Non-Linear Vibration
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Module - 3
Solution of Nonlinear Equation of Motion
Lecture - 7
Generalized Method of Averaging

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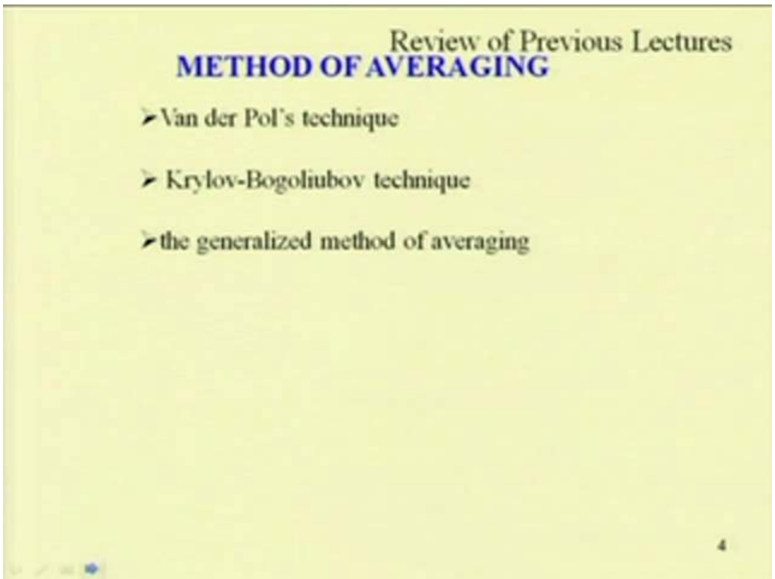

Welcome to today class of non-linear vibration. So, today's class we are going to discuss about the generalized method of averaging. Before that, we will revise what we have studied in case of the averaging method. So, these are different solution techniques for solving the non-linear differential equations. So, previous class we have studied 2 methods. One is Van der Pols technique and other one is this Krylow-Bogoliubov technique, which are part of this averaging method. So, this averaging method is a variational method, in which we are taking slowly varying terms and we are solving the equations.

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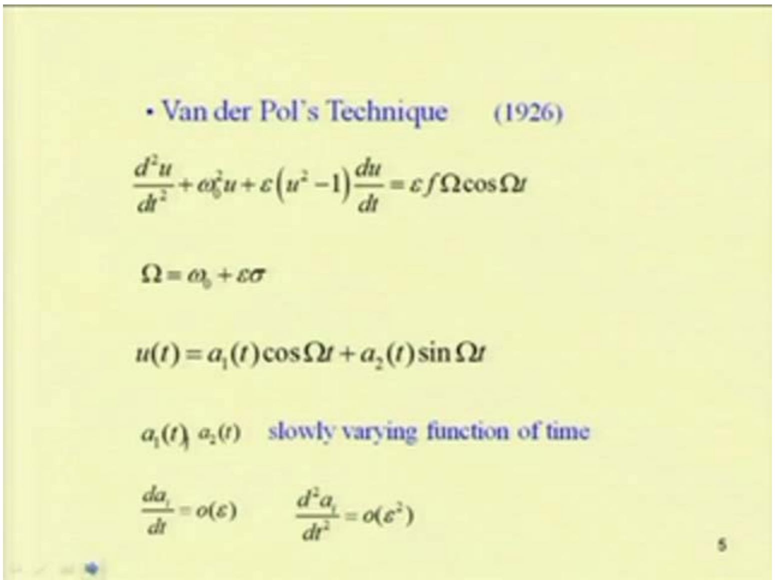

So, in case of the Van der Pol techniques, so which he has developed in 1926. So, this is the equation for the Van der Pol equation, that is d square u by d t square plus omega 0 square u plus epsilon u square minus 1 into d u by d t equal to epsilon f omega cos omega t. So, in this case to find the solution of this equation, so here this term is the non-linear term and to find the equation motion of this non-linear equation motion, so we can assume this omega, that is the external frequency, so that will be equal to this omega 0 plus epsilon sigma, where sigma is the detuning parameter, which express the nearness of this natural frequency to the external frequency.

And also assuming this a u t, u t equal to A 1 t cos omega t plus A 2 t sin omega t, so we can find the solution. So, in this case it may be noted that this, in the absence of this non-linear term, that is the epsilon, putting epsilon equal to 0, the solution of this equation, that is d square u by d t square plus omega 0 square u equal to epsilon f omega cos omega t. So can be written in terms of this A 1 t cos omega t plus A 2 t sin omega t or one can write this equation in terms of x sin omega t plus phi, where x and phi can be obtained.

In this case, by taking this non-linear term, instead of taken this constant A 1 A 2, which are solution of this equation, when epsilon equal to 0, so we can write the same equation. But here, we are taking this A 1 t and A 2 t are slowly varying function of time. So, in case of the Van der Pol method, so we are taking this u t equal to A 1 t cos omega t plus

A 2 t sin omega t, where A 1 and A 2 t are slowly varying function of time. So, as these are slowly varying function of time, so this d A 1 by d t or d A 2 by d t are considered to be order of epsilon and d square A 1 by d t square or d square A 2 d t square are considered to be order of epsilon square.

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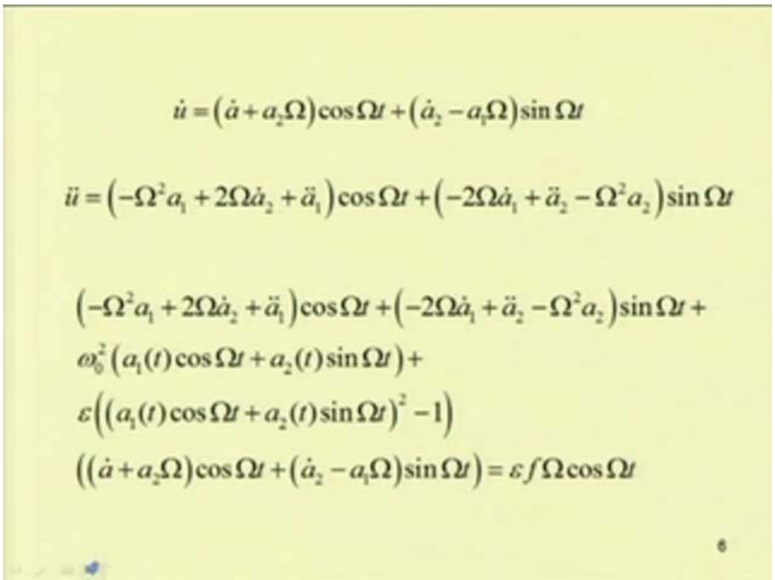

So, last class we have considered this method and by substituting this in the original equation and here also by differentiating this equation, we are taking this u dot equal to a dot plus A 2 omega cos omega t plus A 2 dot minus A 1 omega sin omega t and again differentiating this equation, we have this u double dot equal to minus omega square A 1 plus 2 omega A 2 dot plus A 1 double dot cos omega t plus minus 2 omega A 1 dot plus A 2 dot minus omega square A 2 sin omega t and substituting this u double dot equation and u dot equation and u equation in the governing equation, that is d square u by d t square plus omega 0 square u plus epsilon u square minus 1 d u by d t equal to epsilon f omega cos omega t, so we got this expression.

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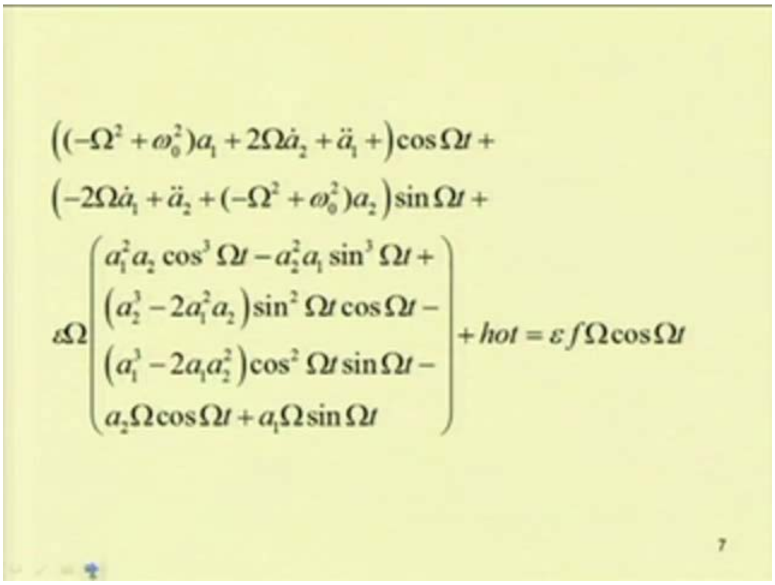

So, in this case, now this equation, so we have to write using these harmonic terms. So, this sin cube and the sin square omega t into cos omega t cos square omega t into sin omega t and similar terms we have to write in terms of the different harmonic terms.

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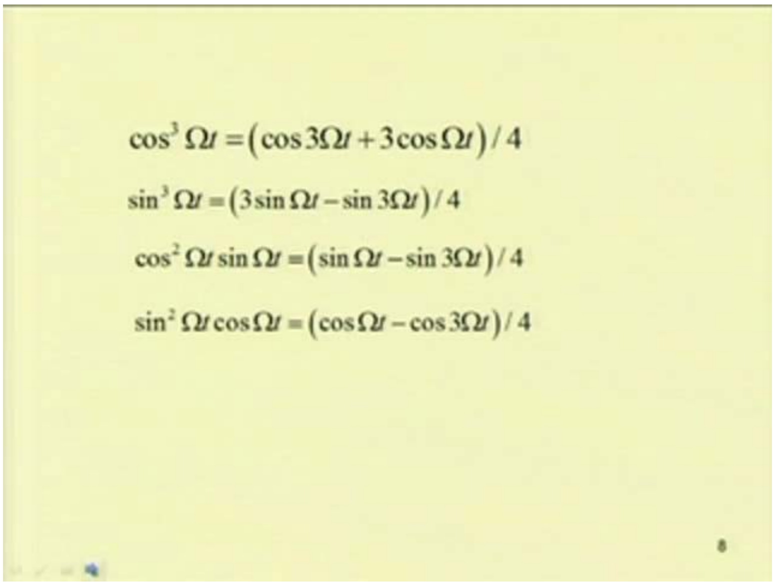

By writing these things, so we can view this transformation, that is cos cube omega t equal to cos 3 omega t plus 3 cos omega t by 4 sin cube omega t equal 3 sin omega t minus sin 3 omega t by 4. Similarly, cos square omega t sin omega t equal to sin omega t minus sin 3 omega t by 4 and sin square omega t cos omega t equal to cos omega t minus cos 3 omega t by 4.

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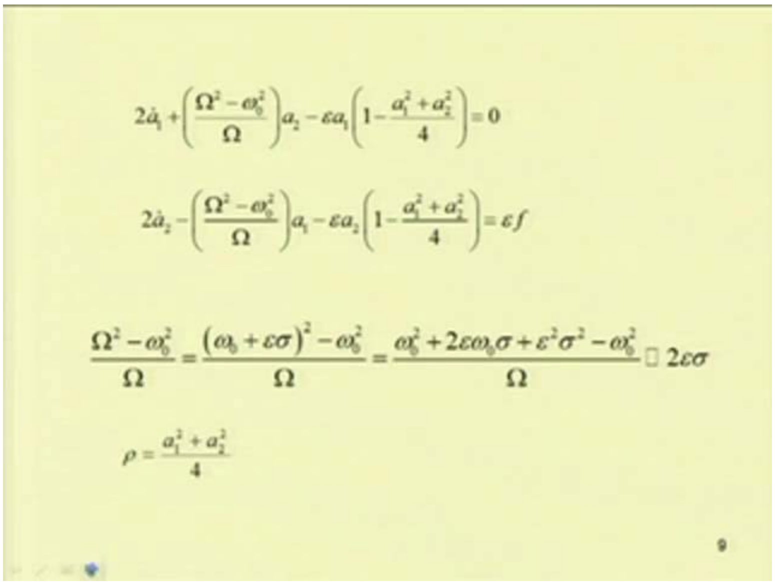

So, by substituting this equation in the previous equation and collecting the coefficient of cos omega t and sin omega t, so we can have this expression. So, these 2 expressions in A 1 dot and A 2 dot, we can solve these equations to find the expression. So, before that, so we can write this omega square minus omega 0 square by omega, nearly equal to this 2 epsilon sigma.

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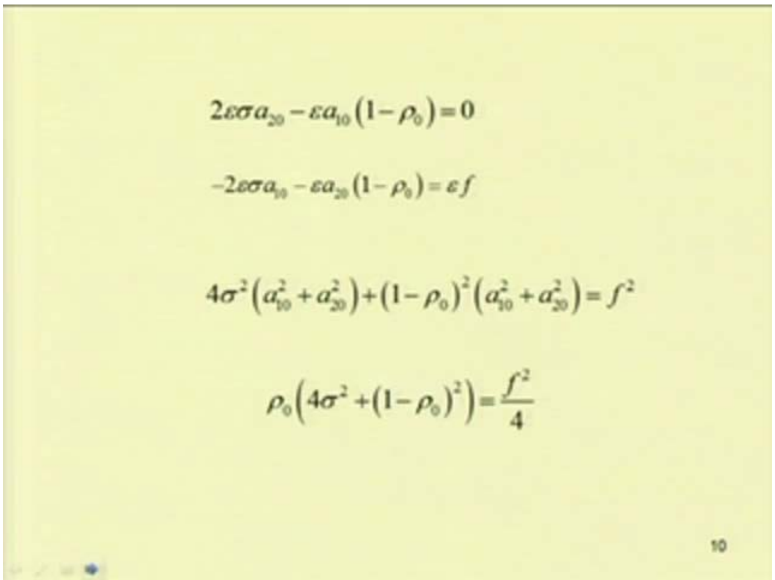

So, this previous equation reduces to this. For steady state, by putting A 1 equal to A 1 0 and A 2 equal to A 2 0, so we can write this equation, the previous equation in this form, from which we can get the frequency response equation. So, in case of the Van der Pol equation or Van der Pol method, so initially we have taken a slowly varying function A 1 t and A 2 t and then substituting that equation in the governing equation and then by collecting the coefficient of the harmonic terms, so we got the frequency response equation. So, from this, by plotting this frequency response equation, one can study different responses of the system. That thing, we will we study in other classes.

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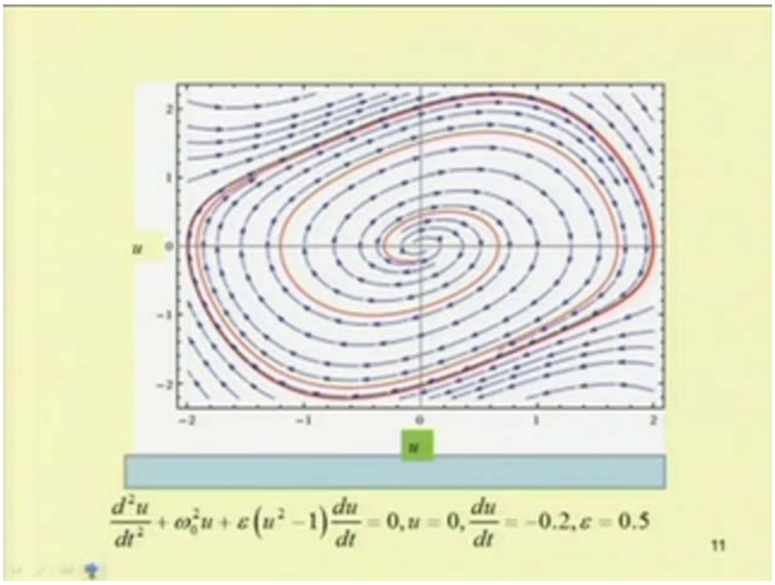

So, this is the flow. That is, the velocity u verses u dot curve. So, one can obtain this thing by solving directly the equation, Van der Pol equation. So, we show clearly a limit cycle solution and also one can plot this equation to find the same unit cycle.

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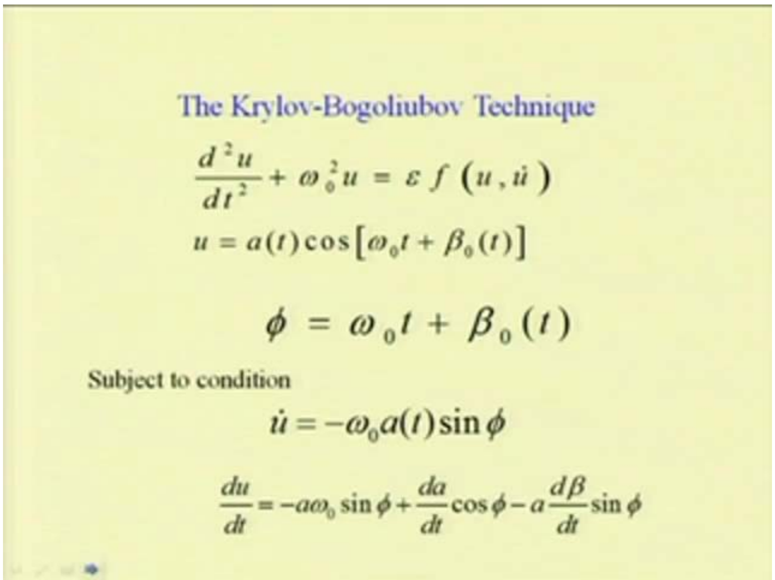

Now, by using this Krylow-Bogoliubov technique, so in this technique, we have, let us take this example of a, simple example and we can solve this thing. So, in this case, one can write the equation. So, let us write the equation in this form also, d square u by d t square plus omega 0 square u equal to epsilon f u u dot, where u equal to a t cos omega 0 t plus b 0 t. So here, we are taking this A as a function of time. Similar to in Van der Pol equation, we have taken A 1 and A 2, two terms. Here, you were also taking 2 terms, 2 unknown variables, that is a and beta. So, this is a t; a is a function of time and also beta 0 is a function of time. So, one can write this equation.

So, this equation is similar to that taken in case of the Van der Pol equation. But, in that case, we have taken A 1 t and A 2 t and in this case, we are taking a and beta. So, a is the amplitude and beta is the phase. So here, by taking this phi equal to omega 0 t plus beta 0 t, so this equation can be written, this u can be written as a, a t cos, so one can write this u equal to a t cos phi t. So, this phi is a function of time. So, this can be written as a t cos phi t. But, in this case, in case of Krylow-Bogoliubov technique, so we are putting one condition. So, we are writing that this u equal to a t cos phi t will be the solution of the first equation subjected to the condition that u dot equal to minus omega 0 a t sin phi. So, but if one differentiate these things, so it will be minus a t into omega sin phi t. So, plus differentiation of this a t term and but neglecting those things, so we are writing that this u will be equal to a t cos phi t, subjected to the condition u dot equal to minus omega 0 a t sin phi.

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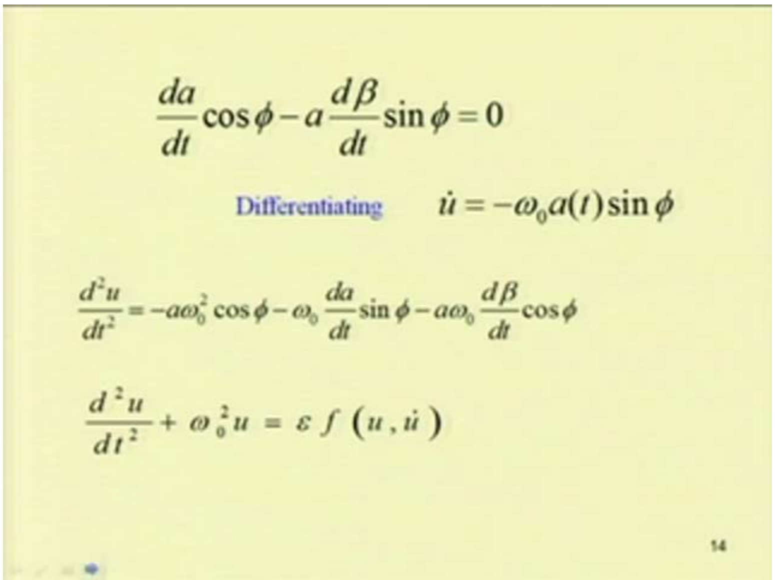

So, to satisfy this condition, so differentiating this we can write d u by d t equal to in this form and by substituting this condition, we can have a condition, that is d a by d t cos phi minus a d beta by d t sin phi. So, it will be equal to 0 and differentiating this thing also we can get this d square u by d t square in this form.

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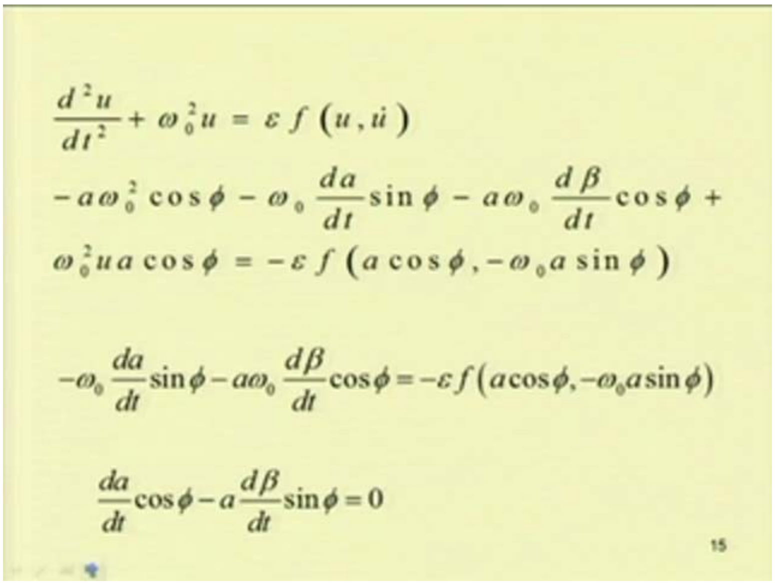

So, substituting this d square u by d t square and u dot and u equation in the original equation, that is in this equation, so we can write this equation in this form, that is d square u by d t square plus omega 0 square u equal to epsilon f u u dot. So, by substituting all these things, so we can and equating the, so by substituting this thing and equating the coefficient of cos phi and sin phi, one can write these 2 equations, that is minus omega 0 d a by d t sin phi minus a omega 0 d a omega 0 d beta by d t cos phi equal to minus epsilon f a cos phi minus epsilon 0 a phi. So, this is for u dot and this is for u. So, one can obtain this equation minus omega 0 d a by d t sin phi minus a omega 0 d beta by d t cos phi equal to minus epsilon f a cos phi minus omega 0 a sin phi.

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$$\left.\begin{aligned}\frac{da}{dt} &= -\frac{\varepsilon}{\omega_0}\sin\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)\\ \frac{d\beta}{dt} &= -\frac{\varepsilon}{a\omega_0}\cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)\end{aligned}\right\}$$

$$\frac{da}{dt} = -\frac{\varepsilon}{2\omega_0}\left[\frac{2}{T}\int_0^T \sin\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)dt\right]$$

$$\frac{d\beta}{dt} = -\frac{\varepsilon}{2a\omega_0}\left[\frac{2}{T}\int_0^T \cos\phi\, f\left(a\cos\phi, -\omega_0 a\sin\phi\right)dt\right]$$

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So, from this equation and the previous equation, one can find a set of equation. So, that is d a by d t equal to minus epsilon omega 0 sin phi f a cos phi minus omega 0 a sin phi and also d beta by d t equal to minus epsilon by a omega 0 cos phi f a cos phi minus omega 0 a sin phi. So, as we know that this right hand side is a periodic function, that is with a period of t, so and these values are slowly varying, d a by d t and d beta by d t are slow varying function, so we can average this equation to find d a by d t equal to minus epsilon by 2 omega 0 2 by t 0 2 t sin phi into f a cos phi minus omega 0 a sin phi d t. Similarly, d beta by d t, so by averaging this thing, one can write minus epsilon by 2 a omega 0 2 by t 0 to t cos phi f a cos phi minus omega 0 a sin phi d t.

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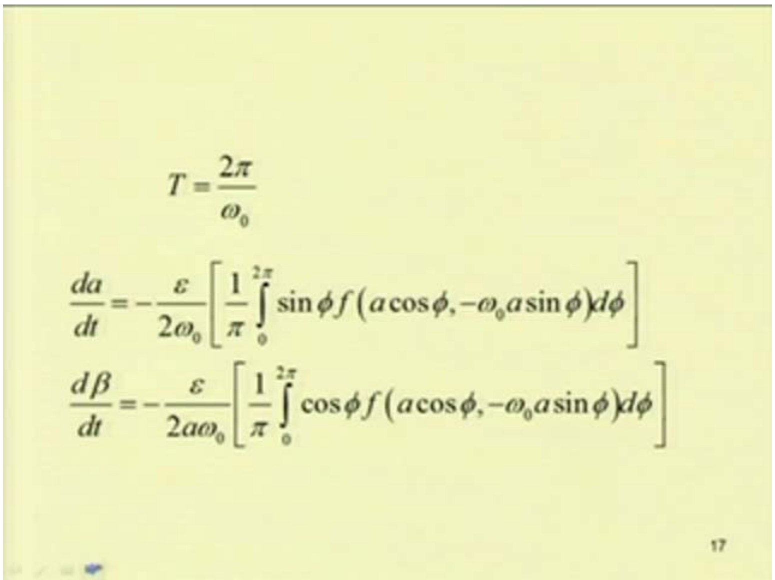

So, as t equal to 2 phi by omega 0, so instead of writing in terms of, in terms of time period t, so one can write in terms of the phase phi, phase angle phi. So, one can write the same equation in terms of this. So, d a by d t equal to minus epsilon by 2 omega 0 1 by phi integration 0 to 2 phi sin phi f a cos phi minus omega 0 a sin phi d phi and d beta by d t equal to minus epsilon by 2 a omega 0 1 by phi 0 to 2 phi cos phi f a cos phi minus omega 0 a sin phi d phi.

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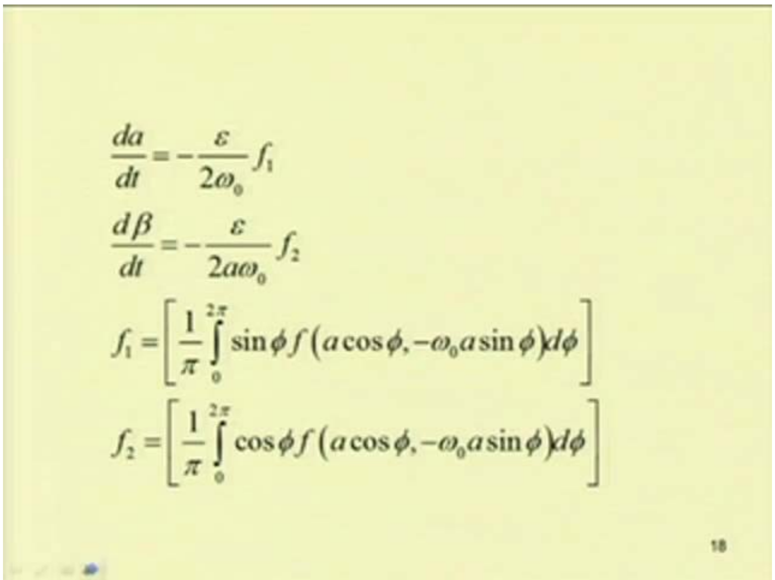

So, one can put this as function f 1 and this function f 2. So, d a by d t can be written as minus epsilon by 2 omega 0 f 1 and d beta by d t equal to minus epsilon by 2 a omega 0 f 2. So, f 1 and f 2 are given. So, from these one can find the, by integrating this thing one can find the equation in terms of d a by d t and from that, by integrating or by solving that equation, one can get the frequency response equation, that is a in terms of omega or detuning parameter and one can find the final solution.

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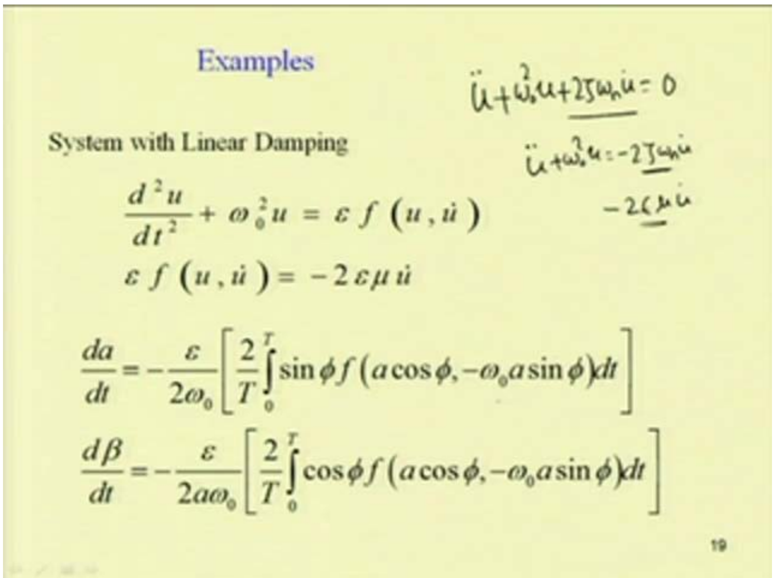

So, last class we have seen this example. That is, let us solve this example once again, that is for a linear damping. So, in case of the linear damping, we can write this d square u by d t square. One can write the equation in this form, u double dot, that is d square u by d t square plus omega 0 square u, so plus 2 zeta omega n u dot equal to 0. So, this 2 zeta omega n u dot can be written. So, if one take it to right end side, then this equation can be written u double dot plus omega 0 square u equal to minus 2 zeta omega n u dot and this zeta omega n, we can write equal to epsilon mu.

So, this equation will be reduced to minus 2 epsilon mu u dot, where this epsilon mu will be equal to zeta omega n, where epsilon is the book keeping parameter, which is very very less than 1 and taking this epsilon value, one can take this mu value. So, one can write this epsilon f u u dot in equal to minus 2 epsilon mu u dot. So, this f will be equal to minus 2 mu into u dot. Now, this d a by d t can be written as minus epsilon by 2 omega 0 2 by t 0 to t sin phi a cos phi minus omega 0 a sin phi d t.

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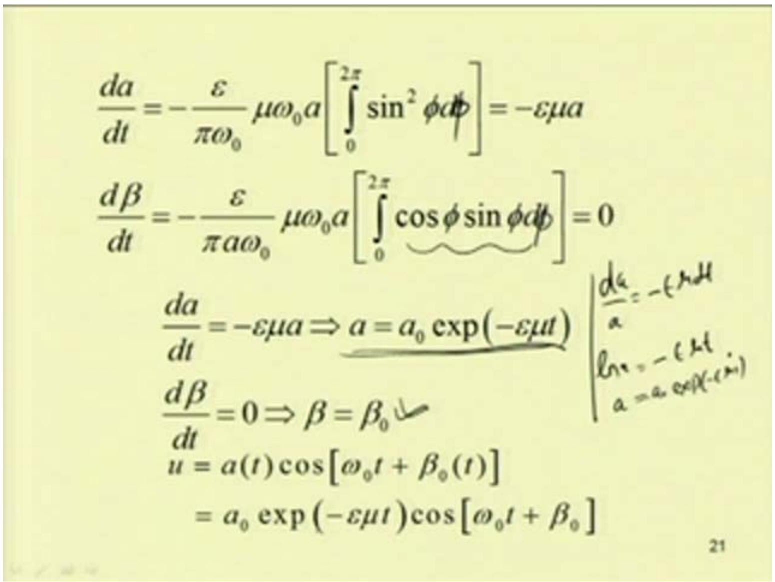

Or, in terms of this, one can write using this period 0 to 2 phi. So, this d a by d t can be written as minus epsilon by phi omega 0 mu omega 0 a 0 to 2 phi sin square phi d d phi. So, this is sin square phi d phi and this is, so from this, by integrating, so one can find, so this term equal to minus epsilon mu a. Similar, but integration of this thing equal to 0, so one can get this is equal to 0, so d beta d t equal to 0. So, beta will be equal to a constant. So, beta is a constant. So, one can write beta equal to beta 0 as d a by d t equal to minus epsilon mu a, so one can find this a, so d a by a will be equal to… So to solve this thing, so one can write this d a by a equal to minus epsilon mu d t. So, by integrating, so this is l n a. So, this will be equal to minus epsilon mu t plus some constant a 0.

So, one can view this constant a 0 or one can write this is l n a equal to minus epsilon mu t. So, a will be equal to, so a, one can write in terms of this e. So, e to the power, so one can write using a constant also. So, it will be a 0 e to the power minus epsilon mu t. So, this a is a function of time. So, a equal a 0 e to the power minus epsilon mu t and beta equal to beta 0 as u equal to a t cos omega t plus beta 0 t, so substituting the value of a t equal to a 0 e to the power minus epsilon mu t, we can write the solution u equal to a 0 e to the power minus epsilon mu t into cos omega 0 t plus beta 0.

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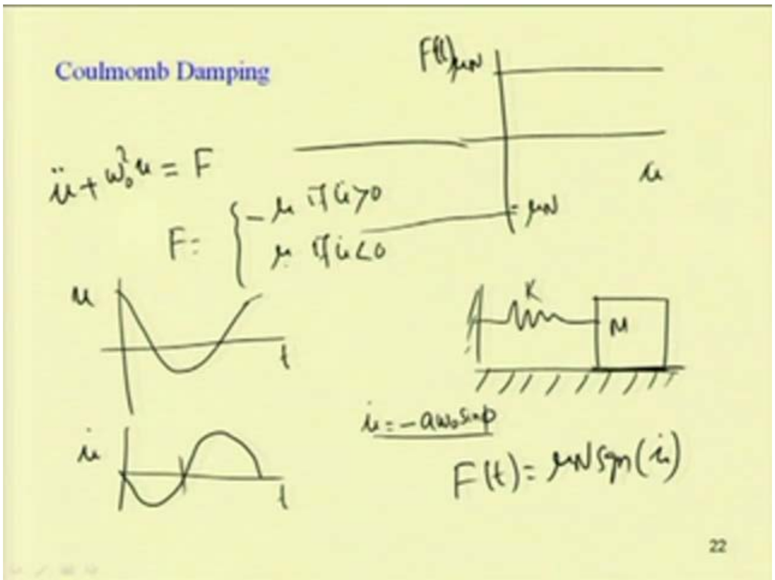

Let us take another example using this Coulmomb Damping. So, in case of Coulmomb Damping, this force f t can be written. So, this force f t is a function of this u dot, that is velocity. So, if velocity is positive, then one can, so one has plus mu n and if velocity is negative, so one has minus mu n. So, this is Coulmomb Damping. So, in Coulmomb damping, for positive u dot, so one can get the force equal mu n and for negative damping, this force equal to minus mu n. So, by considering a system with Coulmomb Damping, for example, let us take the system. So, let us take a simple spring and wire system. So, in this case, this is spring k and so we have Coulmomb Damping between these two surfaces. So, this equation can be written in this form. So, u double dot plus omega 0 square u. So, this is equal to, so this will be equal to this Coulmomb Damping force f. So, this Coulmomb Damping force, in this case, now can be written, so this f can be written equal to minus mu and plus mu.

So, it will be minus mu because, so if we are taking, so already we know that, we are substituting this u dot equal to minus a omega 0 minus a omega 0 sin phi. So, if you plot these things, so if you plot this u and u dot, so this is the plot of u. So, u equal to a cos phi, so this is u versus time or phi you can take. Similarly, u dot versus t, if one plot, so this will be equal to a omega 0 sin omega t. But, one has to substitute this minus a omega 0 sin phi. So, as minus, one has to substitute, so this u dot will be equal to this. So, this is u dot and that is your minus a omega 0 sin phi. So, up to this, so up to phi 0 to phi, so this becomes negative. So, if it is negative then so if u dot less than 0, so this is if u dot greater than 0, so if u dot greater than 0, this force will be equal to minus mu and if it is less than 0, so one can see, so we know this force f t can be written as mu n sigma u dot. So, it will be, so when u dot is positive, so then it will be plus mu n and then it is minus mu n. But, as we are taking this u dot equal to minus a omega 0 sin phi, so now, one can write this f equal to minus u mu, if u dot greater than 0 and equal to mu, if u dot less than 0.

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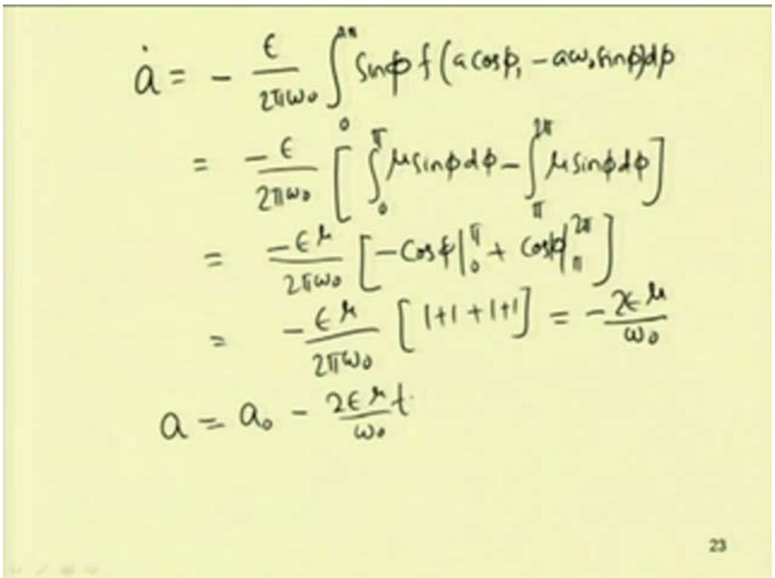

So, by using the previous equation, that is a dot equal to, so a dot equal to minus epsilon by 2 pi omega 0 integration 0 to 2 pi sin phi into f. So, that is our function. So, in this case, this function f a cos phi and minus a omega 0 sin phi d phi. So, this thing can be written in terms of 0 to phi plus phi to 2 phi. So, it can be written minus epsilon by 2 phi omega 0 integration 0 to pi. So, it will be mu sin phi. So, 0 to pi, it will be mu. So, it is 0 to pi. So, for 0 to phi portion, so u dot is less than 0. So, u dot is less than 0 negative. So, as u dot is less than 0, so this becomes plus phi mu. So, mu sin phi d phi, then plus pi to 2 phi. So, one can integrate it pi to 2 pi, so this is plus. Then we have this for pi to 2 pi, so as this value is positive, so u dot is greater than 0, so f will have a value of minus mu. So, f will have a value of minus mu. So, this becomes minus then. So, this is minus and this is minus mu sin phi d phi.

So now, this integration, by integrating this thing, so we can write this. We can take this mu common, so minus epsilon mu by 2 pi omega 0. So, this integration will give you, so this is minus cos phi from 0 to pi, so plus cos phi from pi to 2 pi. So, this becomes minus epsilon mu by 2 pi omega 0. So, this is 1 plus 1. So, it is here also 1 plus 1, so this is 4. So, this becomes minus 2 epsilon mu by omega 0. So, as d a by d t equal to minus epsilon mu by omega 0, so due to presence of omega epsilon, one can see that this is a slowly varying function of time. So, this a will be equal to, so one can write this a equal to, so this part is constant. So, one can write this equal to a 0 minus 2 epsilon mu by omega 0 into t.

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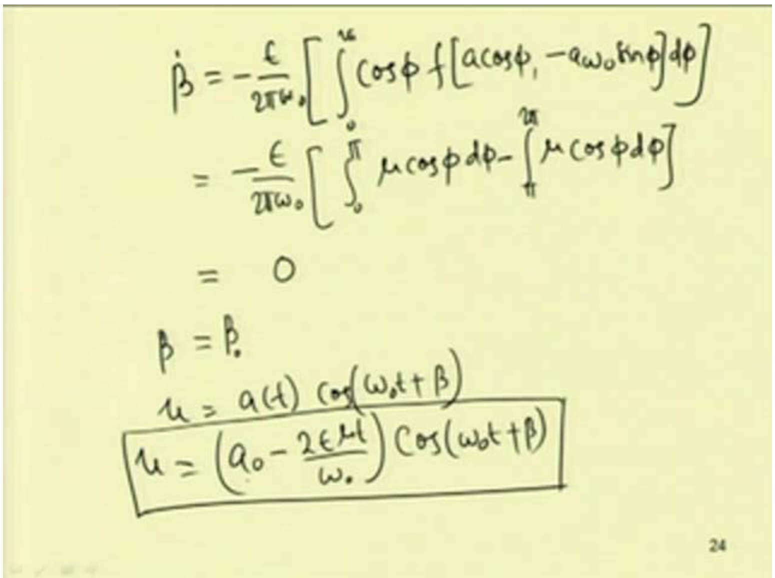

Now, for finding beta, so we know this beta dot equal to, so beta dot equal to minus epsilon by 2 pi omega 0. So here, integration is in terms of cos. So, this integration 0 to 2 pi cos phi into function of u and u dot. So, for u, one can substitute a cos phi and for u dot, similarly one can put a omega 0 sin phi d phi. So, in the present case, so one can write this thing equal to, so it will be equal to minus epsilon by 2 pi omega 0. Like in the previous case, it can be divided into 2 parts. 0 to pi, it will be plus mu, so this is mu cos phi and d phi plus pi to 2 pi. So, minus mu, so this will be minus. So, one can put a minus sign, so this is minus mu cos phi d phi. So, as this integration is 0, so this beta dot equal to 0, so beta d beta by d t equal to 0. So, beta is a constant; beta equal to beta 0.

So, from this, we obtain that a equal to a 0 minus a equal to a 0 minus 2 epsilon mu by omega 0 t and beta equal to beta 0, so this, we can write this u equal to, as u equal to a t cos omega 0 t plus beta, so we can write for this. So, this is equal to a 0 minus 2 epsilon mu. So, minus 2 epsilon mu t by omega 0 whole into this cos omega 0 t plus beta. So, this is the expression for response of the system with Coulmomb Damping. So, one can plot this amplitude, that is a 0 minus 2 epsilon mu t by omega 0. So, this is varying with time. So, one can see that this amplitude varies with time, so this is the property in case of the non-linear system. So, next we will study about the generalized method of averaging.

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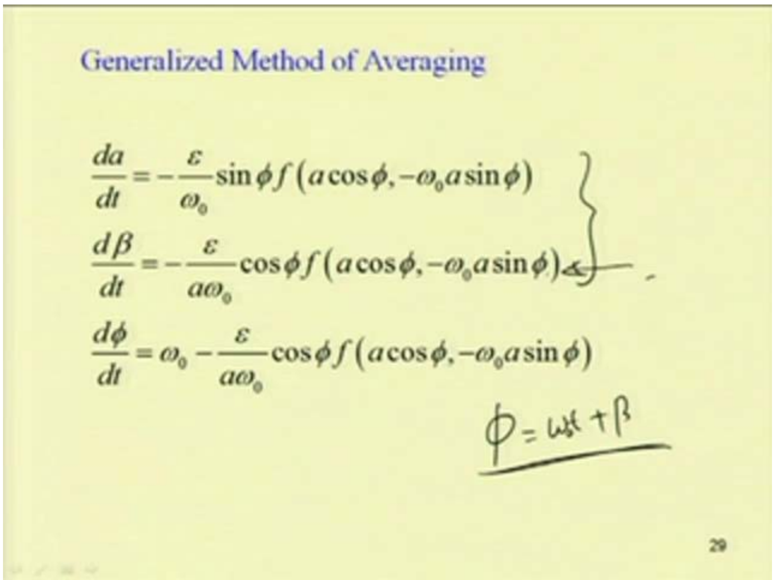

So, in the past 2 methods, that is in Van der Pol's technique and in the Krylow-Bogoliubov technique, so we have taken this a A 1 A 2 or a and phi as the slowly varying function of phi, slowly varying function of time and by substituting this equation in the original equation and we have obtained the frequency response equation or the response equation of the system. So, this generalized method of averaging is the extension of this Krylow-Bogoliubov method. So, in Krylow-Bogoliubov method, we have obtained these 2 equations, that this d a by d t and d beta by d t equal to or this d a by d t equal to minus epsilon by omega 0 sin phi f a cos phi minus omega 0 a sin phi and d beta by d t equal to minus epsilon by omega 0 cos phi f a cos phi and minus omega 0 a sin phi and this d phi by d t equal to minus omega 0, so this d beta by d t, initially we have written, so d beta by d t equal to minus epsilon by a omega 0 cos phi this, so as beta, so we have taken this beta equal to omega or we have taken this phi equal to omega 0 t plus beta.

So, we can write this equation in terms of d phi d t also. The second equation can be written in terms of d phi by d t. So, by writing this d beta by d t equal to, so d phi by d t, so it will be equal to omega 0 minus this term. So, omega 0 minus epsilon by a omega 0 cos phi function of a cos phi minus omega 0 a sin phi. So here, we did the transform from u to a and beta or cos a and phi. So, we have written this u in terms of a and phi. But, in case of the generalized method of averaging, so we have to do further transformation. So, in case of Krylow-Bogoliubov method, so considering this right hand side or this d a by d t and d beta by d t are slowly varying function of time. So, we have integrated these equations to find the solution.

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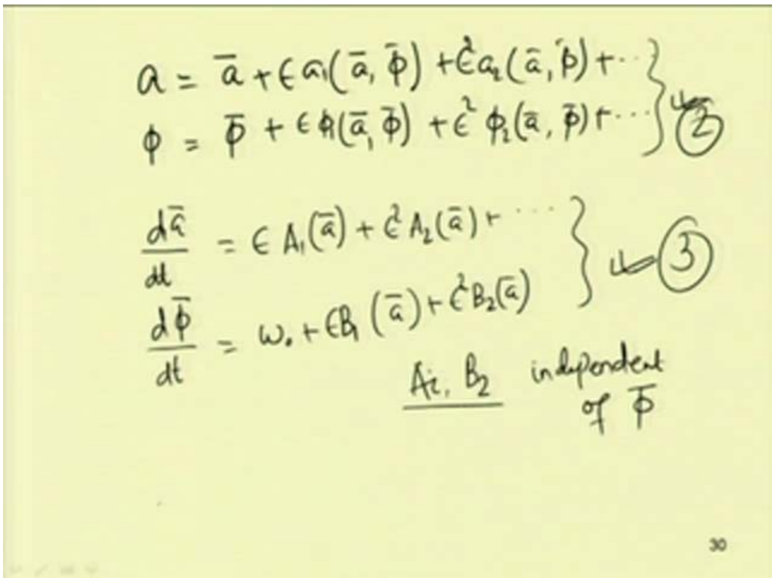

But, in case of this generalized method of averaging, so instead of integrating the solution, so we can do the further transformation by substituting this a equal to a bar plus epsilon A 1, which is a function of a bar and phi bar plus epsilon square A 2, which is a function of a bar and phi bar and one can take the higher order terms of epsilon also. Similarly, phi equal to phi bar plus epsilon phi 1, so which is a function of a bar and phi bar plus epsilon square phi 2 a bar and phi bar. Similarly, one can take higher order terms.

So, now substituting this equation in the previous equation, that is d a and d phi, so one can write this equation in this form, that is d a bar by d t equal to epsilon A 1, so which will be a function of a bar plus epsilon square A 2, which is a function of a bar. Also one can write some higher order terms. Similarly, one can write this d phi bar by d t, so d phi bar by d tm so if you substitute this equation in this equation, so one can write this is equal to epsilon. So, one can write this is equal to, so the first term will be equal to similar to this. So, this will be omega 0. So, it will be equal to omega 0 plus epsilon beta one, which will be a function of a bar plus epsilon square beta 2 or epsilon square.

So, either one can write beta or simply b epsilon B 1 and this is epsilon square b 2 function of a bar. So, it can be noted that this A 1 a i and b I, that is A 1 B 1 or A 2 and b 2, so they are independent of phi bar. So, they are function of a bar only. So, by substituting this a equal to a bar epsilon A 1 a bar phi bar plus epsilon square A 2 a bar phi bar and phi equal to phi bar epsilon phi 1 A 1 bar phi bar plus epsilon square phi 2 a bar phi bar, so in the previous equation, so one can get this d a bar and d phi bar in this way. So now, substituting this equation, so that is this and the previous one in this equation, so one can write and expanding these equations and collecting the coefficient of, like power of epsilon, so substitute these equations, let me put the equation number, so this is equation 1, equation 2, and equation 3. So, by substituting equation 3 and 2 in 1 and collecting the coefficients of like power of epsilon, so one can obtain this equation in this form.

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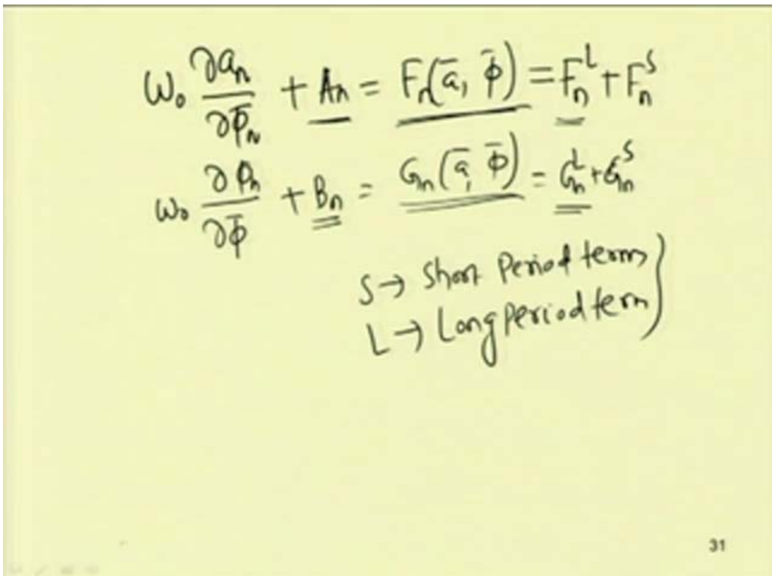

So, it will be omega 0 del a n by del phi n bar plus a n equal to f n a bar phi bar. Similarly, second equation can be obtained; omega 0 del phi n by del phi bar. So, plus b n, so this will be equal to g n a bar phi bar. So, this right hand side, this f n a bar phi bar and this g n a bar phi bar can be written or can be seen as this slowly varying period terms and with, so one can write these things as f n l plus f n s. Similarly, this thing can be written as g n l plus g n s. So, some short period term. So, s for short period term and l for long period term. So, this equation, that is omega 0 del a n by del phi n, where n can be 1 2 3 4 plus a n equal to f n a bar phi bar equal to f n long period terms plus f n short period terms. Similarly, in the second equation, this is g n long period terms plus g n short period terms. Now, we can choose this a n equal to f n long period terms and similarly, b n equal to g n long period terms. So, if you choose in this way, so then these 2 equations can be reduced to omega 0 del a n by, so this will be reduced to omega 0 del a n by del. This is del a n by del phi bar.

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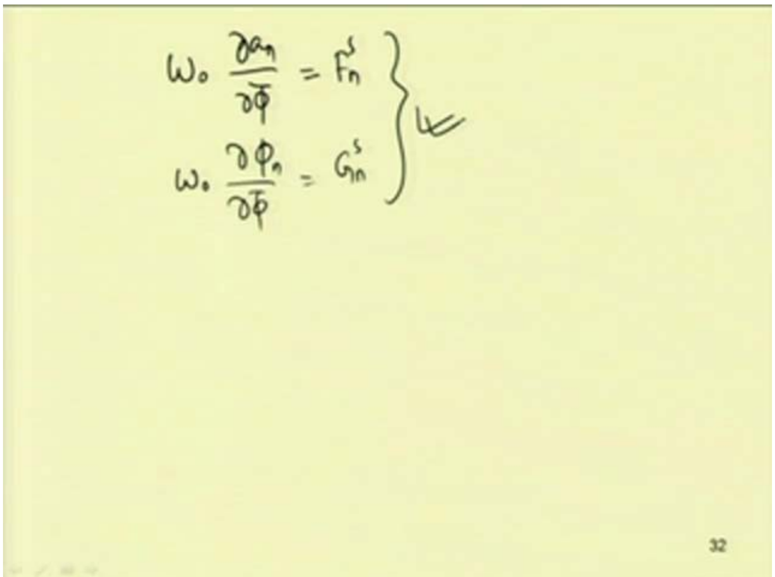

So, this will be equal to f n short period terms. Similarly, omega 0 del phi n by del phi bar, so this will be equal to g n short period terms. As these terms are known, so one can find by solving these equations. So, in this way by using this generalized method of averaging, so one can find the solution of the equation. So, the difference between the previous Krylow-Bogoliubov method and in this method is that, so here we are starting with the equation what we have found from this Krylow-Bogoliubov method. That is, d a by d t equal to minus epsilon by omega 0 sin phi into f. Similarly, d beta by d t equal to minus epsilon by a mega 0 cos phi into f, where we have written this d beta in terms of this d phi.

So, we have written this d phi by d t equal to, so d phi by d t equal to omega 0. In terms of omega 0 we have written. So, after writing these terms in terms of the omega 0 terms, so we can, so d phi by d t equal to minus omega 0 minus epsilon by a omega 0 cos phi f a cos phi minus omega 0 sin phi. Here, instead of integrating these equations to obtain relation between d a by d t and d phi by d t, so another transformation has been carried out. So, a is written as a bar plus epsilon a bar, which are function of a bar and phi bar plus epsilon square A 2 or some higher order terms can be added. Similarly, phi is written as phi bar plus epsilon phi 1 plus epsilon square phi 2 and then substituting that equation in previous equation, so one can write or from these one can write this d a by d t equal to d a by d t as a bar and phi bar can be considered constant.

So, it will be equal to epsilon a capital A 1 a bar plus epsilon square A 2 a bar. Similarly, d phi bar by d t equal to omega 0 epsilon b 1 plus epsilon square b 2. So here, a i and b i, that is i equal to 1 to 2. So, r independent of phi, so that means they are written in terms of a bar and now, by taking, substituting this equation in equation 1, so one can write this omega 0 del a n by del phi n bar plus a n equal to f n a bar phi bar, so where this f n can be written in terms of the slowly varying or long period term with some long period term and with some short period terms.

Similarly, this second equation also can be written using some long period term and short period term. Now, by taking this a n equal to these long period terms, by assuming this a n equal to this long period term, so one can find this omega 0 del a n by del phi n equal to f n s. So, one can solve those things to find a i and or a n. Similarly, in second equation, by taking this b n equal to this long period term and solving this equation, so one can find this a n and phi n. So, after finding this a n and phi n, so as we have taken a equal to a bar this term A 1 A 2 and phi bar equal to phi bar plus epsilon phi 1 plus phi 2, so one can find this a and phi. So, after obtaining a phi, so u equal to a t cos phi. So, that way one can obtain the solution. So, let us take one example, that is the same Duffing equation let us take, and let us take the Van der Pol’s equation and solve these equations.

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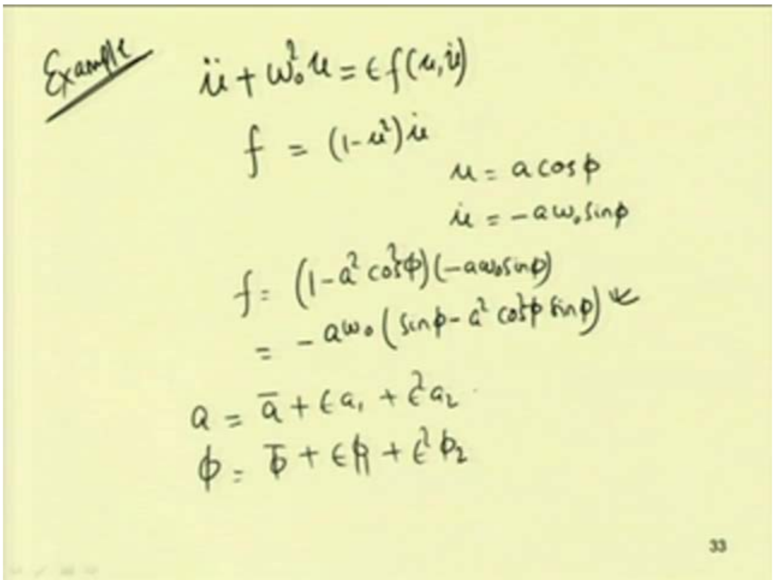

So, in case of the Van der Pol’s equation, so we can write or this equation is, so u double dot plus, so let us take this example and solve this thing using this generalized method of averaging. So, u double dot plus omega 0 square u equal to this epsilon f, so here, epsilon f u u dot. So, in case of Van der Pol’s equation, this f can be written equal to 1 minus u square into u dot. So, 1 minus u square into u dot, one can write. So now, already we know we have to substitute a u equal to a cos phi. So, we have substituted u equal to a cos phi and u dot equal to minus a omega 0 sin phi. So, by substituting this in this equation, so we have this f equal to 1 minus mu square. So, u square equal to a square cos square phi and u square cos square phi into minus a omega 0 sin phi. So, this equation becomes, so minus a into omega 0 sin phi minus a square cos square phi into sin phi. So, this is our f now.

So now, by using this f in equation 1, that is this d a by d t, so we can write this d a by d t and d phi by d t. So, this d a by d t and d phi by d t before writing that thing, so we can substitute this a equal to A 1 a bar. So, by substituting a equal to, so let us substitute a equal to a bar plus epsilon a 1 plus epsilon square A 2. Similarly, this phi equal to phi bar plus epsilon, so we have written this phi equal to epsilon phi 1 and epsilon square phi 2 epsilon phi 1 plus epsilon square phi 2.

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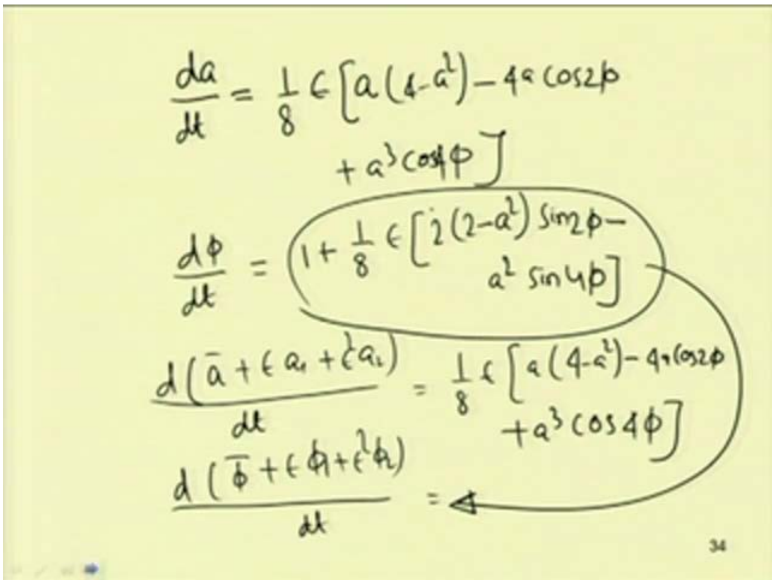

So, this d A 1 by, so by substituting this thing, so we can write this equation d a by d t. So, this will be, this can be written as 1 by 8. So, this equation can be written as 1 by 8 epsilon into a 4 minus a square minus 4 a cos 2 phi plus a cube cos 4 phi. Similarly, this d phi by d t equal to 1 plus 1 by 8 epsilon into 2 into 2 minus a square sin 2 phi minus a square sin 4 phi. So, collecting the order of epsilon, so one can write this equation now. So, d a by d t equal to a bar plus, we have to write this equation in this form or a equal to a bar plus epsilon A 1 plus epsilon square A 2.

So, if I will write that thing, so then this equation can be written as b a bar epsilon. So, substituting this equation in this equation, so this is, let me write this d. So, that is a bar plus epsilon A 1 plus epsilon square A 2 by d t. So, this will be equal to the first equation, that is 1 by 8 epsilon into a into 4 minus a square minus 4 a cos 2 phi plus a cube cos 4 phi. Similarly, I can write for the second equation, that is d phi bar plus epsilon phi 1 plus epsilon square phi 2 by d t will be equal to the second equation, the right hand side of this equation can be written in this place. So now, collecting the coefficients, so from this it will be epsilon d A 1 bar by d t plus epsilon square d A 2 by d t.

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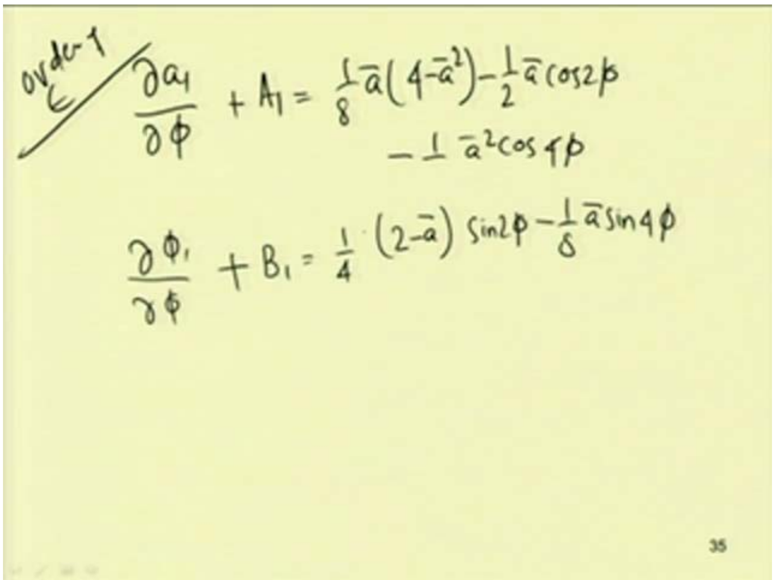

So now, collecting the coefficient or equating the coefficient of epsilon and epsilon, so one can we can write, so we can write this thing d del A 1 by del phi. So, it will be equal to del A 1 del phi. So, plus A 1, so that will be equal to 1 by 8 a bar 4 minus a bar square minus 1 by 2 a bar cos 2 phi minus 1 by 8 a bar square cos 4 phi. Similarly, del phi 1 by del phi can be written, plus this b 1 will be equal to 1 by 4 2 minus a bar sin 2 phi minus 1 by 8 a bar sin 4 phi. Now, taking A 1 and B 1 as the terms with long period, that is A 1 we can take with long period. This is the with long period and with short period, so we can write, so we can write this A 1 equal to with, so this is order of epsilon.

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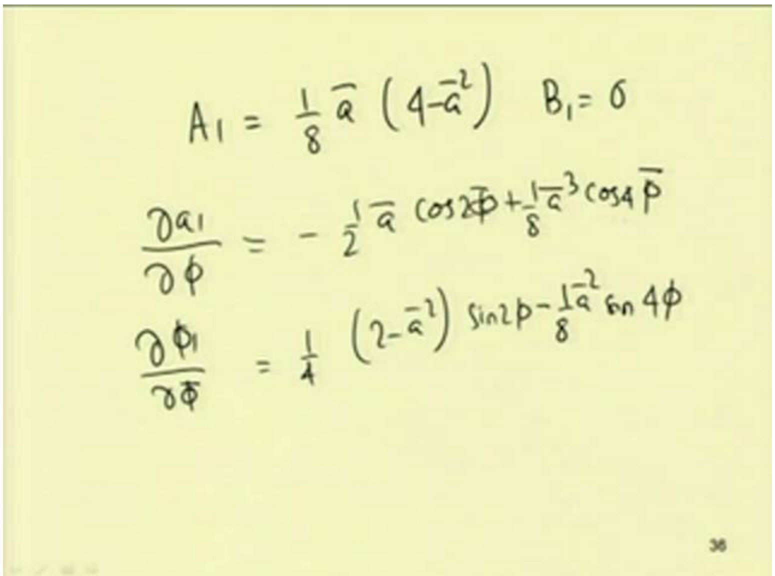

Similarly, order of epsilon square also one can write, so order of epsilon square, writing the order of epsilon square, so one can find this A 1 equal to, so A 1 will be equal to 1 by 8 a bar 4 minus a bar square. Similarly, B 1 will be equal to 0. So, by substituting this thing in this equation, so one can get del A 1 by del phi and del A 1 by del phi will be equal to minus half a bar cos 2 phi bar plus 1 by 8 a bar cube cos 4 phi bar. Similarly, del phi 1 by del phi bar will be equal to 1 by 4 2 minus a bar square sin 2 phi minus 1 by 8 a bar square sin 4 phi. So, the solution now, d A 1 by d phi equal to minus half a bar cos 2 phi cos 2 phi plus 1 by 8 a bar cube cos 4 phi.

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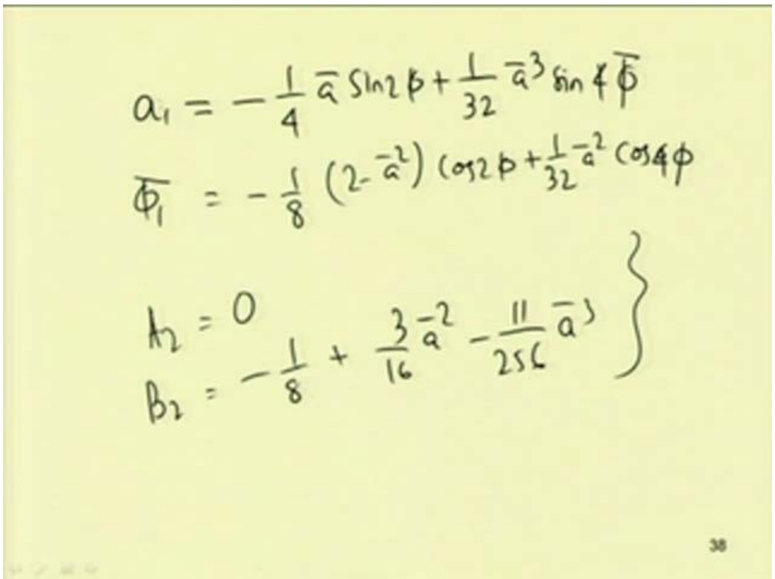

So, one can find the solution, so A 1 and so one can find A 1. So, I write this thing. So, one can find A 1 equal to 1 by 4 a bar sin 2 phi plus 1 by 32 a bar cube sin 4 phi bar. Similarly, phi 1 bar equal to minus 1 by 8 2 minus a bar square cos 2 phi plus 1 by 32 a bar square cos 4 phi. Similarly, one can find for epsilon square term; one can find A 2 and beta 2. So, this A 2 can be obtained to be 0 and b 2 will obtain 1 by 8 plus 3 by 16 a bar square minus 11 by 256 a bar cube. Similarly, one can, so by substituting all these thing, so one can get the solution.

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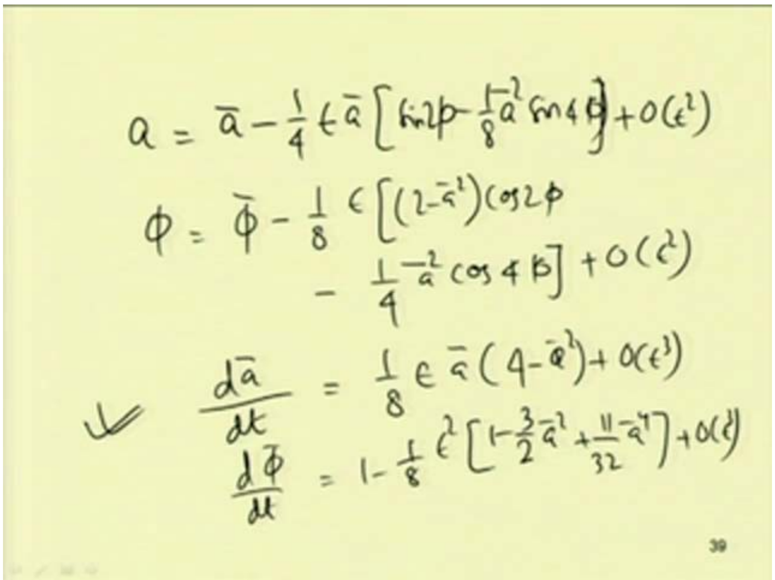

So, this first order solution or one can write the first order solution by using only A 1 and phi 1. Also one can use this A 2 and phi 2. So, all the terms, by using all the terms, one can get this a equal to a bar minus 1 by 4 epsilon a bar sin 2 phi minus 1 by 8 a bar square sin 4 phi. So, if one neglect the higher order terms, that is order of epsilon square one can write the solution in this form. So, a will be equal to this and phi will be equal to phi bar minus 1 by 8 epsilon 2 minus a bar square cos 2 phi minus 1 by 4 a bar square cos phi cos 4 phi.

So, by neglecting the order of epsilon square, one can find the solution a and phi in this way, where this d a bar by d t. So, this is equal to 1 by 8 epsilon a bar 4 minus a bar square plus order of epsilon cube. Similarly, this d phi bar by this d t equal to 1 minus 1 by 8 epsilon square 1 minus 3 by 2 a bar square plus 11 by 32 a bar 4th plus order of epsilon cube. So, the solution matches well with the already obtained solution using this method of multiple scales. So, this way one can use this generalized method of averaging to find the solution of a equation. Particularly, these methods are useful when we do not have weakly varying function. So, for strongly varying non-linear function, one can use this generalized method of averaging. So, next class will study another method of averaging and will solve some more problems as well.

Thank you.