this alpha 2 x square plus alpha 3 x cube or the higher order terms.

So, now substituting expression for omega square like this, so we can have this equation that is 1 plus delta 2 into alpha square plus delta 3 into alpha cube plus the higher order terms into d square x by d tau square plus 1 minus alpha x plus alpha by omega 1 f x equal to 0. So, the so this is the second step so second difference between the Lindstedt Poincare method. So, the first difference, so we have expanding instead of expanding omega here we are expanding this omega square term.

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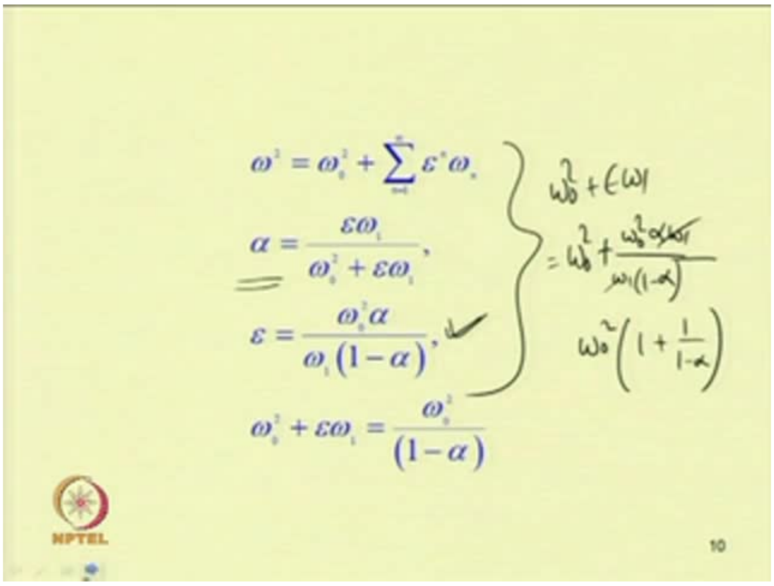

And the second difference is that we have introduced a term alpha which is equal to epsilon omega 1 by omega 0 square plus epsilon omega 1. So, from this we got this expression for epsilon and then we got.

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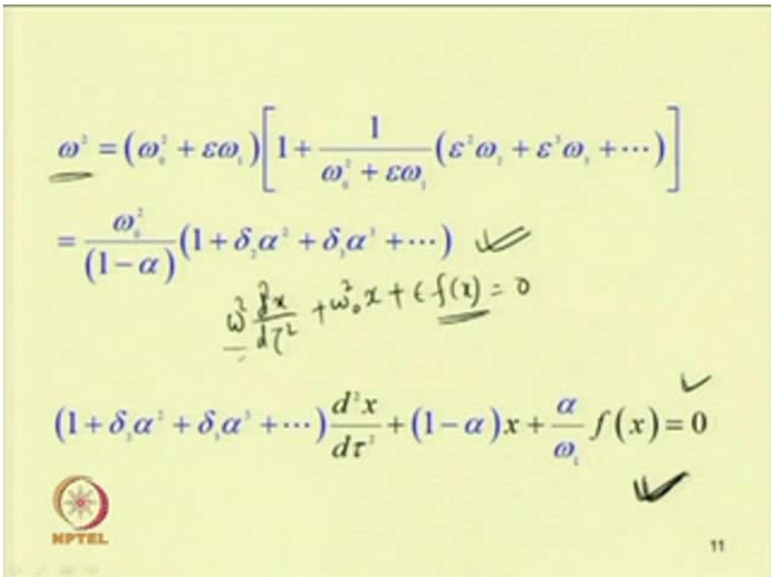
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The expression for omega square in this way and by substituting this expression for omega square, so we got this x expression.

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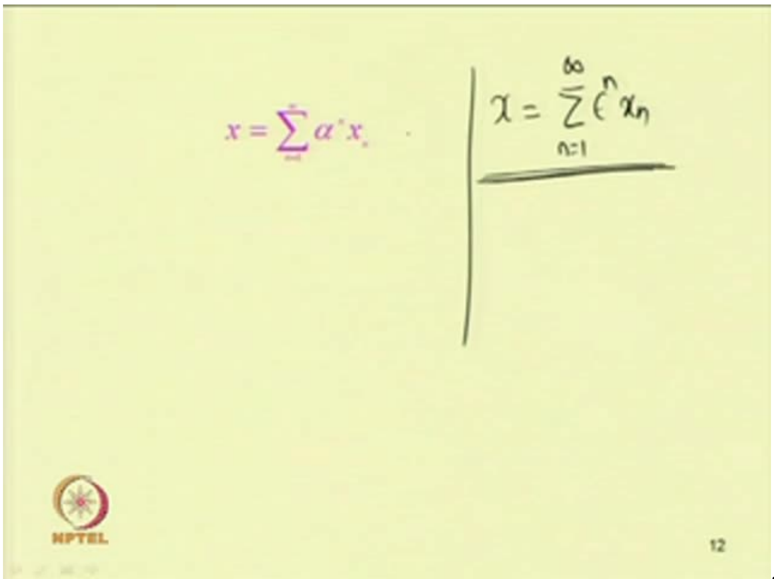


Now, the third difference is that instead of expanding this x so previously, we have written this x in this form so we have x equal to summation n equal to 1 to infinity epsilon to the power n x n. So, in the classical Lindstedt Poincare technique, so we have expanded this term x in terms of epsilon, but in this case instead of expanding in this term in terms of epsilon, we are taking this x equal to n equal to 1 to infinity alpha to the power n x. So, this will be equal to alpha 1 x 1 plus alpha 2 or alpha square x 2 plus alpha cube x 3. So, in this way we can write.

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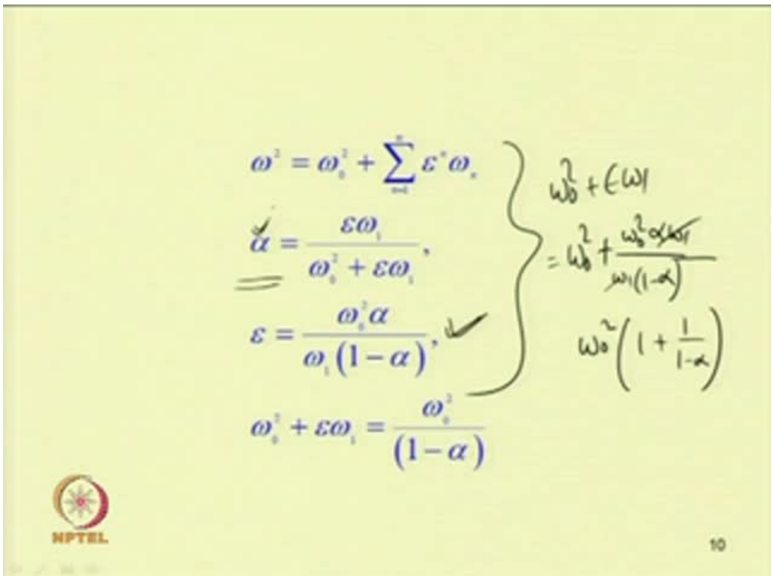

So, here it may be noted that this alpha equal to epsilon omega 1 by omega 0 square plus epsilon omega 1, so for higher value of this epsilon. So, that means when we have this high non-linear terms so in that case also one can obtain that this alpha term can be small very small.

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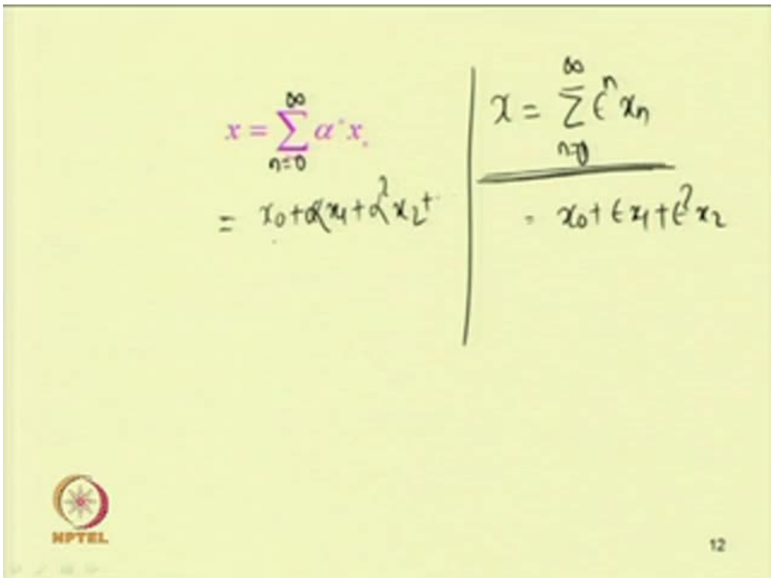


Now, expanding this equation so we can write this so we can first term, so we can put this n equal to so instead of putting x equal to this way, we can put x equal to so n equal to 0 to n equal to 0 to infinity so in that way. So, first term will be equal to x 0 so alpha to the power 0 equal to 1, so this becomes x 0 plus alpha x 1 plus alpha square x 2. So, let me take two terms only so if I will higher terms so then so, me higher order terms also will be there now. So, here we should take n equal to 0 to infinity in classical so that this term will be equal to x 0 plus epsilon x 1 plus epsilon square x 2, so in this way. So, now by in this modified Lindstedt Poincare technique by taking this x equal to x 0 plus alpha x 1 plus alpha square x 2 plus alpha cube x 3.

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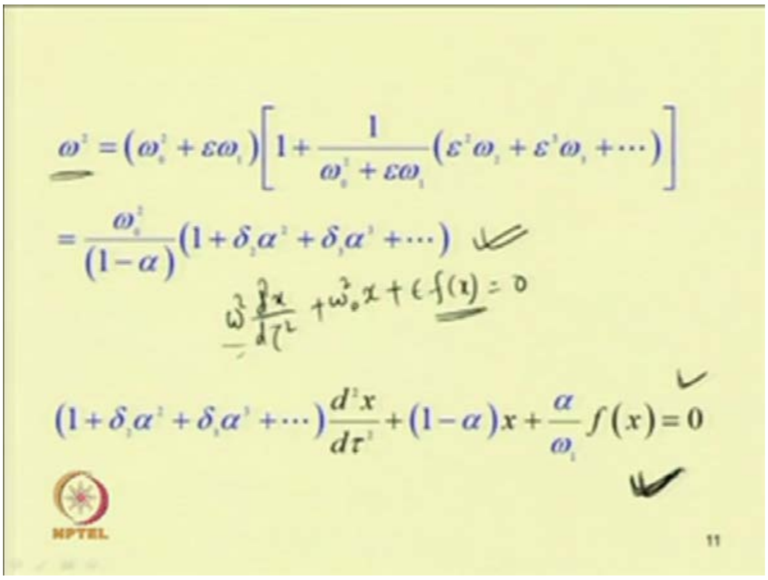


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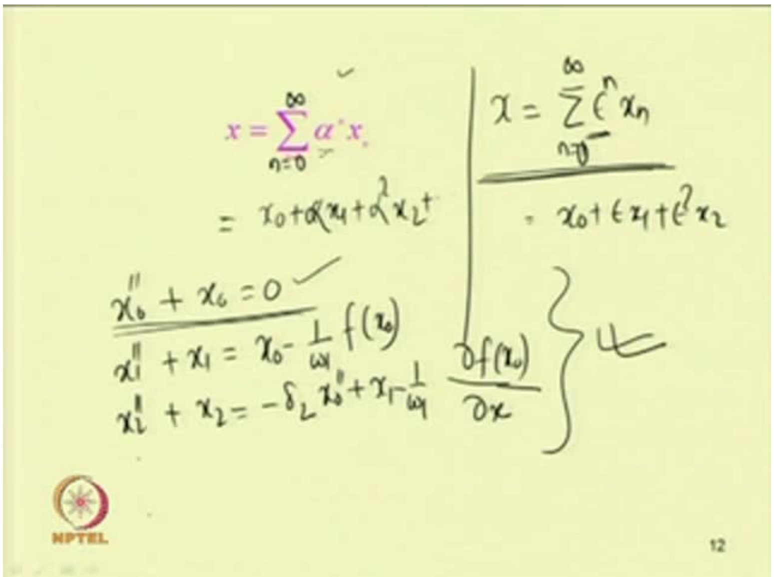


And substituting those expression in this expression and separating the terms with Alpha to the power 0 alpha to the power 1 and alpha to the power 2, then we can write this equation in this form so our first equation will be x 0. So, x 0 double dash plus x 0, so will be equal to 0, so this is the first equation then the second equation will be equal to x 1 double dash plus x 1. So, this will be equal to x 0 minus 1 by omega 1 function of x 0, in the third equation will get x 2 double dash plus x 2 this will be equal to minus delta 2 x 0 double dash plus x 1 minus 1 by omega 1 del f x 0 by del x. So, in this way we can write the terms so higher order terms also can be written. So, we know the solution of the first equation and we can substitute the solution of the first equation in the second equation and kill the secular term.

So, in that way will get the terms with so in this way will get in similar to that of the Lindstedt Poincare technique will get the terms this omega 1 omega 2 and higher order terms. So, one has to follow the similar procedure in this, but here the difference between the modified Lindstedt Poincare technique, and this technique is that here instead of using epsilon we are using this term alpha. So, the forth difference between this two term is that the initial, so how you are using the initial condition. So, we are using the initial condition that is in original equation our differential equation, we know the solution and we have assumed the initial condition.

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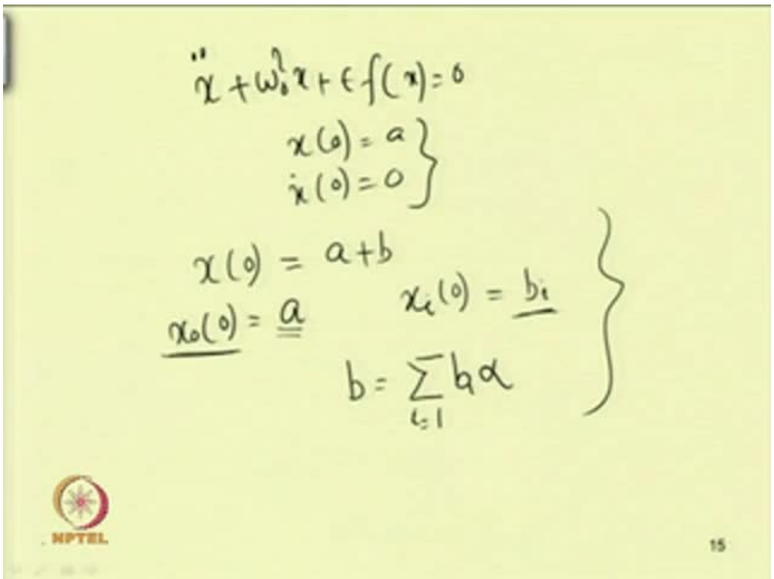


So, here the initial condition what you have assumed in this case that is our differential equation. Let our differential equation is this x original differential equation x double dot plus omega 0 square x plus epsilon f x equal to 0. So, subjected to the initial condition that is x 0 equal to a and x dot 0 equal to 0, so let us assume this thing. So, when we are solving this equation here in the forth step, so we can write this x 0 equal to we can write this x 0 equal to a plus b.

So, here this u or x 0 0 that is we have this x 0 0 will be equal to a and this x i that is i equal to 1 2 1 2 and higher order terms. So, the initial condition we can take equal to b i.

So, the first term, the first term we can take that is x 0 0 equal to a and then this other terms that is x 1 0 equal to b 1 x 2 0 equal to b 2 x 3 0 equal to b 3. So, in this way we can take so where so here we can take this a and b in such way that so this a is the sum of the odd harmonics and b i is the sum of the initial value of the sum of all even harmonics.

So, b i will be the sum of all even harmonic terms and a will be sum of all odd harmonic terms. So, this b can be written in this form so b equal to i equal to 1 to 1, so this is b i alpha i or so alpha. So, by taking in this way so we can we can find the solution, this initial conditions will be used to find the final solution of the equation motion. So, let us take one example for example, let us take the governing difference duffing equation with odd non-linearity so by taking the duffing equation with odd non-linearity.

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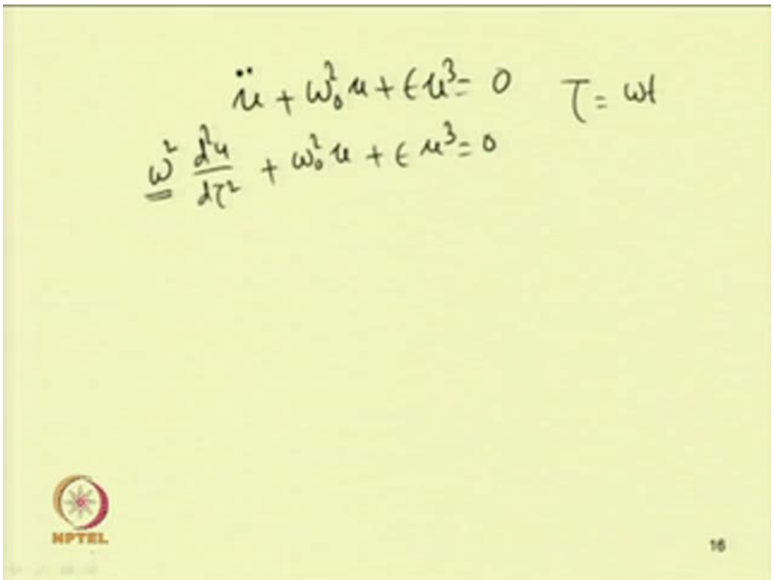


Let us take this equation mu double dot plus omega 0 square u plus epsilon u cube so this is equal to 0. So, in this case now by taking tau equal to omega t. So, this equation can be written in this form so this will be equal to omega 0 square d square u by d tau square plus omega 0 square u, so this is omega square. So, we have taken tau equal to omega t, so this becomes omega square so omega square d square u by d t d tau square plus omega square u plus epsilon, so this is u cube so this is equal to 0. So, now we have to substitute this omega square term, so omega square we have we can write this omega square in this form.

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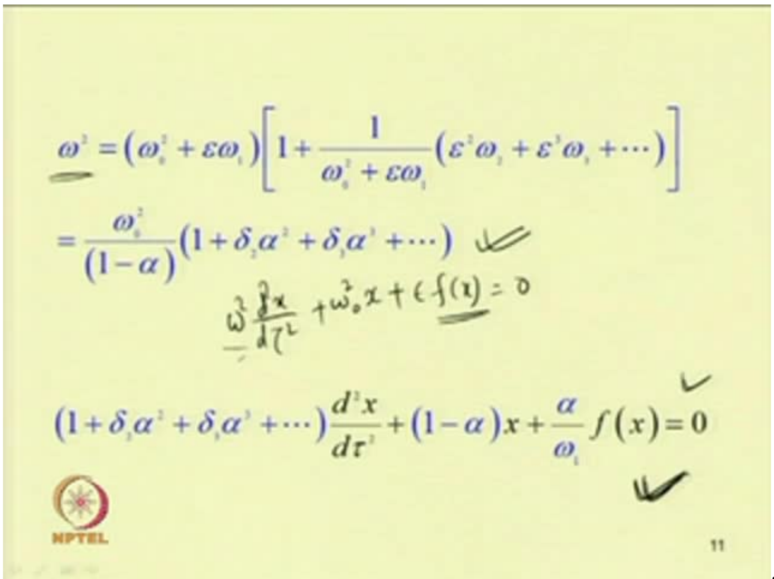


So, omega square equal to omega 0 square epsilon omega 0 square by so let me write this omega 0 square by 1 minus alpha.

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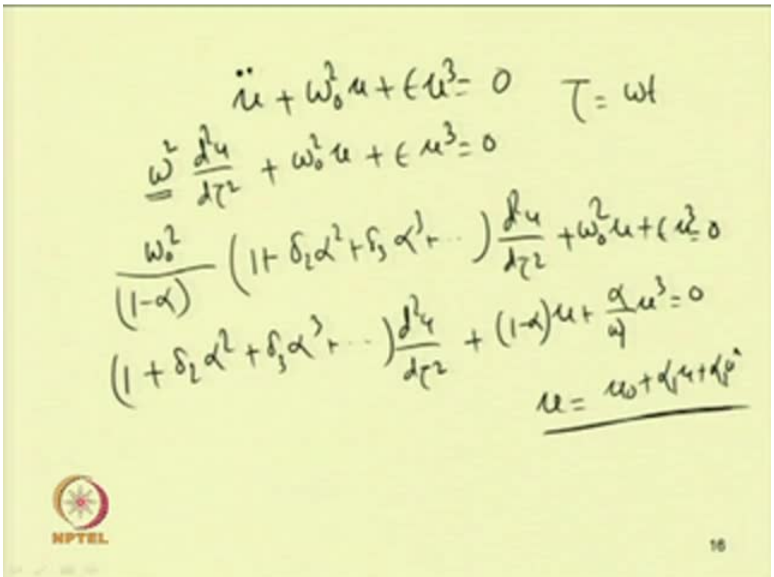


So, this thing can be written as omega 0 square by omega 0 square by 1 minus 1 minus alpha. So, this term can be written as omega 0 square by 1 minus alpha in to 1 plus delta 2 alpha square plus delta 3 alpha cube this higher order terms, or it can be in a simplified form, so this way it can be written into d square u by d tau square plus omega 0 square u plus epsilon u cube equal to 0. So, with further expansion it can be written or by multiplying this thing it can be written 1 plus delta 2 alpha square plus delta 3 alpha cube into d square u by d tau square plus, so this 1 minus alpha will be multiplied there so 1 minus alpha into omega 1 square so 1 minus so omega 0 square has been taken common. So, this becomes 1 minus alpha into u plus this alpha by so one can write this term alpha by omega 1 into, so this will be u cube so f u equal to u cube, so one can write this is u cube so it will be equal to 0. So, this way now so from this equation so one can by expanding this thing this u also in terms of so one can write u equal to u 0 plus alpha 1 u plus alpha 2 u square by writing u in this form so one can obtain this equation .

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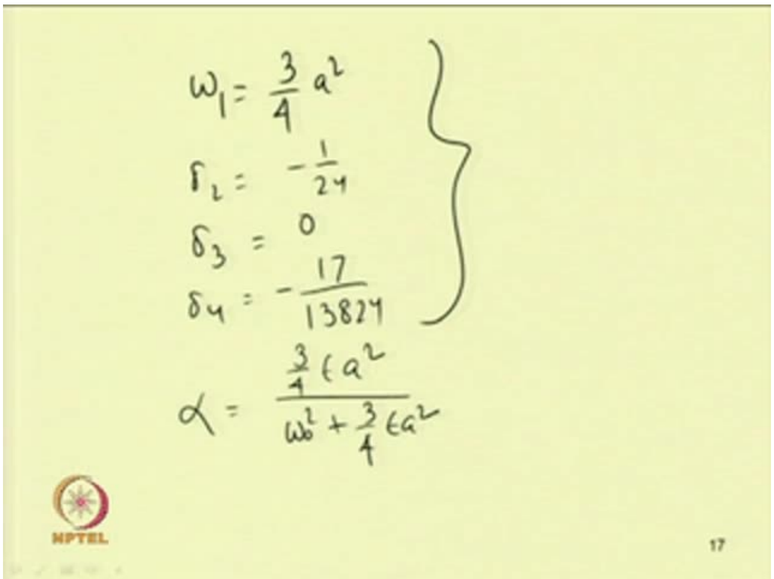


So, in this case one we obtain this omega 1 equal to 3 by 4 a square delta 2 equal to minus 1 by 24. So, delta 3 equal to 0 delta 4 equal to minus 17 by 13824. So, the parameter alpha so already we know this expression so alpha is obtained to be 3 by 4. So, alpha equal to 3 by 4 epsilon a square by omega 0 square plus 3 by 4 epsilon a square. So, the frequency relation frequency amplitude relation can be written in this form.

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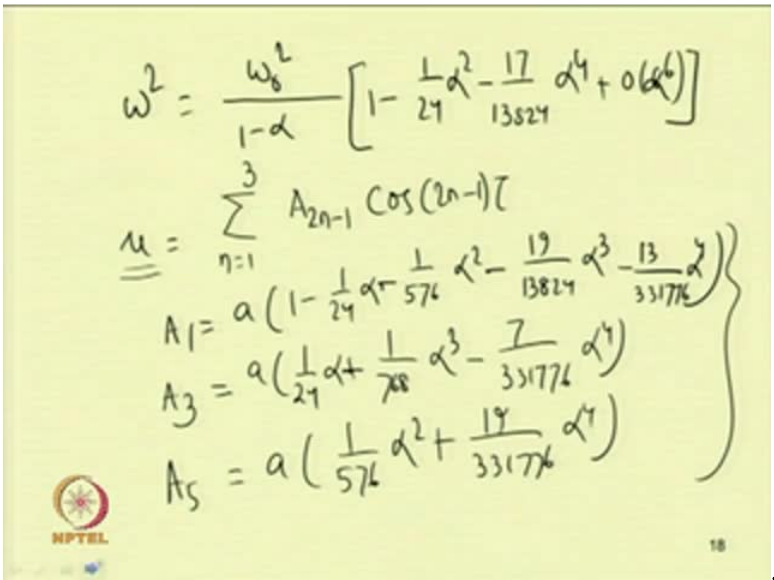


That is omega square so this will be equal to omega 0 square by 1 minus alpha into so already we have written this expression, so this is by using this delta 1 delta 2 delta 3. So, one can write omega square equal to omega 0 square by 1 minus alpha by 1 minus 1 by 24 alpha square minus 17 by 13824 alpha to the power 4 plus order of alpha to the power 6. So, this u can be written in this form so u so let us take up to 3 terms so it will be equal to n equal to if we are taking 3 terms n equal to 1 2 3. So, this will be equal to a 2 n minus 1 cos 2 n minus 1 tau so plus higher order terms.

So, here this a 1 will be equal to so if we are 3 terms so by putting n equal to 1, so we have this a 1 so a 1 we can have obtain to be a into 1 minus 1 by 24 alpha minus 1 by 576 alpha square minus 19 by 13824 alpha to the power cube minus 13 by 331776 alpha to the power 4. Similarly, by putting n equal to 2 so this becomes 4 minus 1 that is 3 so this is a 3. So, a 3 equal to a into 1 by 24 alpha plus 1 by 768 alpha cube minus 7 by 331776 that is alpha to the power 4, so now by putting n equal to 3 so we will have so 6 minus 1 that is a 5, so a 5 equal to a into 1 by 576 alpha square plus 19 by 331776 alpha to the power 4.

So, one can take higher order terms also or one can obtain this higher order terms so one can write this expression for u in or 1 can write this u as a 1 cos. So, this is cos tau then plus a 3 cos 3 tau plus a 5 cos 5 tau. So, one can by using this one can get the solution up to any accuracy, so in this way one can use this modified Lindstedt Poincare technique to solve the differential equation motion. So, in this module so we have studied this we have started with a straight forward expansion method. So, from this straight forward expansion method in the differential governing differential equation, we have only expanded.

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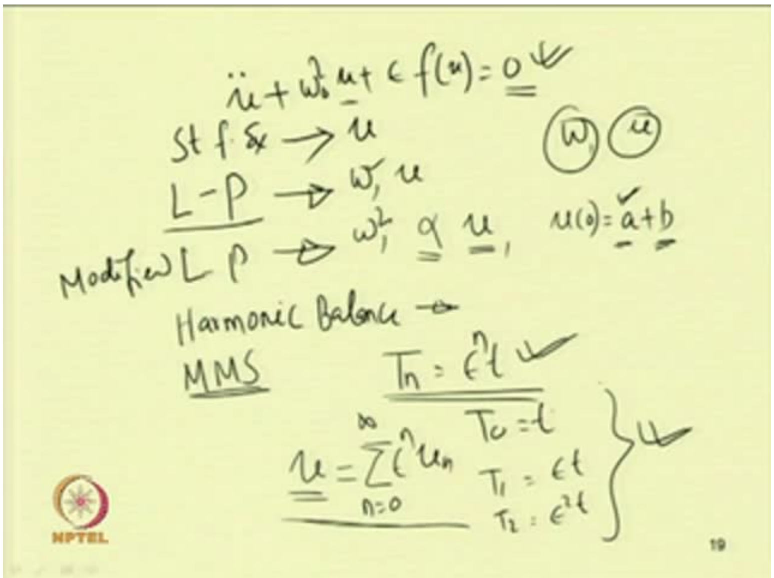


Let us take this equation that is u double dot plus omega 0 square u plus epsilon f u so it is equal to 0. So, this is for a free vibration for a force vibration one can take this forcing term beside. So, in case of this straight forward expansion we have only expanded this term with u and we obtain the solution, which contain unbounded response. So, the straight forward expansion is not useful as it contains the terms, which tends to infinity.
So, to modify that problem we have used this Lindstedt Poincare technique, so in which we have taken the frequency dependence of the amplitude into account and we have expanded both this omega and u, so both u and omega where expanded in case of this L P method that is Lindstedt Poincare technique. So, in case of straight forward expansion so straight forward expansion so expansion of u only takes place.

So, in case of L P method we have of the expansion of omega and u then in this case, the solution of weakly non-linear system can be made, but in case of this modified so modified L P method so in case of modified l p method which differs from this L P method in four different steps. So, in the first step we have expand this omega square.

Second step we have introduced another parameter alpha. In the third step we have expand this term u in terms of so in terms of alpha.

And the forth step before the initial condition so this initial condition u 0 is taken to be a plus b. So, where a is the summation of the initial values of the odd harmonics and b contains the summation of the terms in the even harmonics. So, this modified Lindstedt Poincare method can be used for both odd non-linearity and even non-linearity and up to very high order of accuracy.

So, next method what we have studied that is the simply harmonic balance method, so in case of harmonic balance method, we have taken so a fourier series and by substituting this fourier series in the governing equation will obtain the coefficients of this equations, and we have found the solution. The disadvantage of this harmonic balance method is that we should know the solution apiary, that is the solution should be a periodic solution then only we can apply or then only we can use this harmonic balance method. And the other difficulty in this case of harmonic balance method is that the we cannot take up to a higher order harmonics due to cumbersome in solving the equations so, but one can use symbolic software to solve or to go for higher harmonics.

So, then this have then we have use this method of multiple scale, method of multiple scales so in which we have taken this time. So, this T n equal to epsilon n t so here we have taken different times scales that is so we have taken this T 0. In this case T 0 becomes so, t and higher order times so that is T 1 equal to epsilon t and T 2 equal to epsilon square t and in this way by taking different time scales. So, we have solved the governing equation of motion.

So, in the first step we have substituted the time step different time step and then we have expanded this u, so u we have written in terms of or u equal to 1 can write in a different way also, or one can put this thing epsilon to the power n, n equal to 0 to infinity u n by taking this u equal to n equal to 0 to infinity epsilon to the power n u n and taking different time scales, and substituting this thing in this governing equation.

So, one can get a set of equations by separating in the order of epsilon then by the killing the secular terms. So, one can get a set of first order differential equations, so which can be solved for the steady state solutions. So, for steady state solution so for example, in this case so cos by substituting this equation so one can get a set of first order equation so let the set of or this reduced equation, let me write in this form.

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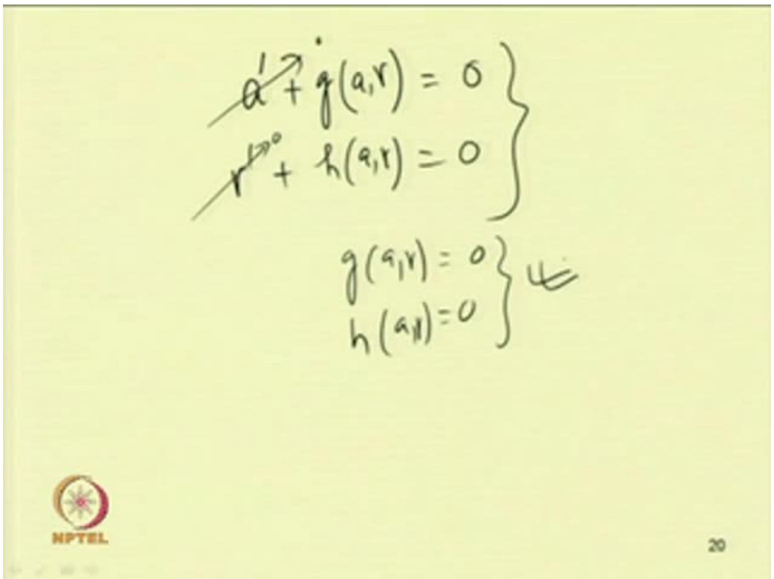


That is a dash plus let me write this a dash plus g, so which is a function of a and gamma a and gamma, gamma is the phase. So, a dash a g gamma equal to 0 where a dash is a function of t 2 or higher order terms higher order higher order times. So, this gamma dash plus so let me write this h a gamma equal to 0. So, for steady state so one can substitute this or one can as this a and gamma will no longer the function of time. So, this one can put this term equal to 0 this term equal to 0 and get a set of algebraic or transcendal algebraic equation. So, a gamma equal to 0 and h a gamma equal to 0 so by solving the set of equations. So, one can obtain the solution of the governing equation.

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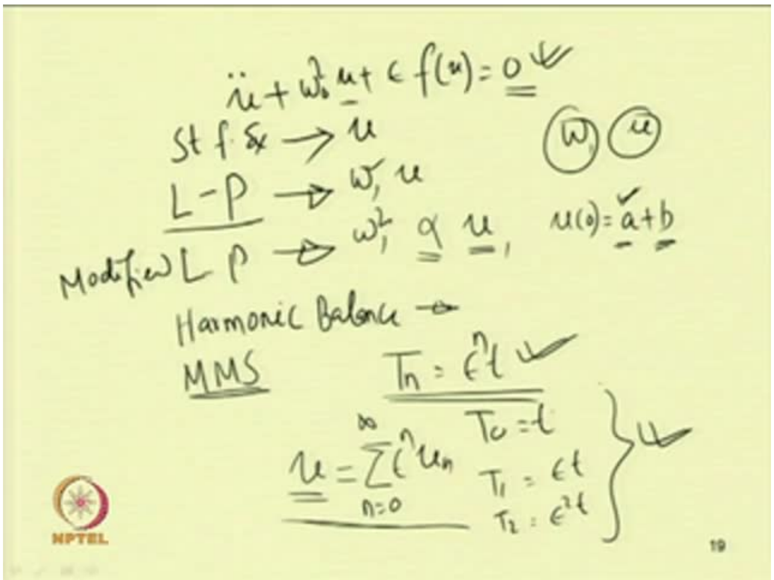


So, in case of method of multiple scale so we are reducing the second order differential equation to a set of first order differential equation.

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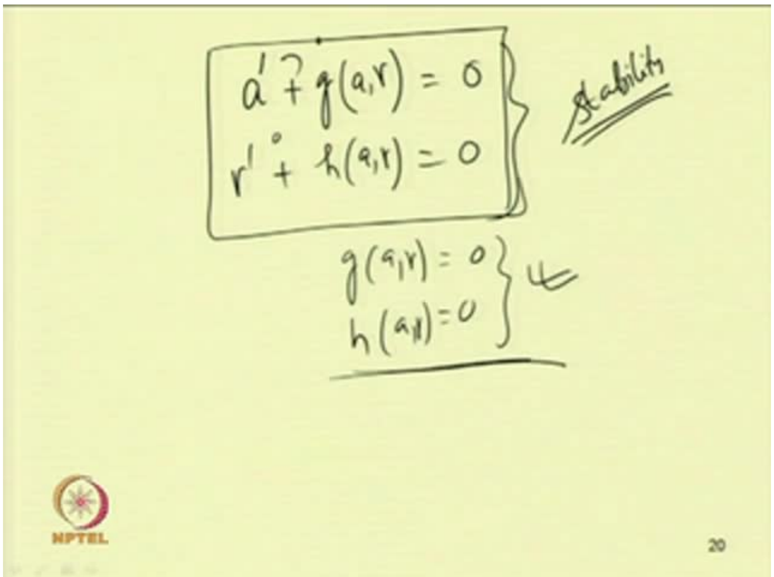


So, the advantage of this harmonic this method of multiple scale is that by reducing this equation to a set of first order equation. So, in addition to getting the steady state response which we obtain by solving this algebraic, or transcendal equation so we can study the stability of the systems also, the stability study can be directly done from this first order equation. So, by perturbing the set of first order equation now we can get the

Jacobian matrix and from this Jacobian matrix by finding the Eigen values. So, we can study the stability of the system, but in case of this harmonic balance method.

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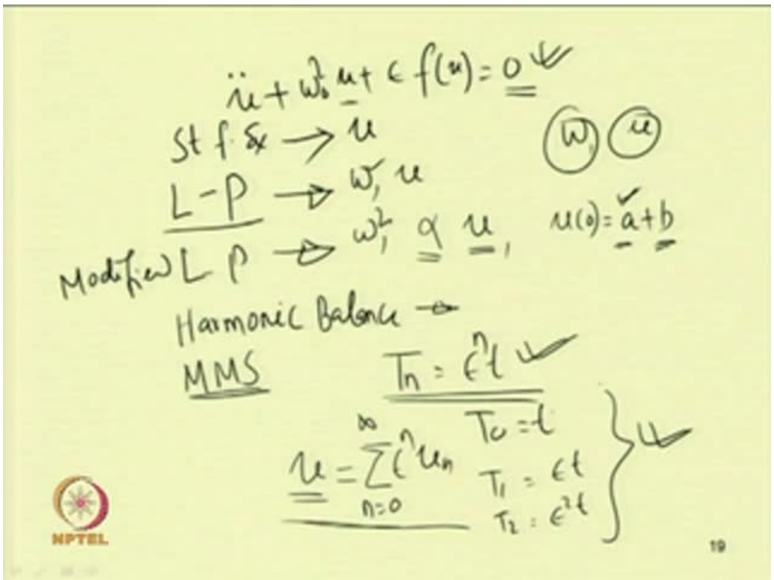


As we are not reducing the equation to its first order form, so here only we are getting this solution of the system. And now by perturbing this equation again by perturbing the second order equation and substituting the solution, we can locally find this are find the stability of the resulting solution. So, the disadvantage of this harmonic balance method is that we cannot go for this higher order terms also, for studying the stability of the system. So, we have to do the stability analysis again on the second order equation.

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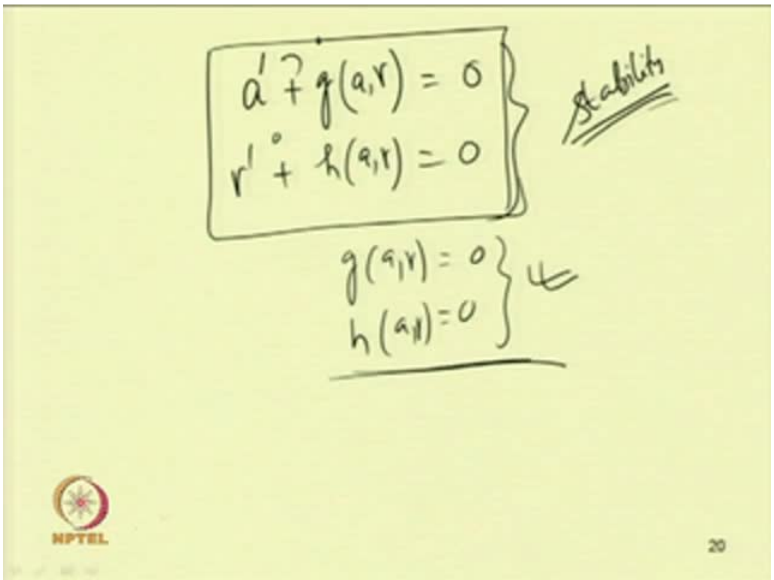

But in case of method of multiple scale from the reduced equation directly will get the solution and also, we can steady its stability by finding the Eigen values of the Jacobian matrix. So, in addition to this classical methods, we have studied this incremental harmonic balance method and also this intrinsic multiple scale harmonic balance method by adding this harmonic balance method and the multiple scale method. So, this way we have studied different also in addition to this we have studied the method of averaging.

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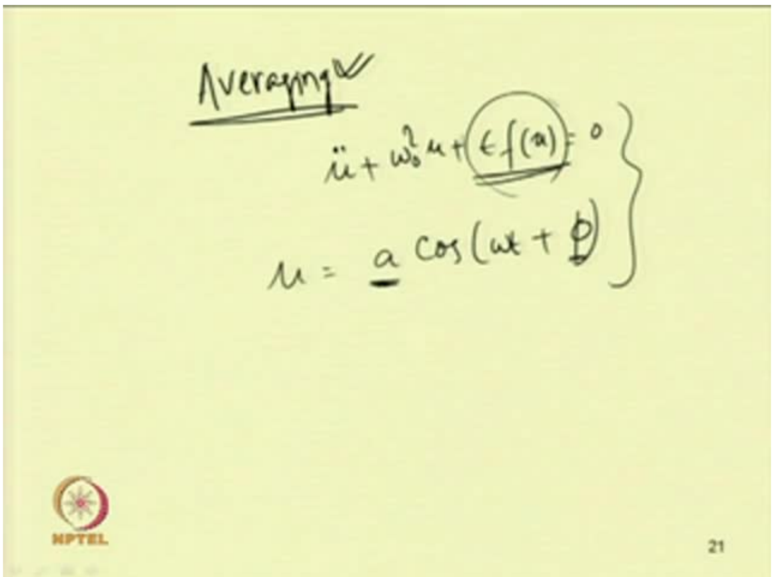


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So, in case of averaging so in case of averaging initially we have taken the solution of the homogenous part. That means, let us instead let us take this equation u double dot plus omega 0 square u plus epsilon f u equal to 0 so here this is the non-linear term. So, in method of averaging initially we have neglecting this term. So, let us find the solution so the solution u can be written as a cos omega t plus phi. So, in this method in method of averaging instead of taking this a and phi to be constant. So, we have taken this a and phi has function of time. So, by taking a phi as function of time, we have different we have different sets of averaging method.

So, first we have studied this Vander Pol's method then we have studied this Krylov Bogoliubov method and then we have studied this generalized harmonic balance method and further, we have studied this Krylov Bogoliubov Mitropolski method. So, in all this methods so the way we take this a and phi are different so by taking so these are function of time.

So, by introducing this a and phi as function of time instead of taking them as constant, one can get a set of equations and or one can reduce this second order differential equation to a set of first order differential equation. And one can obtain the frequency relations by killing the secular terms. So, in this way we have studied the different methods that is straight forward expansion method, Lindstedt Poincare technique averaging method, then method of multiple scale.

And in addition to that this incremental harmonic balance method and intrinsic harmonic time, multiple scale harmonic balance method, but all these methods we have studied for single degree of freedom systems. So, this analysis of single degree of freedom systems can further be applied for this multi degree of freedom system in a similar way. For example, let us take this take a two degree of freedom system that means let this equation.

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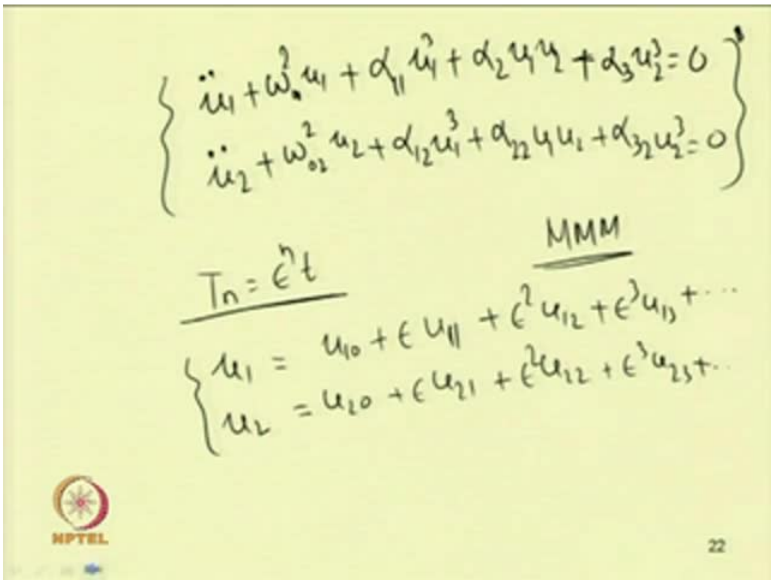


Let me write in this form that is u 1 so u 1 double dot plus omega 0 square u 1 plus let me write this alpha 1 u 1 cube, let me add another non-linear term that is alpha 2, let there is a couple term between u 1 and u 2 then alpha 3 u 2. So, u 2 cube so let me write this alpha 3 plus alpha 3 u 2 cube equal to 0, so this is first equation and second equation let me write this form so this u 2 double dot plus omega. So, let me put this is 0 1 so omega 0 2 square u 2. So, this is alpha 1 0 so or alpha I can put it 1 1. So, let me put this alpha 1 2 alpha 1 2, so that is u I can write this is u 1 cube also I can write same non-linear terms I can add here u 1 cube plus alpha 2 2 u 1 u 2 plus alpha 3 2 u 2 cube equal to 0.

So, let these are set of non-linear differential equation so let us apply this method of multiple scale by while applying this method of multiple scale. So, I can proceed in a similar way I can take this time scale T n equal to epsilon to the power n t and I can take this u 1 in this form. So, u 1 I can write u 1 equal to u 1 0 plus epsilon u 1 0, so this is u 1 0 then u 1 1 plus epsilon square u 1 2 plus epsilon cube u 1 3 in this way I can write the more terms.

Similarly, I can put this u 2 equal to u 2 0 plus epsilon, so u 2 1 plus epsilon square u 2 2 plus epsilon 2 3. So, in this way I can expand this term u 1 and u 2 and substitute in this governing equation, and by substituting this equation and separating the order of epsilon

I can get a set of equations. So, in this case of second two differential two degree of freedom system, so I will get 4.

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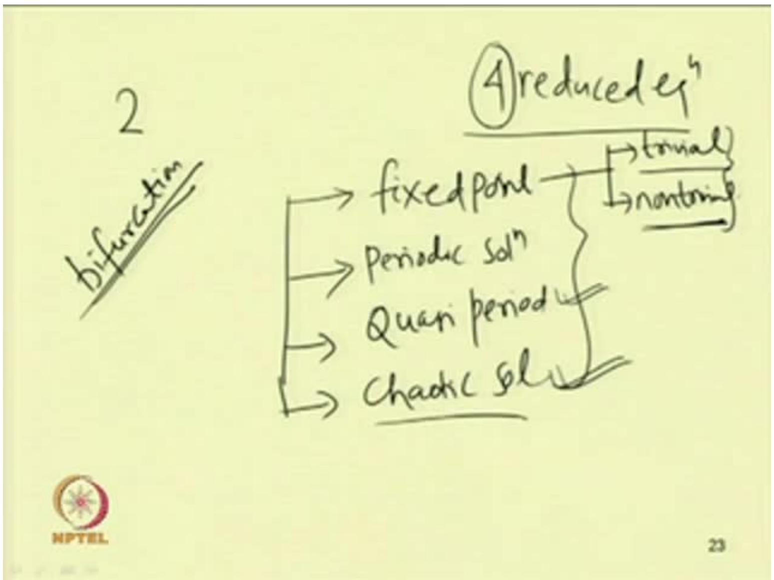


So I will get 4 reduced equation, so in the first case, so when I have taking only a single equation I, I was getting a I was getting only 2 equation 2 first order equation and in those 2 equations by neglecting or by putting this terms, which are function of time equal to 0 for steady state solution. I have a set of algebraic equation and those algebraic equation one may not that they may not be linear, so they may contain this couple terms also, so by 1 may not solve this equations easily or one may go for numerical solutions to solve those equations. Similarly, here so instead of solving 2 equation one can solve a set of four equations, so by solving a set of 4 equations so from this reduced equation one can find the response of this solution.

So, after getting the response of the solution so one should be interested to find the stability of those solutions so in this response also so during our introduction class, we know we can have different type of solutions. So, this solutions may be a fixed point solution fixed point solution in which it will have a fixed value, or it may be a periodic solution or it may be quasi periodic solution or it may be chaotic solution. So, in case of this fixed point also the solution may be trivial solution that is it has a 0 value, or non trivial solution.

So, in this case of fixed point this trivial solution and non trivial solution there may be different types of bifurcation, so as we are going to get multiple solutions unlike in case of linear system where we are getting single solution. In case of this non-linear case will get multiple solution and in this multiple solutions by changing the stability of the solution, one can obtain different type of bifurcations.

So, in the next class we will be studying or in the next module, will be studying the stability of this fixed point response, periodic solution, quasi periodic solution and or will first study about the different type of responses that is this fixed point response and their stability and bifurcation then periodic solution their stability and bifurcation. Then will have some knowledge about this quasi periodic solution and chaotic solution also, will study how to control this chao's or chaotic solution of the system. And one can take different assignments to find the or one to use all these methods what we have studied, so one can take this.

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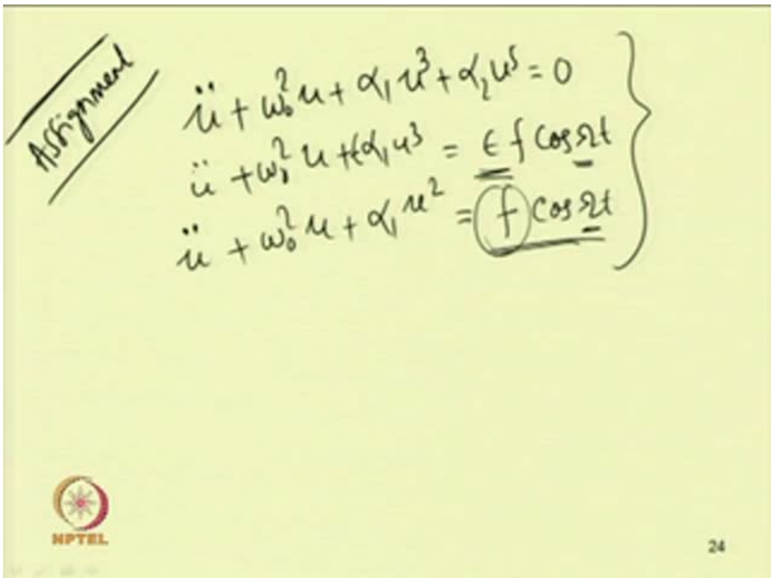


Duffing equation that is u double dot plus epsilon or u double dot plus omega 0 square u plus alpha 1 u cube plus alpha 2 u to the power 5 equal to 0. So, this is for free vibration of non-linearity and by taking so this left side equal to keeping this left side equal to constant as a or let me write it again. So, this is omega 0 square u plus alpha 1 u cube let us take the right side. So, this will be equal to epsilon f cos omega t, so this is the external frequency then let me take another equation also u double dot plus omega 0 square u plus alpha 1 u square.

So, let me take a force this is f cos omega t, so in the previous case I have taken this is this forcing term to be very small that is order of epsilon and in this case or here also I can put a term epsilon. So, in this case this forcing is not this is not small. So, this forcing is not small one can take the duffing equation with less amplitude of forcing with large amplitude of forcing, and omega is the excitation.

So, by using the different methods what we have studied so one can find the solution of this so this is the assignment problems. So, one can solve this problems to have a better knowledge about the methods, what we have studied and one can compare the solutions obtain by different methods. In the last module actually we are going to study all this equation and will study this equation and their bifurcation, and will study how this equations are applied to different science and engineering problems.

Thank you.