Non-Linear Vibration
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Module - 3
Solution of Non-linear Equation of Motion
Lecture - 9
Incremental Harmonic Balance method
Intrinsic Multiple Harmonic Balance method

Welcome to today's class of non-linear vibration. So, in today's class, we are going to study this solution of non-linear equation of motion by using this incremental harmonic balance method and intrinsic multiple scale harmonic balance method. Also, we may study about the modified Lindstedt Poincare method which is also used for the solution of non-linear equation of motion. So, in our previous classes, we have studied many classical methods.

So, we have started with the straight forward expansion method, then we have studied this Lindstedt Poincare method, then method of multiple scales harmonic balance method and also we have studied this averaging method.

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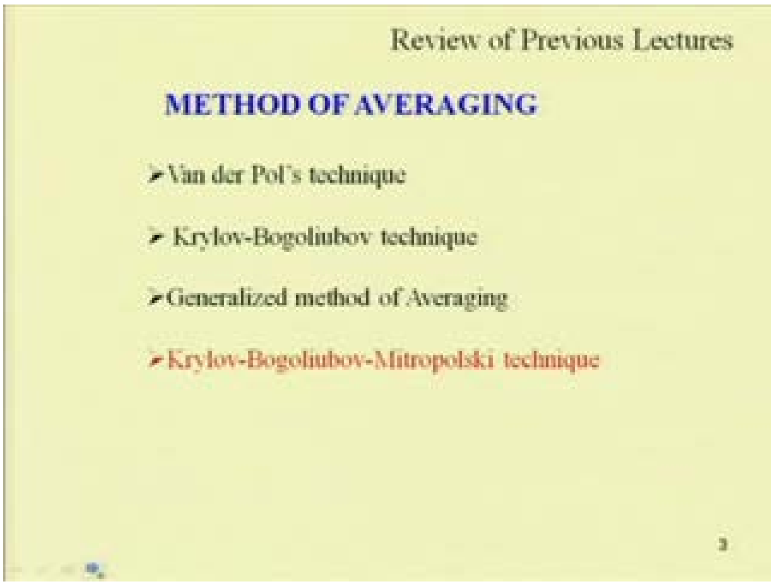

In case of averaging method, we have studied this Van der Pol's technique, then Krylov-Bogoliubov technique, generalized method of averaging and in the last class; we have studied about this Krylov-Bogoliubov-Mitropolski technique. So, in all these techniques, we can solve the non-linear equation of motion where the non-linearity appears in a weak form. So, for strongly non-linear systems, these methods have to be modified and in today's class, we are going to study about this incremental harmonic balance method where one can use this method for both weak and strongly non-linear equations.

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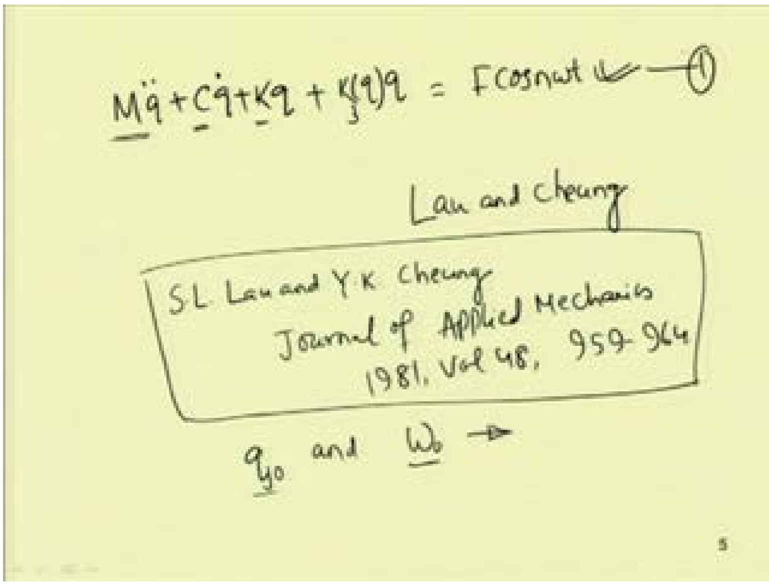

So, let us take the governing equation, general governing equation of motion in this form that is M q double dot plus C q dot plus K q. So, let us take a non-linear term which I can write in this form. So, this is a function of K 3 q. So, this is a function of non-linearity and then let me put a forcing term which I write in this form F cos n omega t.

So, this incremental harmonic balance method was developed by Lau and Cheung. So, S. L. Lau and Y. K. Cheung which has published in the journal of applied mechanics. So, it is in 1981 volume 48 and page number 959 to 964. So, according to this incremental harmonic balance method, so this is a two-step method. So, in the first step, we use this increment of the variables and then in the second step, we will use this Galerkin method to find a set of equations, set of algebraic equation which we can solve using this Newton Robertson method or any iterative method.

So, in the first step we can assume this q. So, let this q let me assume this q j 0 and omega j 0. So, this is the starting point. I can take this q j 0 and omega j 0. So, we can start. Let us consider this is the jth state of vibration of the system for frequency omega 0. So, q j 0 is the jth state of frequency or we can assume in other way also. So, let this equation, I can assume this is for a multi-degree of freedom system equation or for a single degree of freedom system. So, in case of multi-degree of freedom system, this M will be mass matrix, C is the damping matrix, K is this stiffness matrix and K 3 will contain this non-linear terms. So, it may be cubic non-linear terms and f is a force vector and for single degree of freedom systems. So, this M will represent the mass of the system, C the damping, K is the stiffness and K 3 is a non-linear term and F is the forcing. So, depending on whether we are taking a multiple degree of freedom system or single degree of freedom system, one can consider this q j 0 and omega 0 as to denote a state of vibration of the system. So, the neighboring state also in this equation, this first equation let this is equation number 1.

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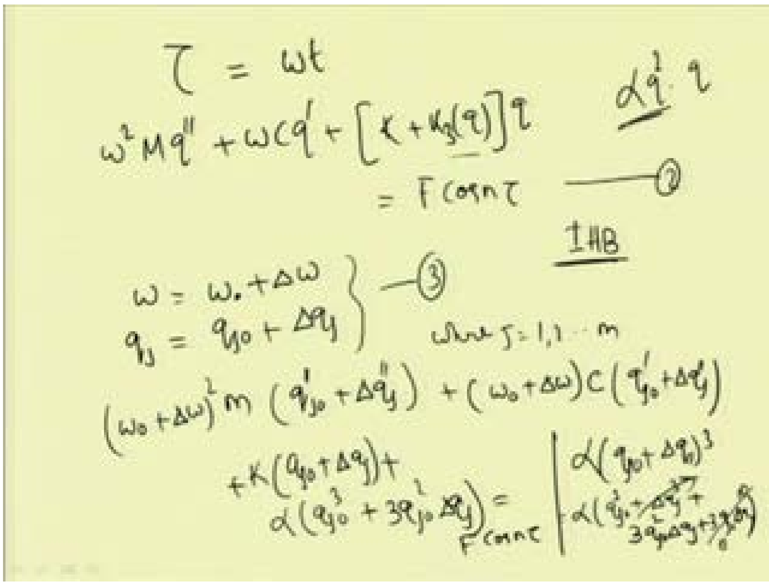

So, we can write this equation by using this non-dimensional time tau equal to omega t. So, in this case I can write that equation in this form, omega square M q double dash plus omega C q dash plus K plus K 3 q q. So, this is equal to F cos n tau. So, by using this non-dimensional time tau, I can write this first equation in this form. So, in the IHB method that is incremental harmonic balance method. I stands for incremental, H for harmonic and B for balance. So, in case of this incremental harmonic balance method, in the first step I will find the neighboric state this omega I can write. So, initially we have started with omega 0, the neighboring state I can find by incrementing this by delta omega. Similarly, this q j 0. So, I can or next stage this is q j, I can write this will be equal to q j 0 plus delta q j, where this j equal to 1 2 m. So, up to m i can take now by substituting this equation in this original.

So, in this equation, let this is equation number 2. So, by substituting this 2 in this equation and neglecting the higher order incremental term, one can write this equation in this form. So, in this case, it will be written in this form, omega 0 square. So, by substituting this omega 0 plus delta omega this whole square, you can substitute. So, let me take for a single degree of freedom system, and then in this case, it will be omega 0 plus delta omega whole square m and for q double dot double dash. So, I can write this will be equal to q j 0. If it is q j 0 double dash plus delta q j double dash plus for omega, I can substitute this omega 0 plus delta omega and then c and for q dash, I can write q j 0 dash plus delta q j double dash. Similarly, for this thing I can substitute. So, this will be k plus k q. So, if I will take a Duffing type of equation, in this case it will be alpha q square into q.

So, this term will be equal to alpha q square. So, this q square term I can substitute this q equal to q j 0 plus delta q j. So, square term and in addition to that I have 2 multiplied with this q. So, it will be alpha. So, in that case, it will be alpha into q square q that is q cube. So, it will be alpha into q j 0 plus delta q j whole cube. So, this will be equal to alpha, then I can q j 0 cube alpha q j 0 cube, then plus delta q j cube plus 3 into q j 0 square into delta q j plus 3 q j 0 into delta q j whole square. So, in this I can neglect this delta q 0 cube, and delta q 0 square term. So, the remaining term will be alpha q j 0 cube plus alpha into 3 q j 0 square delta q j. So, in general if this is a cubic non-linear term, in this position I can have this 3 into. So, I can write these terms in this way. So, this will be 3 k 3 into delta q plus 3 k 3 into delta q.

So, if I will or I can expand this equation and I can write in a better way also. So, by expanding this thing I can write k into, for example in this case of Duffing equation, it can be written k into for q q k 0 plus delta q j plus. Similarly, for this cubic non-linear term it will be these terms will be there only. So, it will be alpha into q j 0 cube plus 3 into q j 0 square delta q j. So, this will be equal to, so this term will be written as will be equal to f cos n tau. So, now, already we know that. So, if q j 0 is the present state of this system, then it will satisfy the governing equation or this equation 2 as omega 0 1 q j 0 are the current state of the system. So, it will satisfy this equation. So, if it is satisfying this equation, then this term that is omega 0 square m into q j double dash plus omega 0 c into q j dash plus k into q j 0 plus alpha into q j 0 cube will be equal to F cos n tau.

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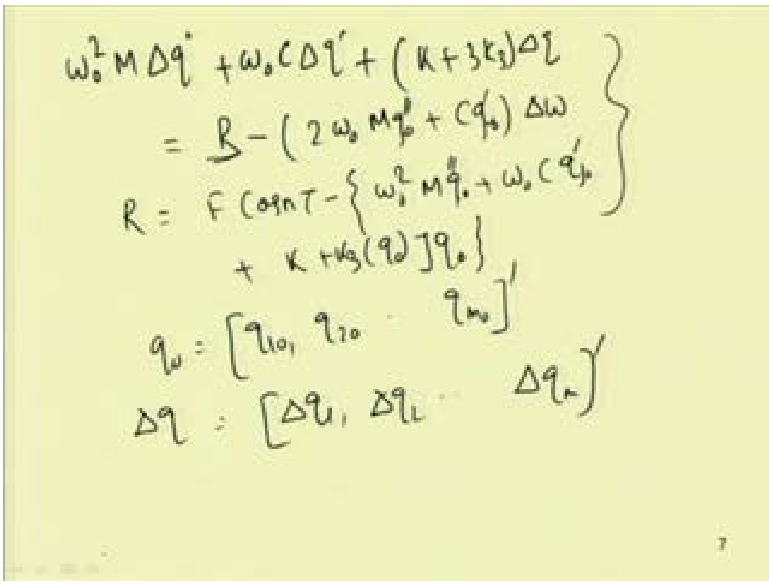

So, the remaining term by neglecting this higher order in increment can be written in this form that is omega 0 square M delta q double dash plus omega 0 C into delta q dash plus K plus 3 K 3 into delta q. So, this will be equal to some residue minus 2 omega 0 M q 0 double dash plus C q 0 dash into delta omega q j 0. So, this is j 0. So, we have written j. So, this will be the thing. So, here R can be written as the residue that is F cos n tau minus omega 0 square M q 0 q j 0 double dash plus omega 0 into C q j 0 dash plus k plus K 3 q 0 into q 0. So, here for multiple degree of freedom system, one can write this q 0 will be nothing but your q 1 0 q 2 0 and q m 0. So, if we will take m degree of freedom system, then this q 0 can be written in this formula that is q 0 equal to q 1 0, q 2 0, q m 0 and for the single degree freedom system. So, this will be equal to q only. One can write this q or q 1 0 and here this delta q also this is equal to delta q 1 delta q 2 and delta q m. So, this is transpose of this. So, here R is the residue or this is the residual or quality factor term which will be equal to 0 when the solution we are having is the exact solution.

If q is the exact solution, then this residue will be equal to 0. So, this is the first step. So, in the first step, we have substituted one increment that is considering the present state to be omega equal to omega 0 and q j equal to q j 0. Then we did some increment. So, we did increment in the frequency that is my delta omega and also, in the state of the system q j by delta q j and substituting this in the governing equation. So, now, we have reduced that equation to this form. So, in this second step, we will use Fourier series or we will use the harmonic balance method.

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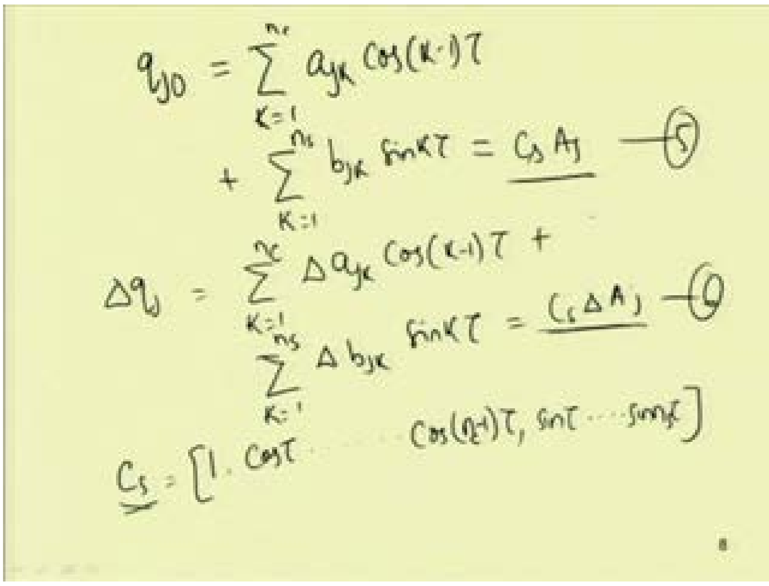

So, in this case, so in harmonic balance method, we can use this q j 0. So, we can write this q j 0 in terms of the Fourier series. So, we can write this is equal to k equal to 1 to n c. This is cosine terms. So, let n c represent the number of cosine terms. So, it is a j k cos k minus 1 tau plus k equal to 1. Similarly, I can write k equal to 1 to n s. N s represent the number of sin terms we are considering. So, that will be b jk sin k tau. So, this thing can be written in matrix form also using the c s into a j, where a j are the coefficients or c j a j are the Fourier coefficients. So, q j 0, one can write in this form. Similarly, delta q j can be written in this form same way. So, k equal to 1 to this is n c. So, it will be delta a j k cos k minus 1 tau plus k equal to one to n s. So, delta b j k sin k tau.

So, this thing similarly can be written using the c s delta a j where this c s can be written. So, one can write this c s equal to 1. So, c s is nothing but for the first term it is by substituting k equal to 1. So, this becomes 0 and this becomes. So, cos 0 equal to 1. So, this becomes a j k. So, it is coefficient equal to 1 similarly for the second term that is k equal to 2. So, one can find this is equal to cos 2 minus 1 that is 1. So, cos tau. So, the coefficient of a j 2 will be equal to cos tau. So, the c s will be equal to 1 and then next term will be equal to cos tau. So, the term similarly one can find the other terms also. So, this will become n c minus 1 tau. So, this is for the cosine terms and then sin terms also one can write similarly. So, this will be equal to sin tau the first term in case of sin part. So, it will be equal to k equal to 1. So, this coefficient of b j 1 will be equal to sin tau. So, this is first term in sin tau and then sin 2 tau and one can find the last term equal to sin n s tau. So, this is c s. So, one can write this q j 0 using this form that is c s into a j and delta q j term also c s into delta a j.

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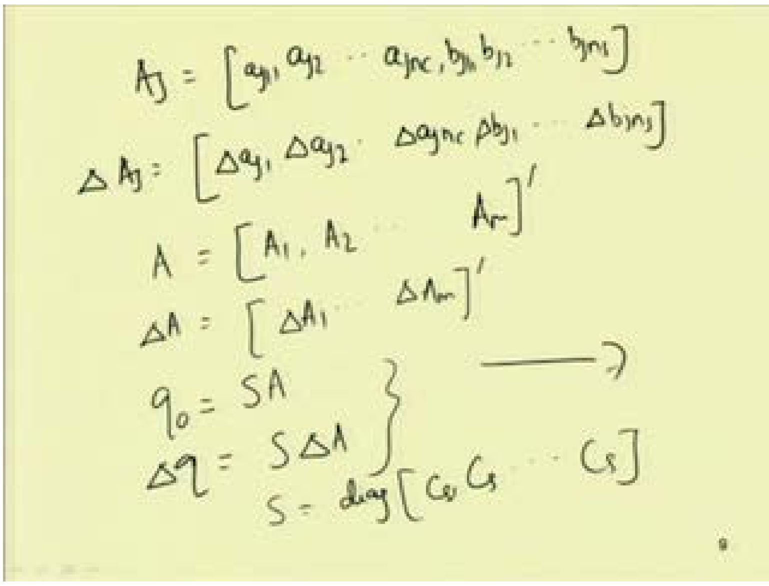

So, by using these two, here a j is nothing but the coefficients. So, a j can be written as a j 1, a j 2. So, it will be equal to a j n c. Then it will start with b j 1 b j 2 and b j n s. So, n s is the number of terms in the number of sin terms. So, one can write this way. Similarly, this delta A j equal to delta a j 1 delta a j 2 delta a j n c and b delta b j 1 and last term will be delta b j n s. So, this is delta A j and the first term is A j. So, in this way, now writing this equation that is q j 0 and delta q 0 using this Fourier series. So, one can now substitute this equation. So, before substituting this equation in the governing equation or the equation what we have obtained here, that is equation let me put this is equation number 4 and let me put equation number 5 and this is equation number 6.

So, now this q j 0 and delta q j 0 we can write in this way and we can by using this A as A 1 A 2. So, A transpose. So, here we have written this c s a j where a j will be equal to your A 1, A 2, A 3 and A m. So, we are taking m terms. Similarly, this is c s into a delta a j, now we can write this A equal to this transpose and delta A equal to delta A 1 delta A m transpose. So, we can write this q 0 using this form. So, we can write this as S into A and this delta q can be written as S into delta A. So, in this form we can write where this S can be written as diagonal. So, this is c s and cs. So, in this form we can write this q 0 and delta q. So, by writing this form that is 6, now this we can put it equation 7.

So, now, using 6, this equation 7 in equation 4. So, we can use this thing in equation 4 and then by substituting this in equation 4 and applying this Galerkin method for a cycle, so we can write this equation. So, we can obtain a set of algebraic equation now by substituting this equation 7 in equation 5. So, equation 7 in equation 4. So, by substituting equation 7 in equation 4, we can get, so this will reduce this equation to that of algebraic equation. So, those algebraic equation while substituting equation 7 in equation 4 and applying Galerkin method over a cycle.

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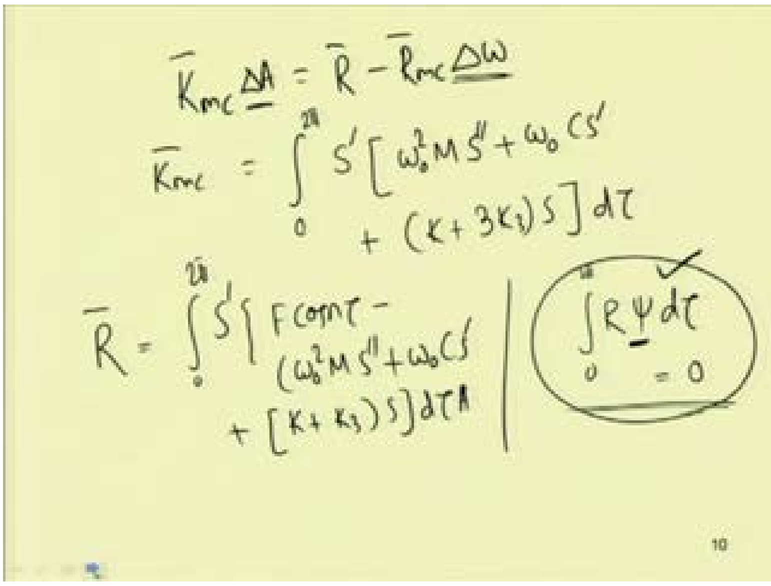

So, we will have this equation that is K m c delta A, this is equal to R bar R mc into delta W. So, this is increment in the amplitude and this is increment in the frequency where this K mc bar can be written as we are applying this Galerkin method over a cycle. So, this is 0 to 2 pi. So, this will be equal to S transpose into omega 0 square MS plus omega 0 CS dash plus K plus 3 k 3 into S into d tau using Galerkin method over the cycle. So, in Galerkin method, if we have a residue, then in case of the Galerkin method; let we have a differential equation. In this differential equation by applying the admissible function, we will get a residue. So, let R be a residue.

So, if by using this residue and let me use weighing function if I will integrate over a cycle. So, it will minimize let over the cycle, let me find it over the cycle. So, in this case it is over this 2 pi. So, if I will put this equal to 0, then I will get a equation which will give me the solution of the equation or this will reduce this equation to its weak form. So, by using this Galerkin method, I can reduce this equation to its weak form and in this case also by using this Galerkin method and minimizing, I can find a set of algebraic equation. So, here this K mc bar equal to S dash into, so this is the residue part. So, in this residue part, it is this weighing function that is S is multiplied and it is integrated over the cycle and one can get this R dash also in this form. So, R dash will be also be 0 to 2 pi S dash into F cos n tau minus omega 0 square M into S double dash plus omega 0 CS dash plus K plus K 3 S d tau A.

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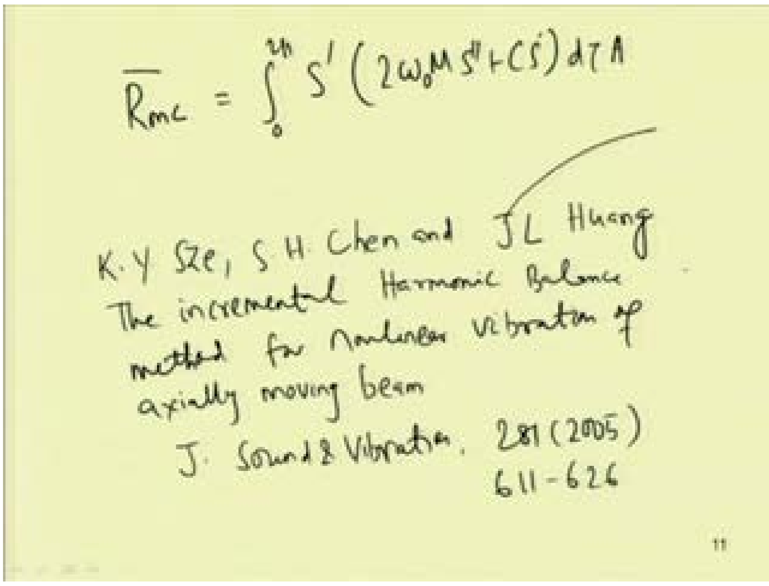

Similarly, this R mc bar will be equal to integration 0 to 2 pi S dash. So, there is a transpose of S multiplied by 2 omega 0 M S double dash plus CS dash into d tau into A. So, here after getting this equation, now the objective is to find this increment delta A and find the next stage of the solution and when it converge, then we get the actual solution. So, the solution process gives a gauge solution and the non-linear frequency amplitude response curve is obtained then by solving point by point. By implementing either the frequency or the amplitude or one can implement both the amplitude and the frequency and this in this case, one may use this Newton Robertson iterative method to solve this equation. So, in case of this incremental harmonic balance method, one follows two steps. So, in the first step, one takes the state vector and then increment this and after getting the incremental equation, one applies this harmonic balance method and using this Galerkin method. So, it reduces to that of a set of algebraic equation which is solved iteratively by using this Newton Robertson method.

So, now, there are many modifications of this incremental harmonic balance method also. So, one such modification is carried out by Cheung Etal, where they have taken this in this Incremental method, they have taken this arc length and they have found the solution and one more paper one may follow in this incremental harmonic balance method by KY Sze, S H Chen and J L Huang. So, this paper is on the incremental harmonic balance method for non-linear vibration of axially moving beam. So, here this incremental harmonic balance method is used for a axially moving beam and this paper is published in this journal of sound and vibration in volume 281 in 2005 and page number 611 to 626. So, there are many other methods many other papers are published and also they are on this incremental harmonic balance method.

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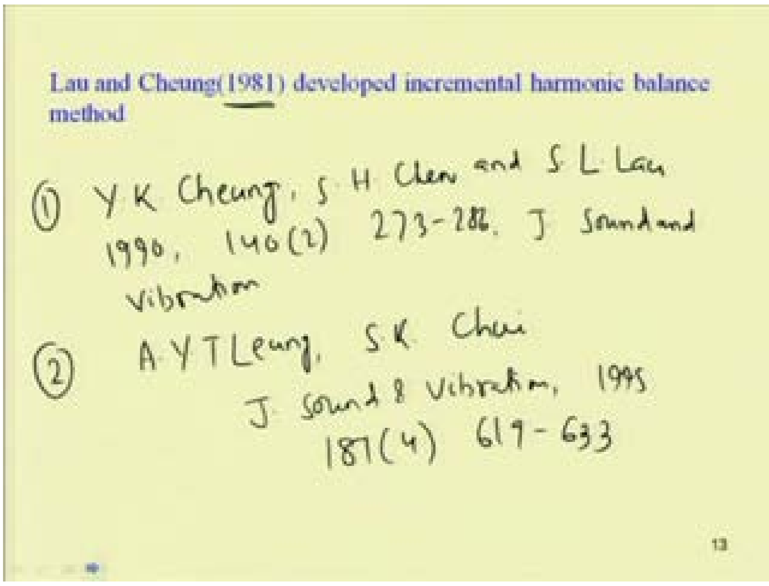

So, the first paper is by Lau and Cheung which was published in 1981, already I told. So, the other one can check by Y K Cheung and S H Chen and S L Lau. So, in this paper, it is in 1990 volume 142. So, this is 273-286 in the journal of sound and vibration. So, in this paper, they have used this incremental harmonic balance method to cubic non-linear systems. So, for further study one can see another paper by this a Y T Leung and S K Chui. So, in this paper, this non-linear vibration of coupled Duffing equation has been solved using this incremental harmonic balance method. This is also published in journal of sound and vibration in 1995, volume 187, pages 619 to 633.

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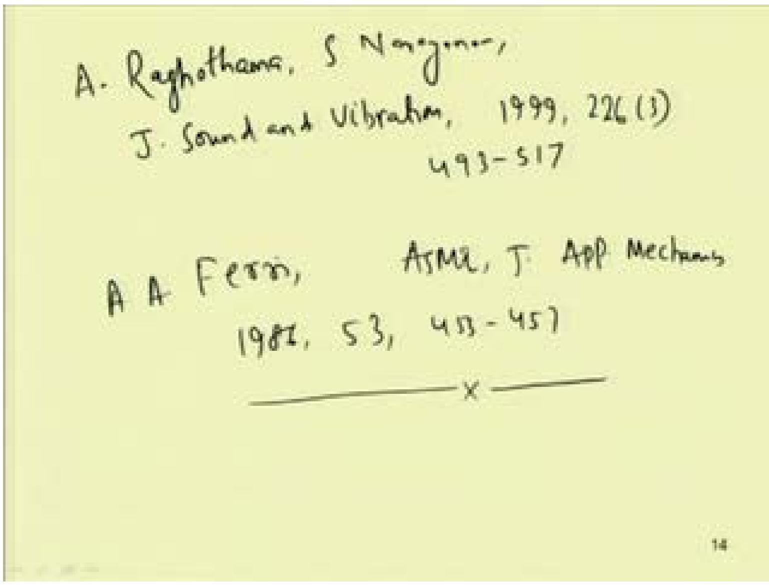

Another paper is also one may refer by A. Raghotham and S Narayanan. So, this is published in journal of sound and vibration in 1999 volume 276 is to 3. This is pages 493 to 517. So, here this two-dimensional aerofoil problem has been solved using incremental harmonic balance method. Also, one may see another paper by A A Ferri. So, here the covalence of incremental harmonic balance method and harmonic balance Newton Robertson method had been discussed. So, in this case, this is published in this A M S E Journal of applied mechanics in 1986 volume 53, pages 455 to 457. So, one can study these papers related to incremental harmonic balance method and use this method for effective analysis of the non-linear systems. So, next we will study about this intrinsic harmonic balance method.

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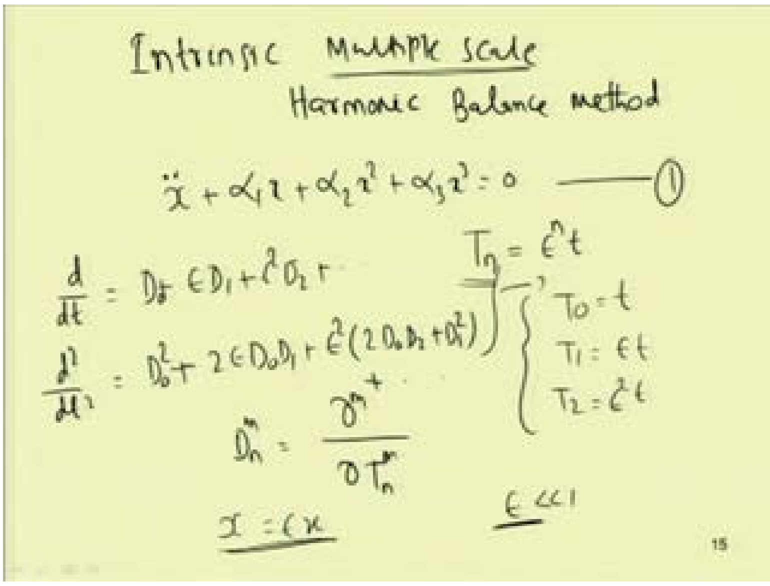

So, in case of this intrinsic harmonic balance method, we will study about this intrinsic multiple scale harmonic balance method. So, we have started with ordinary harmonic balance method in which we have used this Fourier series in the governing equation, and by separating the coefficients of the cosine and sine terms, we get a set of algebraic equations which we solve to find the solution of the non-linear equation, but with increase in order of the equations or with increase in the number of harmonic terms in the harmonic balance method, this method becomes very cumbersome and also, we have used this method of multiple scales for solving the non-linear equation motion previously.

So, now in this intrinsic multiple scale harmonic balance method, we will combine these two methods that is harmonic balance method and the multiple scale method to find the solution of the non-linear equation of motion. So, let us take a governing equation in this form. So, that is x double dot plus alpha 1 x plus alpha 2 x square plus let us take this alpha 3 x cube equal to 0. So, in case of the method of multiple scale, generally we take a time term that is at T n which is equal to epsilon n t. So, these are the time scales. So, different time scale for example, this T 0 n equal to 0. So, this becomes T 0. So, T 0 equal to tau T 0 equal to T and T 1 equal to epsilon T. Similarly, T 2 equal to epsilon square t. So, by using this method of multiple scale, we use different time scales and these derivative terms we write in this form that is d by d t equal to D 0 plus epsilon D 1 plus epsilon square D 2 plus the higher order terms we can write.

Similarly, the second derivative d square by d t square we can write equal to D 0 square plus 2 epsilon D 0 D 1 plus epsilon square into 2 D 0 D 2 plus D 1 square plus higher order terms we can write. So, here this D 0 is d by d t 1. Similarly, D 1 will D 0 equal to d by d t 0, D 1 equal to d by d t 1 and similarly, this d we can write this D nm. So, this is equal to del m by del T nm. So, in this way one can write now by using these terms in the governing equation. So, one can use this method of multiple scale to solve this equation, but in case of this intrinsic multiple scale harmonic balance method, we will combine both this multiple scale method and harmonic balance method.

So, before applying this harmonic or multiple scale method, let us first reduce this equation or let us use this book keeping parameter to write this equation again. So, by writing this x equal to epsilon x, one can write this equation in terms of the book keeping parameter. Now, substituting x equal to epsilon x this equation becomes epsilon x double dot plus epsilon into alpha 1 x plus alpha 2 epsilon square into alpha 2 x square plus epsilon cube into alpha 3 x cube.

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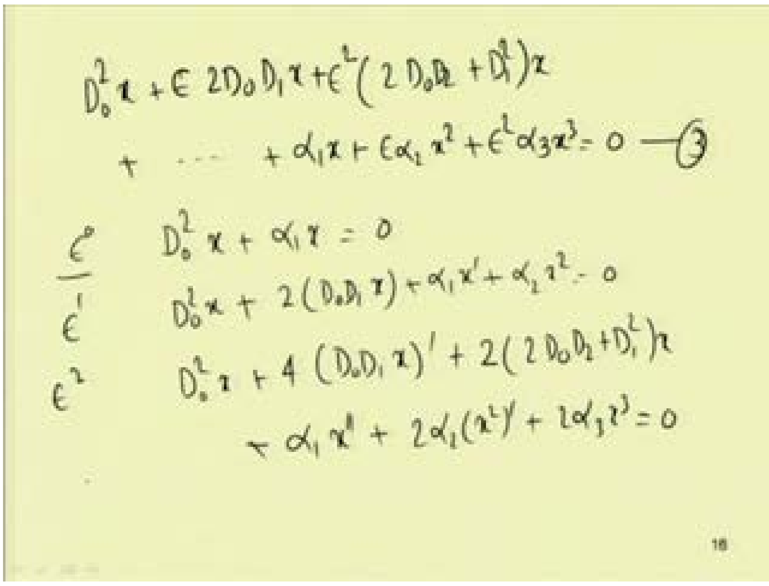

So, we one can write taking this epsilon common and as epsilon not equal to 0. So, one can write this equation in this form. So, it will be D 0 square x plus epsilon into 2 D 0 D 1 x plus epsilon square into 2 D 0 D 2 plus D 1 square x plus higher order terms plus alpha 1 into x plus epsilon alpha 2 into x square plus epsilon square alpha 3 into x cube equal to 0. So, by substituting this book keeping parameter epsilon where epsilon is the positive number which is very less than 1. So, by writing or using this book keeping parameter, we can write this governing equation in this form where we have used different time scales. So, this is the equation 3. I can put these two as equation 2. So, in this way I can write the equation.

So, now, setting epsilon equal to 0 or we can order with different order of epsilon. So, we can write this equation in this form. So, order of epsilon 0, this equation can be written in this form that is D 0 square x plus alpha 1 x equal to 0 in the order of epsilon, one I can write this is equal to D 0 square x plus 2 into D 0 D 1 x plus alpha 1 x dash plus alpha 2 x square equal to 0. Similarly, order for epsilon square can be written as D 0 square x plus 4 D 0 D 1 x dash plus 2 into 2 D 0 D 2 plus D 1 square x plus alpha 1 x double dash plus 2 alpha 2 x 2 dash plus 2 alpha 3 x 3 equal to 0. So, in this way we can write by different order of epsilon.

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$$x = \sum_{m=0}^{M} a_m(\epsilon, T_1, T_2)\cos m\left(\omega_0 T_0 + \theta(\epsilon, T_1, T_2)\right) + b_m(\epsilon, T_1, T_2)\sin m\left(\omega_0 T_0 + \theta(\epsilon; T_1, T_2)\right)$$

$$\omega_0 = \sqrt{\alpha_1}$$

$$\begin{cases} a_m = \underline{a_m^0(T_1, T_2)} + \epsilon a_m^1(T_1, T_2) + \epsilon^2 a_m^2(T_1, T_2) \\ b_m = b_m^0(T_1, T_2) + \epsilon b_m^1(T_1, T_2) + \epsilon^2 b_m^2(T_1, T_2) \\ \theta_m = \theta^0(T_1, T_2) + \epsilon \theta^1(T_1, T_2) + \epsilon^2 \theta^2(T_1, T_2) \end{cases}$$

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Now, by using this harmonic balance method, we can write this x equal to the solution x we can write in this form. So, this will be summation M equal to 0 to m. So, let us take M number of terms a m. So, in this case instead of using a constraint coefficient, we are using this coefficient which is a function of epsilon or parameter epsilon. Then T 1, T 2 it will take only up to two terms. One can write this way and then cos m omega 0 T 0 plus theta. So, theta is also we are writing function of epsilon T 1 T 2 plus this b m. So, this is cosine term and then we can write the sin terms also b m epsilon T 1 T 2. So, this is sin m omega 0 T 0 plus theta epsilon T 1 T 2.

So, here this omega 0 is nothing but the square root of this alpha 1. So, in comparison to our original harmonic balance method where we have taking this a m and b m to be constant, here we are taking this a m and b m as function of epsilon and the time scales in case of method of multiple scales. Also, we have taken this a m and b m or we have taken these coefficients as function of different time scale only. So, we have not considered that thing to be a parameter of epsilon, but in this intrinsic multiple scale harmonic balance method, we are considering this a m and b m to be a parameter of epsilon and also function of this different time scales T 1, T 2. So, if we are taking only two terms, we can take T 1 T 2 and if you take this higher order terms, then we can take as many term as we required. So, this a m can be written by using this epsilon.

So, we can write this a m equal to a m 0. So, this a m 0 is not a function or is not a parameter of epsilon. So, this a m 0 can be written as a m 0 T 1 T 2 plus epsilon a m 1 T 1 T 2 plus epsilon square a m 2 T 1 T 2 plus higher order terms. Similarly, this b m can be written as b m 0 T 1 T 2 plus epsilon b m 1 T 1 T 2 plus epsilon square b m 2 T 1 T 2 plus the higher order terms. Similarly, this theta m can be written. So, theta m can be written as theta 0 T 1 T 2 plus epsilon theta T 1 T 2 plus epsilon square theta T 1 T 2. So, in this way we have written this a m, b m and theta m by using this book keeping parameter epsilon.

So, in contrast to the method of multiple scale where we have taken only this term, this a m was only a m 0 that is a function of T 1 and T 2. Similarly, b m is a function of T 1 and T 2 and theta m as a constant here we are taking this a m b m and theta m are function of or using this book keeping parameter epsilon also in case of harmonic balance method. So, these terms a m and b m we have considered as constant. So, in this intrinsic harmonic balance method, we are taking this term x to be a m is a function of T 1 T 2, that is scaling parameters and also, the book keeping parameter. Similarly, b m is also a b m also contain epsilon and the scaling terms and theta also contain.

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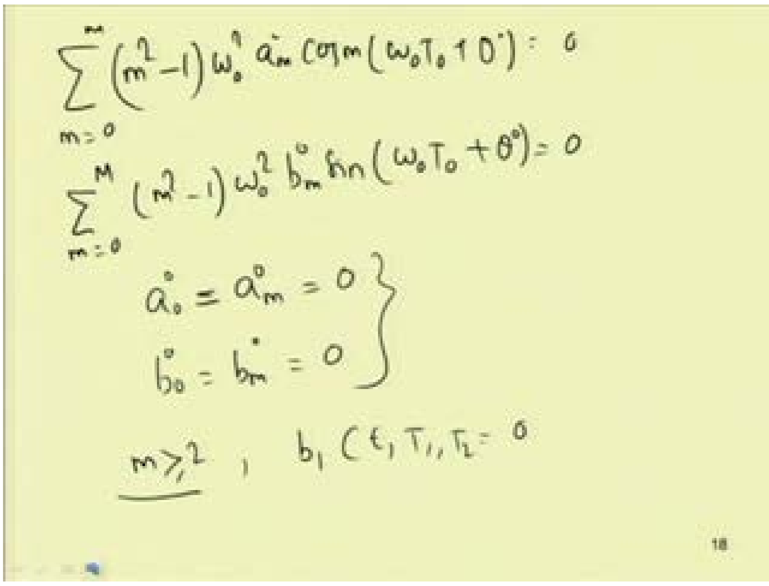

Similarly, by substituting this equation, this x in the previous equation that is order of epsilon to the power 0 and higher order, we can write this equation in this form, where it can be written in this that is it will be m equal to 0 to m. So, this becomes m square minus 1 into omega 0 square into a m 0 cos m into omega 0 T 0 plus theta 0 equal to 0 and also, this is m equal to 0 to m square minus 1 into omega 0 square b m 0 sin omega 0 T 0 plus theta 0 equal to 0. So, we can find this now for this particular case. So, we can find this a 0 0 equal to a m 0. So, this will be equal to 0 and also by substituting this for different coefficients. So, this is equal to 0. Now, we will equate different coefficient of cos and sin to 0 and we can get these equations. So, in this case we can get this a 0. So, this will be equal to a m 0 equal to 0.

Similarly, this b 0 0 will be equal to b m 0. So, this will be equal to 0. So, for m greater than 2 as the system is autonomous, so we get this b 1 epsilon T 1 T 2 equal to 0. So, in this way by getting a set of equations, now we can solve these equations to find this coefficient and after these coefficients, we can find the frequency amplitude relation. So, in this case from this equation, we can get this frequency amplitude relation to be in this form.

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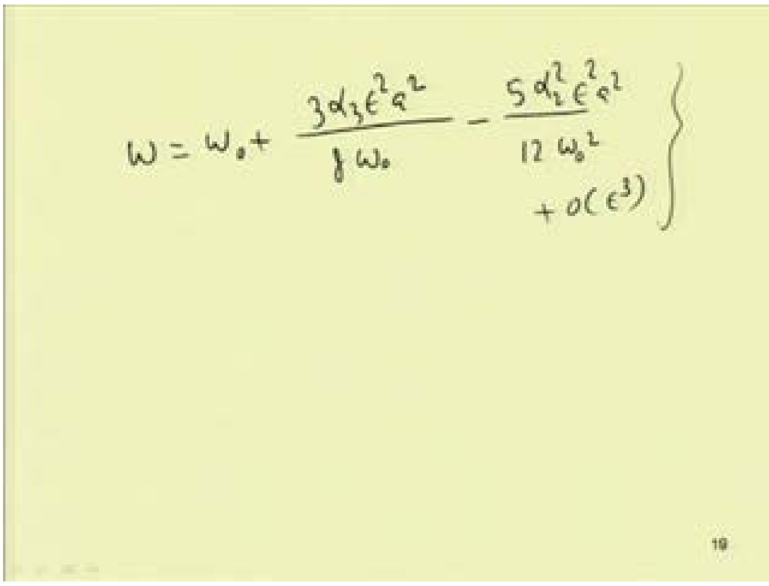

So, that is omega equal to omega equal to omega 0 plus 3 alpha 3 epsilon square a square by 8 omega 0 minus 5 alpha 2 square epsilon square a square 12 omega 0 square plus the higher order terms. So, by using this intrinsic multiple scale harmonic balance method also, one can solve different types of non-linear equations.

So, in today's class we have studied two methods. One is incremental harmonic balance method which is a two step method and also, we have studied this intrinsic multiple scale harmonic balance method. So, by using this method, we can solve a set of non-linear equations. So, in the next class we will study about this modified Linstedt Poincare method. Already you know in case of the Lindstedt Poincare method which is extension of the straight forward method. So, it has been modified for the strongly non-linear systems. So, we will study that modified Lindstedt Poincare method in the next class and also, we will study some other methods which are currently used to solve this non-linear equations.

Thank you.