Non-Linear Vibration
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Module - 3
Solution of Nonlinear Equation of Motion
Lecture - 6
Method of Averaging

So, welcome to today's class of non-linear vibration. So, we are continuing with the solution of non-linear equation of motion, and already we have covered different methods; and today class we are particularly study about this method of averaging. Before studying this method of averaging, let me briefly review the other methods what I told you in the previous classes.

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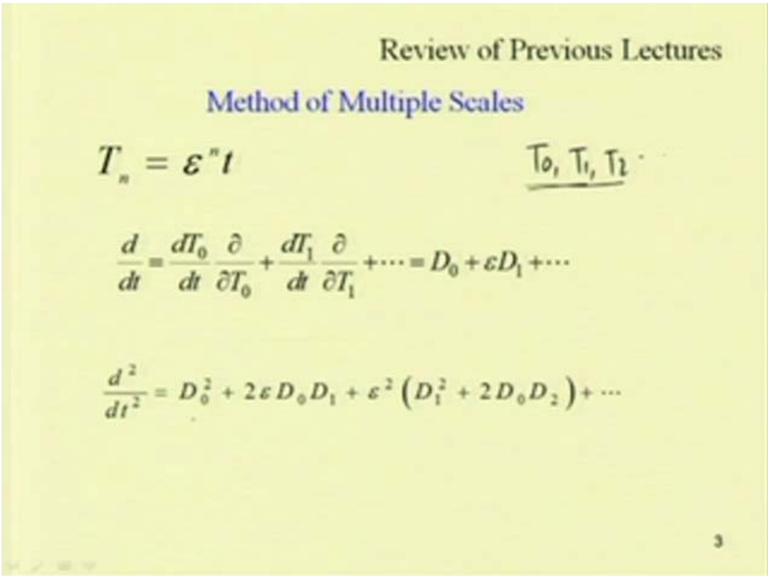

So particularly, this method of multiple scales and harmonic balance method, so in case of method of multiple scales, we are taking different time scales which can be represented by using the time terms T n equal to epsilon n to the power t, where t is the time, epsilon is the book keeping parameter and these time scales are T 0, T 1, T 2 like this. So, where this T 0 equal to t and T 1 equal to epsilon t and T 2 equal to epsilon square t. So, by substituting these different time scales in the original differential equation of motion and separating the terms with different order of epsilon then, we can get a set of equations. So, from those equations we can eliminate the secular terms to get the frequency equations.

So, in this case initially taking this T n equal to epsilon to the power n t, so one can write this d by d t equal to D 0 plus epsilon D 1, and d square by d t square equal to D 0 square plus 2 epsilon D 0 D 1 plus epsilon square D 1 square plus 2 D 0 D 2 plus the higher order terms.

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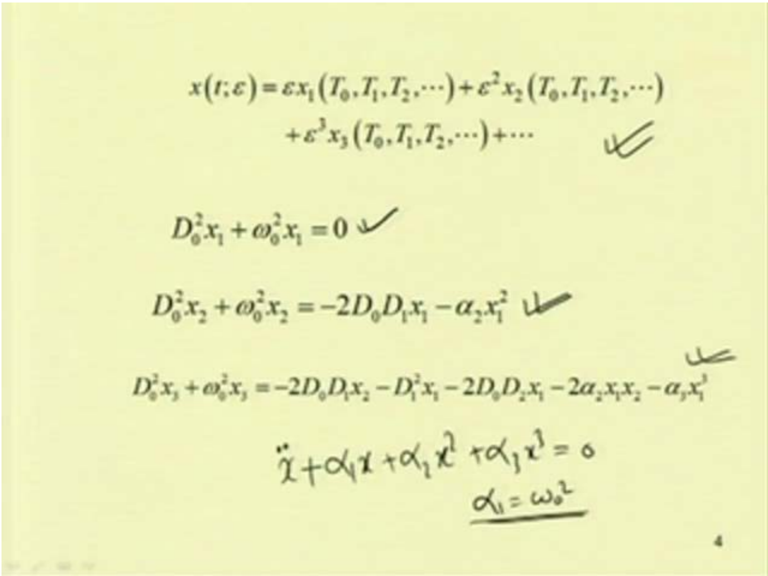

By taking these terms and assuming that the equation in this form x equal to epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3, and by substituting it in the original equation. So, the equation what we have studied in this case is x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube. So, we have taken three terms and we have equated that to 0. So, in this case this alpha 1 equal to omega 0 square by substituting this equation, in this equation we and we got these three equations, so by taking the different order of epsilon. So, we know the solution of this equation D 0 square x 1 plus omega 0 square x 1.

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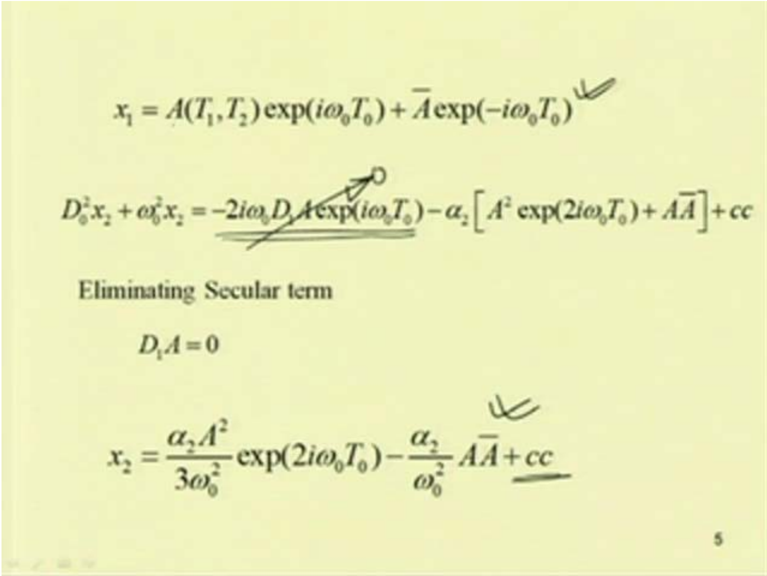

So, this solution by substituting it in this equation second equation and getting the solution of x 1 and x 2, and substituting those two in the third equation, we can get different frequency response equations. So, for example, in this case by the solution of this equation equal to a 1 e to the power i omega 0 t 0 plus a 1 bar e to the power minus i omega 0 t 0. So, the solution by substituting in the second equation, so you got the equation to be written in this form, and in this case one can see that this is the secular term; that means, presence of this term will leads to a infinite will leads to infinite response. So, one has to eliminate this term. So, to eliminate this term one can find that this D 1 A equal to 0. So, by taking D 1 A equal to 0, so that means, making this term equal to 0.

So, one can find this expression for x 2. So, x 2 can be written as alpha 2 A square by 3 omega 0 square e to the power 2 i omega 0 T 0 minus alpha 2 by omega 0 square AA bar plus it is complex conjugate. So, now, we know the solution of solution in terms of x 1 and x 2, but we do not know what is the exact value of a and b.

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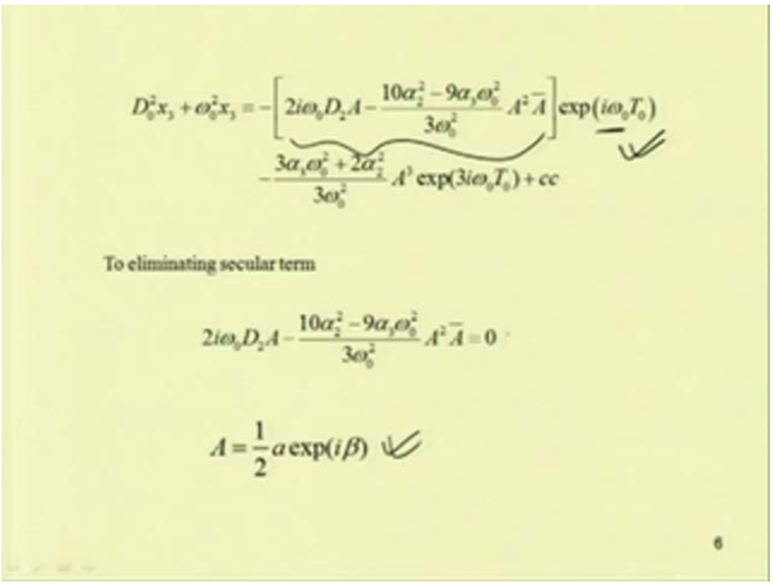

So, by substituting these two x 1 and x 2 in the third equation, so one can find this expression, so here also one can note that this term is a secular term, because the coefficient of e to the power i omega T 0. So, one has to eliminate this term. So, to eliminate this term one can write this 2 i omega 0 D 2 a minus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 3 omega 0 square A square A bar equal to 0, now substituting this A equal to half a e to the power i beta in this equation.

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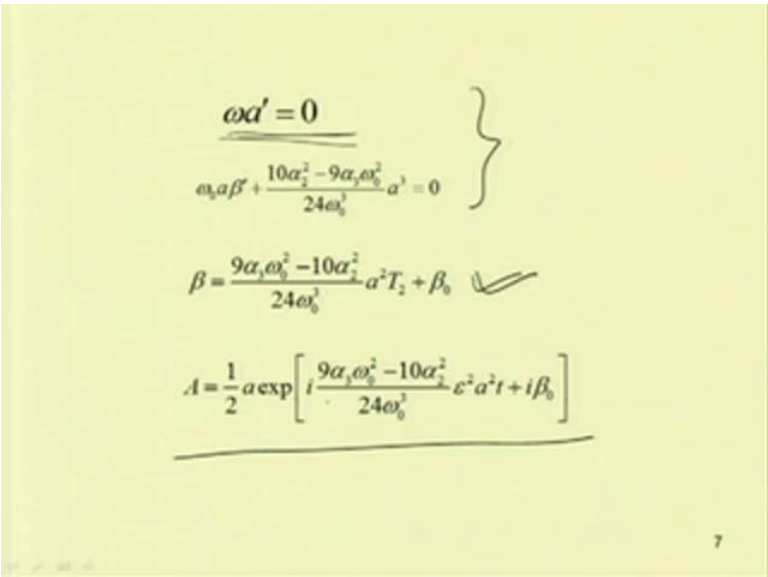

So, one can separating the real and imaginary parts, so one can get these two equations that is omega a dash equal to 0 and omega 0 a beta dash plus 10 alpha 2 squared minus 9 alpha 3 omega 0 square by 24 omega 0 cube a cube equal to 0. So, for steady state, so one can substitute this a dash and beta dash equal to 0. So, from this equation one can known thus that this a dash equal to 0. So, a is not a function of t 0 and t 1. So, now, from this equation one can find, so in this case as a dash is d a by d t 2, so it is equal to 0, so a dash is not a function of t 0 t 1 and t 2.

But beta one can find this beta dash equal to minus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 24 omega 0 cube and integrating that thing so one can get this expression. So, from this expression of a and beta, so one can write a equal to half a e to the power i 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 cube epsilon square a 2 square t plus i beta 0.

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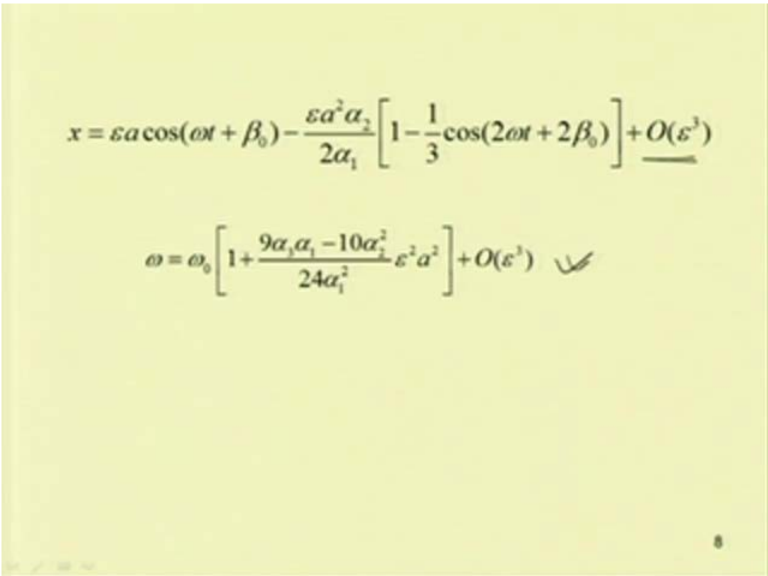

So, one can find finally, the solution in this form. So, x equal to epsilon a cos omega 0 t plus beta 0 minus epsilon a square alpha 2 by 2 alpha 1 into 1 minus one-third cos 2 omega t plus 2 beta 0 and these are the higher order terms, so neglecting these higher order terms the solution can be written in this form where this omega equal to omega 0 into 1 plus 9 alpha 3 alpha 1 minus 10 alpha 2 squared by 24 alpha 1 square epsilon square a square. So, in this case of method of multiple scale by taking different time scales one can find the solution of the equation motion and the frequency response curve.

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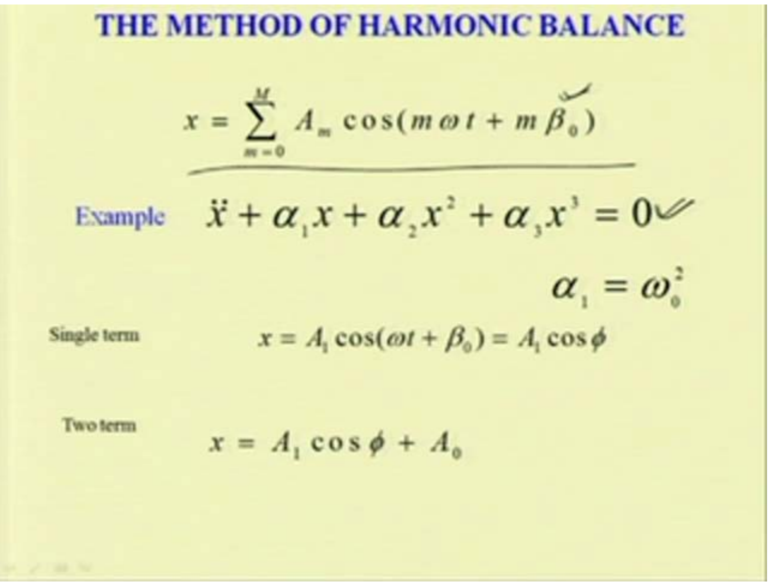

Similarly in case of the harmonic balance method, one can take the solution in this form, so x equal to summation of different harmonics, so either one can write in terms of cos and sin or one can use this cos and this phase difference beta, so one can write x equal to m equal to 0 to M A m cos m omega t plus m beta 0. So, last class we have studied this method by taking this example of x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube equal to 0, so one can take a single term or two terms or multiple terms. So depending on the complexity of the problem, so one can consider different terms or different order of terms by taking single term and multiple terms.

Last time we have shown that depending on the number of terms the accuracy of the solution can be achieved. So, by taking a single term that is x equal to A 1 cos phi or by taking a two terms A 1 cos phi plus A 0 or taking three terms that is A 0 plus A 1 cos phi plus A 2 cos 2 phi. So, one can find the solution of this equation, the disadvantage of this harmonic balance method is that one has to known the solution or up to what order one has to take one should have the idea or but in case of method of multiple scale, one can one enough have to know the solution of theory and one more advantage of method of multiple scale above this method of harmonic balance is that one can study the stability of the systems by taking the reduced equations.

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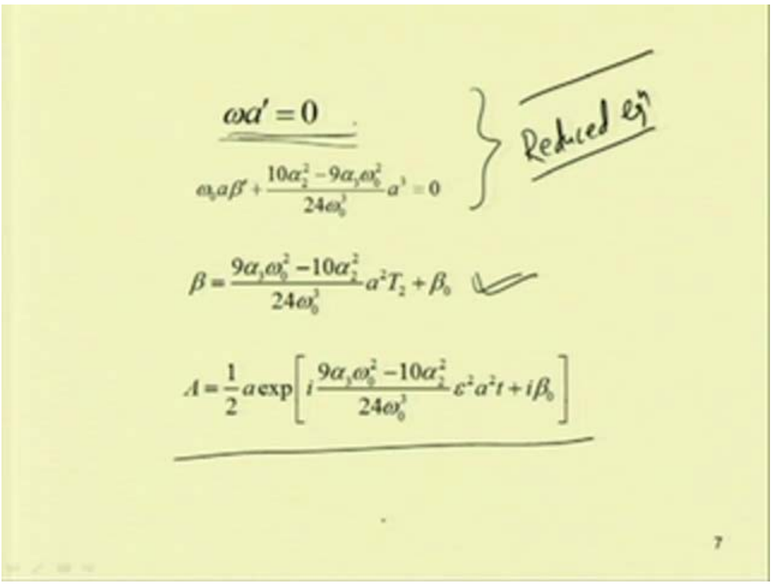

So, in this case taking these two as the reduced equation, so one can study the stability of this reduced equation and one can study the stability of the whole system, but in case of this by using this harmonic balanced method after getting the solution one has to perturb the original equation of motion to study the stability of the system.

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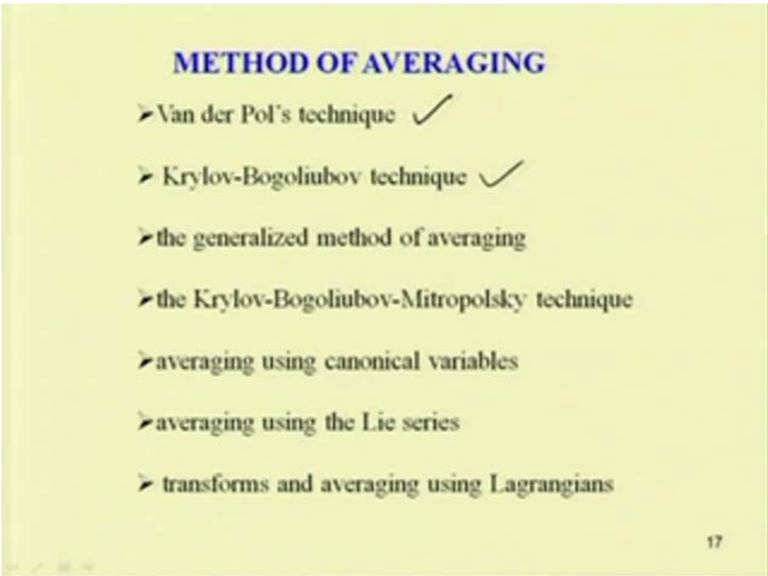

So, today class we will study about the method of averaging, so in case of method of averaging. So, which is a variational technique, so there are several techniques available to study or to find the non-linear solution of the non-linear differential equation, so in this case one can use this Van der Pol technique or Krylov-Bogoliubov technique, the generalized method of averaging the Krylov-Bogoliubov-Mitropolsky technique, averaging using canonical variables, averaging using the Lie series, transform and averaging using Lagrangians. So, there are many methods available and we are going to study today about this Van der Pol's technique and the Krylov-Bogoliubov technique and next class we will study about this generalized method of averaging and Krylov-Bogoliubov-Mitropolsky technique.

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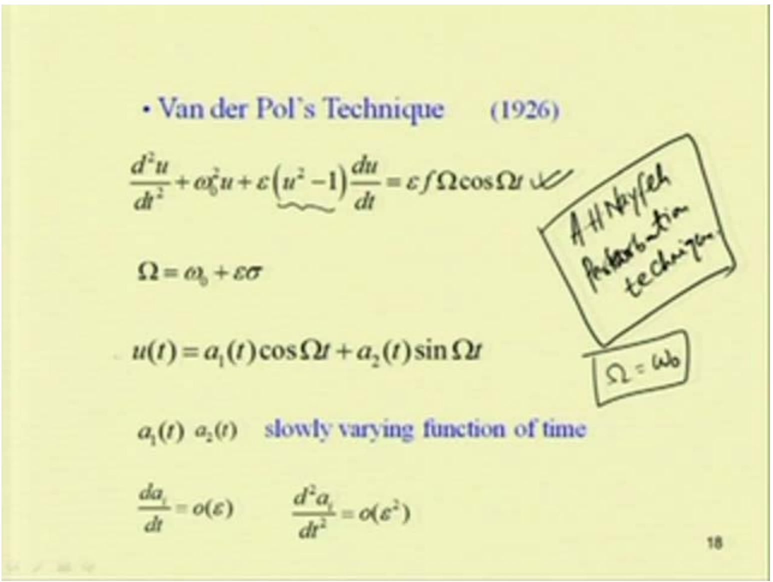

So, in 1926 Van der pol propose this method to solve this type of equation. So, this equation is generally known as the Van der Pol equation in which one can write the equation in this form that is d square u by d t square plus omega 0 square u plus epsilon u square minus 1 into d u by d t equal to epsilon f omega cos omega t, here f is the amplitude of the excitation external excitation, omega is the frequency of external excitation and this omega 0 is the natural frequency of the linear system, and then this epsilon is a book keeping parameter, so depending on the value of this u. So, if u square is greater than 1, so this term will be positive, so this may be this term will be positive and, but when u square is less than this term then this may be negative; that means, this bracketed term if it is negative.

So, this equation is similar to that of a equation in which we have negative damping and if u square is greater than 1 then we have a simple equation similar to that of a single degree of freedom system, but one can use or one can add a non-linear terms also to this equation. So, in this equation was originally solved in 1926 by Van der pol using the following technique, this method is adapted from the book by A H Nayfeh Perturbational Perturbation technique, so it is adapted from this book perturbation technique by A H Nayfeh technique, so in this method, so one can assume the solution u to be a 1 t cos omega t plus a 2 t sin omega t, so in case of method of variation technique, so one by putting this epsilon equal to 0.

So, here we are assuming this epsilon to be small and if one can take this epsilon equal to 0, the equation will reduce to d square u by d t square plus omega 0 square u equal to 0. The solution of that linear equation equal to u equal to a 1 cos omega t plus a 2 sin omega t, so where this a 1 and a 2 are constant; that means, if we are considering epsilon equal to 0 then one can write the solution equal to u t equal to a 1 cos omega t plus a 2 sin omega t, where this omega is nearly equal to this omega 0 or in that case if you are taking only the linear term that is d square u by d t square plus omega 0 square u equal to 0 then the solution will be u t equal to a cos omega t plus a 1 cos omega t plus a 2 cos omega t, where this omega equal to omega 0.

That is the natural frequency of the system, but if we are taking a variation; that means, if we are taking non-zero epsilon, but a very small value of epsilon still this equation will be valid, but in this case, this a 1 and a 2 can be considered as function of time so this. So, one has to consider this a 1 and a 2 in such way that they are slowly varying function of time. So, that this d a i by d t; that means, d a 1 by d t and d a 2 by d t will be of the order of epsilon and this d square a 1 by d t and d square a 1 by d t square or d square a 2 by d t square will be further slowly varying term that is it should be of the order of epsilon square. So, by taking this a 1 t and a 2 t as slowly varying function of time and using this u t equal to a 1 t cos omega t plus a 2 t sin omega t and substituting this equation in the original equation.

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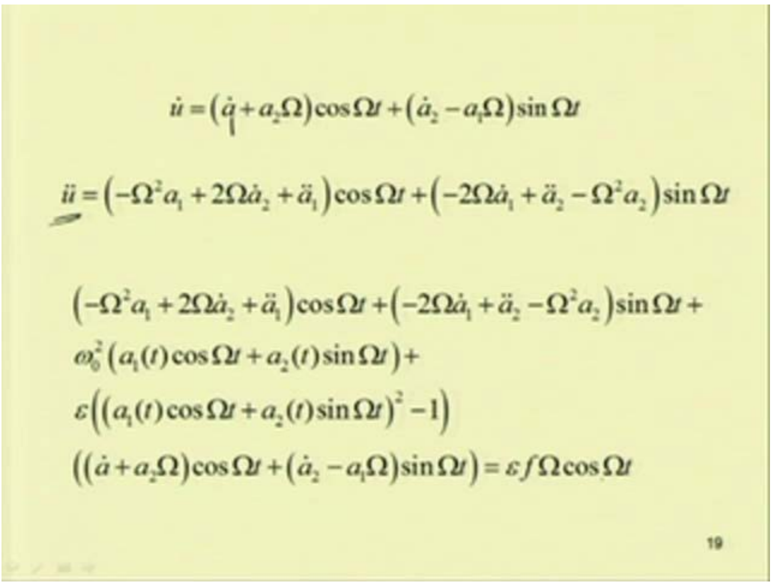

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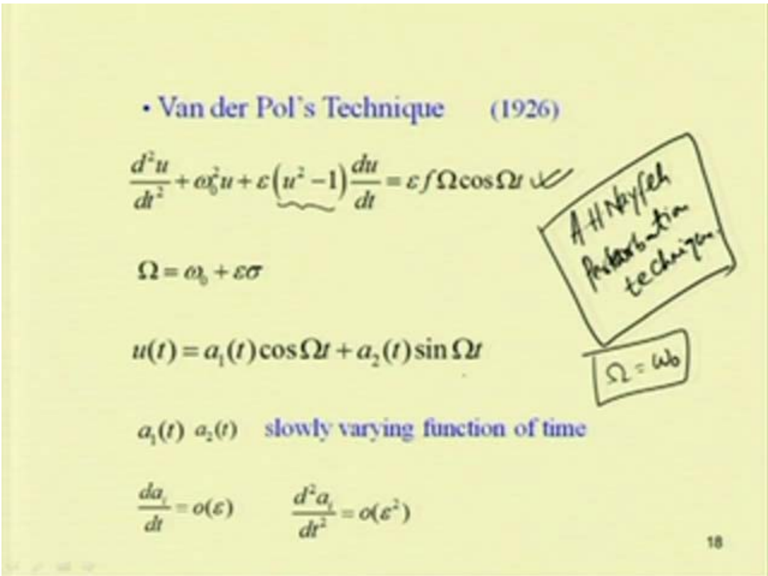

So, one can write this equation in this following form. So, this u as we have taking this u equal to a 1 t cos omega t plus a 2 t sin omega t. So, this u dot by differentiating this thing, so one can get so differentiation of a 1 will be a 1 dot. So, it will be a 1 dot cos omega t and then minus a 1 omega sin omega t, similarly differentiating this term this a 2 t sin omega t 1 can write. So, this will be a 2 dot sin omega t plus a 2 omega cos omega t. So, we have four terms for this u dot t, so u dot can be written as by combining this coefficient of cos and sin one can write this u dot equal to a dot a 2 omega into cos omega t plus a 2 dot minus a 1 omega sin omega t.

So, again differentiating this equation one can write this u double dot equal to, so by differentiating this inside term one can write this will be a 1, so this is a 1. So, a 1 double dot plus a 2 dot omega and into cos omega t, now differentiating this keeping this constant and differentiating this cos omega t 1 can write this as minus a 1 dot plus a 2 omega into sin omega t, similarly differentiating this term one can write this is a 2 double dot minus a 1 dot omega into sin omega t plus a 2 dot minus a 1 omega into omega cos omega t. So, by collecting the coefficient of cos omega t and sin omega t 1 can write this u double dot in this form, so this is equal to minus omega square a 1 plus 2 omega a 2 dot plus a 1 double dot into cos omega t plus minus 2 omega a 1 dot plus a 2 dot minus omega square a 2 into sin omega t.

Now, substituting this equation of u double dot and u t in this original equation that is u d square u by d t square plus omega 0 square u plus epsilon into u square minus 1 into d u by d t equal to epsilon f omega cos omega t, so one can write this equation in this form. So, this will be this for u double dot initially it is written then the term containing this u square, so this is u square minus 1 and this term is for d u by d t, so this is equal to epsilon f omega cos omega t.

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\begin{aligned}
&\left((-\Omega^2+\omega_0^2)a_1+2\Omega\dot{a}_2+\ddot{a}_1+\right)\cos\Omega t+\\
&\left(-2\Omega\dot{a}_1+\ddot{a}_2+(-\Omega^2+\omega_0^2)a_2\right)\sin\Omega t+\\
&\varepsilon\Omega\begin{pmatrix} a_1^2a_2\cos^3\Omega t-a_2^2a_1\sin^3\Omega t+\\ \left(a_2^3-2a_1^2a_2\right)\sin^2\Omega t\cos\Omega t-\\ \left(a_1^3-2a_1a_2^2\right)\cos^2\Omega t\sin\Omega t-\\ a_2\Omega\cos\Omega t+a_1\Omega\sin\Omega t \end{pmatrix}+hot=\varepsilon f\Omega\cos\Omega t
\end{aligned}
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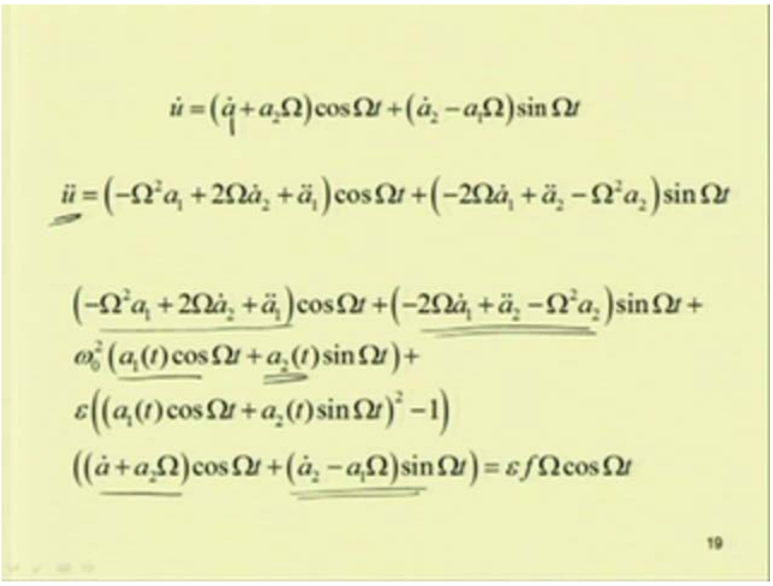

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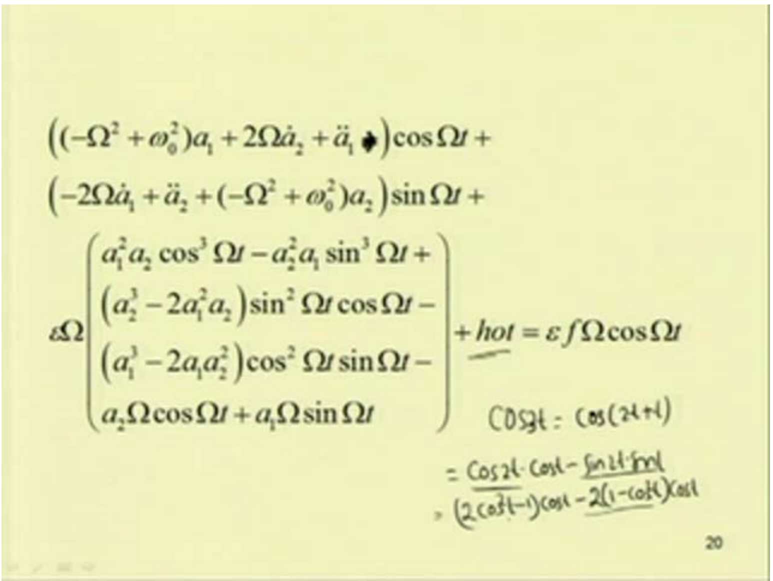

So, now, one can arrange these terms and one can write the equation in this form. So, arranging the terms with cos omega t, so this is the term with cos omega t, and there are some other terms also. So, this term also contain cos omega t, and one can find as minus 1 into this term is there, so this is also cos omega t, so similarly by collecting the terms with sin omega t, so with double line. So, this is sin omega t this term is also for sin omega t and. So, with minus 1 if it is multiplied then this term also contain sin omega t, so one can write these terms in that way.

So, this becomes minus omega square plus omega 0 square into a 1 plus 2 omega into a 2 dot plus a 1 double dot, so a 1 double dot into cos omega t plus minus 2 omega a 1 dot plus a 2 dot plus minus omega square plus omega 0 square into a 2 into sin omega t plus epsilon omega into a 1 square a 2 cos cube omega t minus a 2 square a 1 sin omega cube sin omega t a 2 square a 1 sin omega cube omega t sin cube omega t then a 2 cube minus 2 a 1 square a 2 into sin square omega t into sin omega cos omega t then a 1 cube minus 2 a 1 a 2 square into cos square omega t into sin omega t minus a 2 omega cos omega t plus a 1 omega sin omega t and plus these higher order terms.

So, this will be equal to epsilon f omega cos omega t, now we have to convert this cos cube omega t term sin cube omega t terms and the sin square omega t cos omega t and this cos square omega t sin omega t into the respective cosine and sine terms. So, for example, this cos cube omega t can be broken down by using this cos 3 t. So, we know this cos 3 t equal to cos 2 t plus t, so we can expand this thing in this form. So, cos a plus b, so it will be equal to cos a plus b, so it will be equal to cos a into cos b minus sin a into sin b. So, in that way one can write or this will be equal to cos 2 t into cos t minus sin 2 t into sin t.

So, now, again breaking down this cos 2 t, so cos 2 t equal to cos square t minus sin square t, so or one can write this equal to 2 cos square t minus 1 into cos t, so minus this becomes 2 sin square t 2 sin square t. So, the sin square t 1 can write equal to 1 minus cos square t 1 minus cos square t into cos t, so this becomes 2 sin t into cos t into sin t, so that means, this becomes 2 sin square t into cos t, so this 2 sin square t one can replace it by 2 into 1 minus cos square t into cos t. So, by expanding this thing, so one can get this becomes 2 cos square t and plus again 2 cos square this is 2 cos cube t and again plus 2 cos cube t that is this becomes 4 cos cube t minus.

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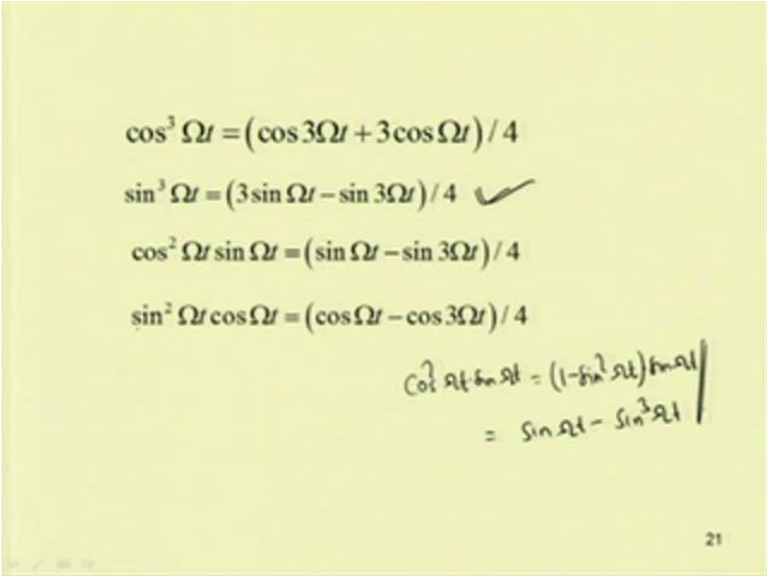

So, this becomes 4 cos cube t, so 4 cos cube t minus 3 cos omega t. So, one can write this cos cube sin omega t equal to cos 3 omega t plus 3 cos omega t by 4. Similarly one can find the sin cube omega t, so by expanding this sin 3 omega t, so one can find this expression. So, the sin omega cube sin cube omega t will be equal to 3 sin omega t minus sin 3 omega t by 4, similarly this cos square omega t sin omega t can be written as sin 3 t minus sin t minus sin 3 t by 4, similarly cos square omega t into cos omega t equal to cos omega t minus cos 3 omega t by 4. For example, this cos square omega t cos square omega t into sin omega t can be written as, so for this cos square omega t one can write this is equal to 1 minus sin square omega t into sin omega t.

So, this becomes sin omega t minus sin cube omega t and already we have found the expression for the sin omega cube t equal to 3 sin omega t minus sin 3 omega t whole divided by 4. So, by substituting this thing in this equation, so one can get the expression for this cos square omega t into sin omega t equal to sin omega t minus sin 3 omega t by 4

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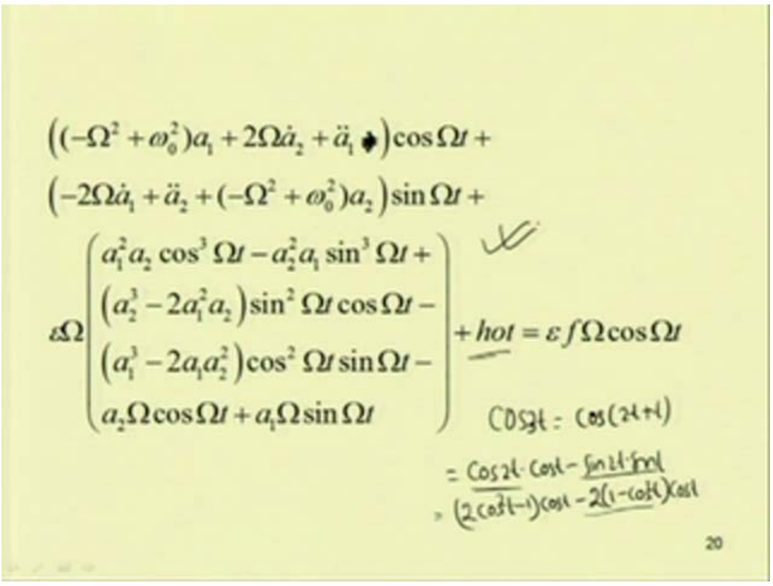

Similarly, one can get the sin square omega t into cos omega t equal to cos omega t minus cos 3 omega t by 4. So, substituting these expressions for cos cube omega t sin cube omega t cos square omega t sin omega t and sin square omega t into cos omega t in the previous equation, so in this equation if you substitute this thing and separate the terms with cos and sin.

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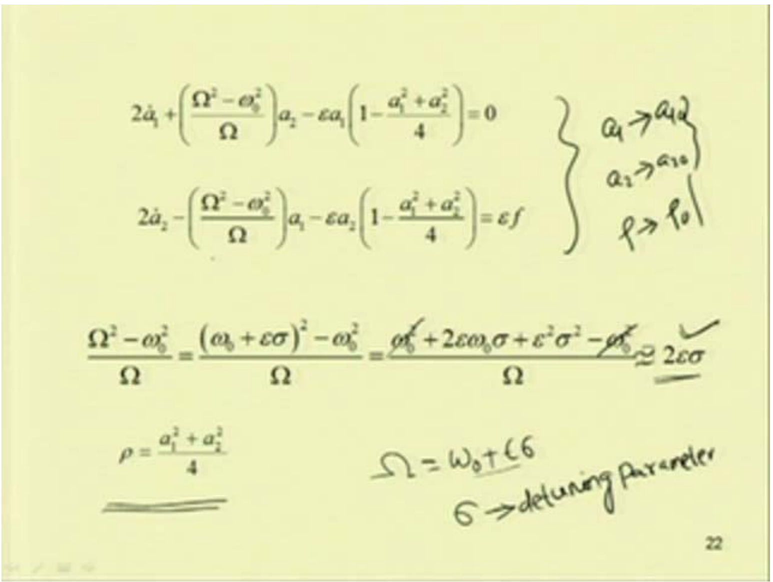

One can obtain two equations, so by collecting the terms with coefficient cos omega t and by collecting the terms with coefficient sin omega t, so we can get these two equations. So, these two equations can be written or these are two a 1 dot plus omega square minus omega 0 square by omega into a 2 minus epsilon a 1 into 1 minus a 1 square plus a 2 square by 4 equal to 0, and the second equation becomes 2 a 1 a 2 dot minus omega square into omega 0 square by omega into a 1 minus epsilon a 2 into 1 minus a 1 square plus a 2 square by 4 equal to epsilon f. So, in this case one can note that we have consider the equation or we have finding the solution near omega equal omega 0; that means, this external frequency nearly equal to the natural frequency of the system that is omega 0 and in this case we know that the solution to be periodic.

So, this u t whatever we what we have considered, so it is equal to a 1 t cos omega t plus a 2 t sin omega t. So, this amplitude of vibration will be equal to a 1 square plus a 2 square root over, so or one can write this equation u t equal to a 1 t cos omega t plus a 2 t cos omega t or in terms of a single term a cos omega t plus phi. So, in that case the amplitude one can find from these two expressions, so instead of writing a 1 a 2. So, one can write using a single term a also or one can use this term rho to represent this a 1 square plus a 2 square by 4, so this omega square that is external frequency square minus omega 0 square by omega can be approximately can be written like this. So, we are using a detuning parameter to express the nearness of this external excitation to that of the natural frequency.

So, if it is very close to this natural frequency then this sigma will tends to 0, so we are writing this omega that is external frequency equal to natural frequency plus this epsilon sigma, where sigma is known as the detuning parameter, so detuning parameter, so using this detuning parameter, so one can write this omega equal to omega 0 plus epsilon sigma, so this detuning parameter shows the nearness of this external frequency towards the natural frequency of the system, so this omega square minus omega 0 square by omega can be written as by substituting this expression here, so omega 0 plus epsilon omega whole square minus omega 0 square by omega, so which is equal to, so this becomes omega 0 square plus 2 epsilon omega 0 sigma plus epsilon square sigma square minus omega 0 square by omega, so this omega 0 square omega 0 square get cancelled and one can write this term is nearly equal to 2 epsilon sigma, so by putting this term nearly equal to 2 epsilon sigma.

So these two equations can be written as 2 a 1 dot plus 2 epsilon sigma a 2 minus epsilon a 1 into 1 minus, so this term is written as rho equal to 0 similarly the second equation can be written as 2 a 2 dot minus 2 epsilon sigma a 1 minus epsilon a 2 into 1 minus rho equal to epsilon f, so from these two equations. So, for steady state one know that this a 1 dot and a 2 dot will no longer be function of time, so this a 1 dot and a 2 dot will be equal to 0 as the response or equilibrium position equilibrium point a 1 and a 2 as a 1 and a 2 can be for steady state can be written as a 1 can be written as a 1 0 and a 2 can be written as a 2 0, so this rho can be written as rho 0, so for steady state when they are not function of time this a 1 dot will be equal to 0 and a 2 dot also will be equal to 0.

So, by substituting this expression here, so one can write these two equation in this form, so one can write 2 epsilon sigma, so this equation becomes 2 epsilon sigma a 2 0 minus epsilon minus epsilon a 1 0 into 1 minus rho 0 equal to 0, similarly the second equation becomes this terms 0, so this becomes minus 2 epsilon sigma into a 1 0 minus epsilon a 2 0 into 1 minus rho 0 equal to epsilon f the purpose of taking this f. So, instead of taking a forcing amplitude f, in this case the forcing amplitude is taken as epsilon f omega, so that one can cancel from omega from both side and one can get a simpler expression, so one can write this equation in this form.

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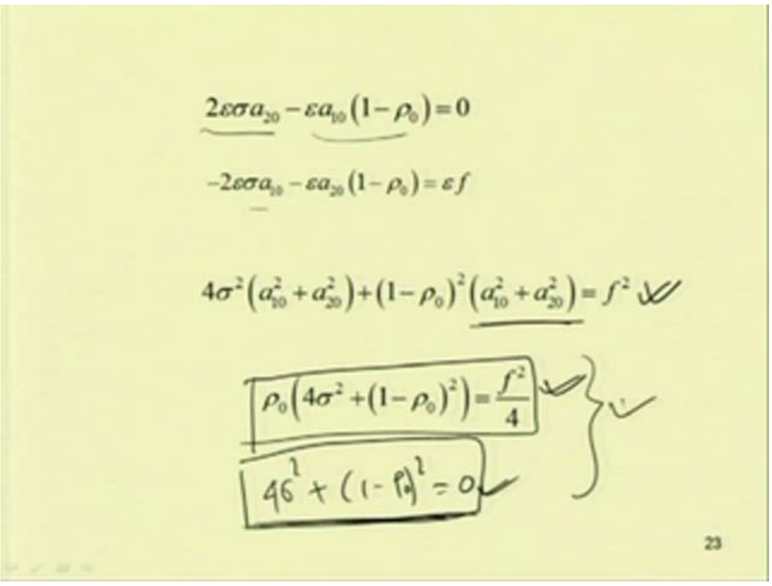

So 2 epsilon sigma a 2 0 minus epsilon a 1 0 into 1 minus rho 0 equal to 0 and second equation becomes minus 2 epsilon sigma a 1 0 minus epsilon a 2 0 into 1 minus rho 0 equal to epsilon f. Now by squaring this two equations and adding, so one can get, so by squaring this. So, it becomes 4 epsilon square sigma square a 2 0 square minus 2 into 2 epsilon sigma a 2 0 into epsilon a 1 0 into 1 minus rho 0 plus epsilon square a 1 0 square into 1 minus rho whole square, so by adding to the second equation, it becomes so plus 4 epsilon square sigma square a 1 0 square plus epsilon square a 2 0 square into 1 minus rho 0 whole square plus 2 into 2 that is 4 epsilon sigma a 1 0 into epsilon a 2 0 1 minus rho 0.

So, right hand side it will be equal to epsilon square f square, so this equation, so from this equation, so by cancelling the term that is 2 into 2 this into this with 2 into this term into this term, so one can finally, get this equation. So, this equation becomes 4 sigma square into a 1 0 square plus a 2 0 square plus 1 minus rho square into a 1 0 square plus a 2 0 square equal to f square, now taking this term common that is a 1 0 square plus a 2 0 square, so one can write this is a 1 0 square plus a 2 0 square into 4 sigma square plus 1 minus rho 0 whole square equal to f square, but we have we can use this equation rho 0 equal to rho 0 equal to a 1 0 square plus a 2 0 square by 4, so by using this expression, so one can write this equation in this form. So, this becomes rho 0 into 4 sigma square plus 1 minus rho 0 whole square equal to f square by 4, so this equation shows.

So, for with the external amplitude equal to 0, so this equation again reduce to 4 sigma square plus 1 minus rho 0 whole square equal to 0, so this is the equation of a circle with center at 1 0 so by. So, for the case of external forcing equal to 0, so one can find one can find this frequency equation, so for f not equal to 0. So, this represent the frequency response curve, so from these one can plot the phase portrait of the system. So, this shows the response to be periodic as it is represent the equation of a circle with this one equation of a circle with center 1 and 0, so this is circular or in this case one can get some harmonics, so one can solve the equation directly by using Runge-Kutta method or use some numerical techniques to find the solution also and one can compare the resulting solution with the solution.

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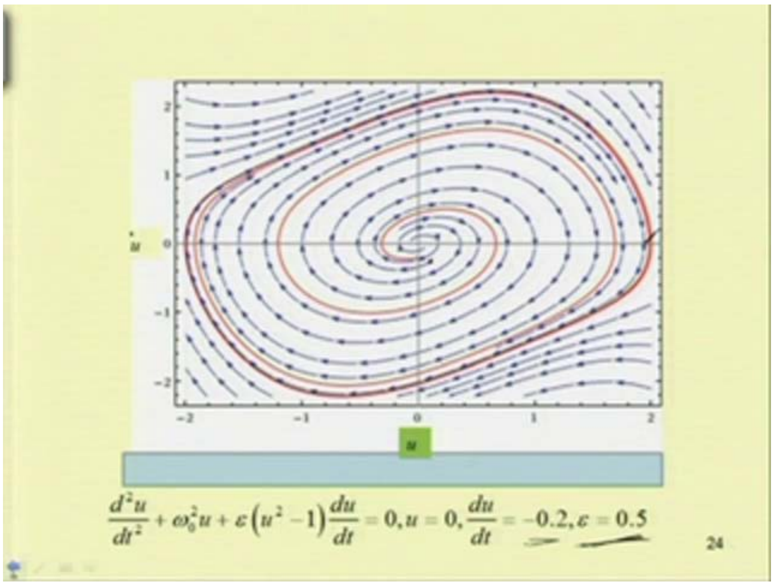

So, for example, using the Runge-Kutta method with taking different initial conditions, so one can see the response to be periodic or one can obtain a limit cycle, if one consider this equation that is that is d square u by d t square plus omega 0 square u plus epsilon into u square minus 1 d u by d t equal to 0 by substituting u equal to 0 and d u by d t equal minus 0.2 and epsilon equal to minus epsilon equal to 0.5, so one can get this plot this phase portrait, so this phase portrait clearly show the existence of the limit cycle limit cycle means. So, if you start from interior point, let this is the interior point, so one can see that finally, it will trace this curve and finally, it will come to this limit cycle, so by taking different initial points, so finally, one come to this limit cycle similarly by taking a point outside of this limit cycle also one can land of with this limit cycle, so by using this Van der pol equation. So, in this case by using this equation one can find the solution of the system, so another method that is this k b method.

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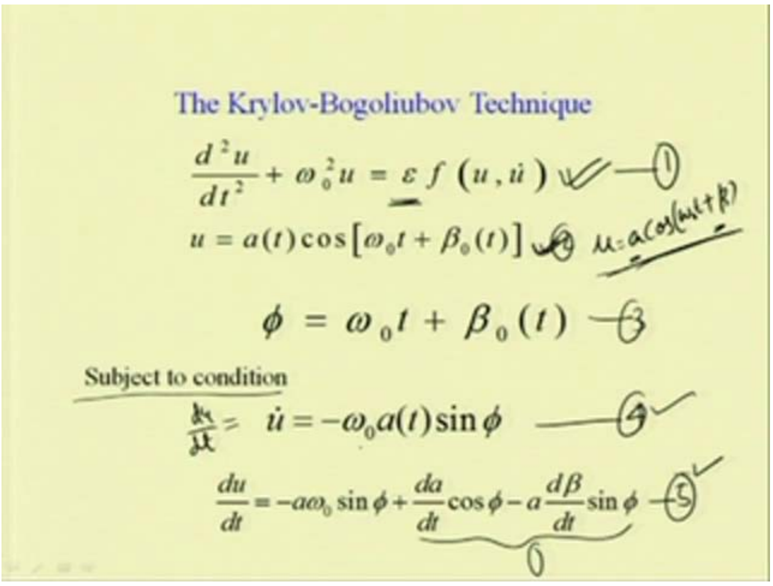

That is Krylov and Bogoliubov technique also will study now to find the equation find the solutions of the non-linear systems, so in case of this method or in this technique. So, let us consider a general equation. So, where d square u by d t square plus omega 0 square u equal to epsilon f u u dot. So, in this case will assume the solution similar to the previous case, in the previous case we have taken two terms, in this case also we are taking two terms, but we are writing the solution in a different way; that means, we can write the solution equal to a u equal a t cos omega 0 t plus b 0 t in the previous case that is in the Van der pol equation.

We have assume the solution of the form u equal to a 1 t cos omega 0 omega t plus a 2 t sin omega t, but one can use this phase difference to show or write this equation in this form also, so by writing the equation or assuming the equation in this form that is u equal to a t cos omega 0 t plus beta 0 t. So, here we are assuming that when this epsilon equal to small or when the epsilon equal to 0 the solution is u equal to a cos omega 0 t plus beta, where this a and beta are constant; that means, for small value for 0 value of epsilon the solution equal to a cos omega 0 t plus beta. So, here this a and beta are constant, but when you are considering the terms other than 0 for this epsilon; that means, epsilon is assume to be small, in this case one can assume the solution to be this same as that of when epsilon equal to 0, but here a and beta are the function of time.

So, here we are assuming a and beta are slowly varying function of time. So, one can take this omega 0 t plus beta 0 t as phi, so one can write this u equal to a cos phi so, where a and beta are function of time. So, in this case Krylov and Bogoliubov propose that, so this will be the solution; that means, u will be the solution a t cos omega 0 t plus beta 0 t subjected to the condition that is this d u by d t or u dot equal to, so u dot will be equal to minus omega 0 a t into sin phi so; that means, this is differentiation of this thing is assume in this form that is omega 0 a t into sin phi, because it is assume that this a dot equal to 0. So, the differentiation of this thing will be.

So now, differentiating this equation one can write this d u by d t equal to minus a omega 0 sin phi plus d a by d t cos phi, so minus a b beta by d t sin phi, so as we have a t, so u equal a t which is a is a function of time. Similarly this cos inside is a function of time and this beta 0 is a function of time, so we have this three terms. But Krylov and Bogoliubov, so they have made this assumption that this equation will be valid or this equation with the solution of the original equation that is d square u by d t square plus omega 0 square u equal to epsilon f u u dot subjected to the condition that this u dot will be equal to minus omega 0 a sin phi, so comparing this two equation.

So, let this is equation 2, so this is equation 3 and this equation 4 and 5, so comparing equation 4 and 5, so we can write we can write these two terms should be equal to 0, so here we have putting this condition that is d u by d t equal to minus omega 0 a t sin phi, so to satisfy this equation that is u equal to a t cos omega 0 t plus beta 0 t, so the differentiation which comes to be minus a omega 0 sin phi plus d a by d t cos phi minus a into d beta by d t sin phi, so comparing this 4 and 5. So, we can take this term to be equal to 0 so; that means, we can take this d a by d t cos phi minus a d beta by d t sin phi equal to 0, so now, differentiating this u dot equal to minus omega 0 a t sin phi.

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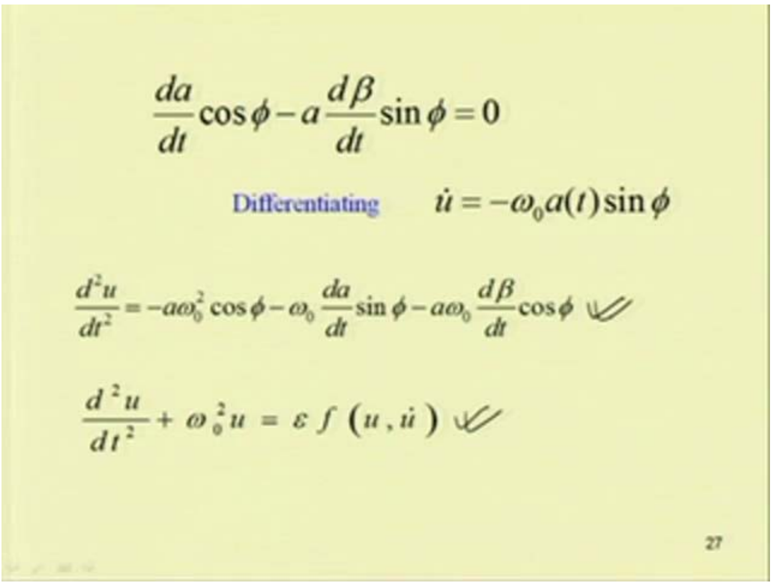

So, one can write, so this becomes, so a is a function of time, so if you differentiate it, so this d square u by d t square will be equal to a omega 0 square. So, a omega 0 square cos phi minus a omega 0 square cos phi into minus omega 0 into d a by d t sin phi minus a omega 0 into d beta by d t cos phi, so substituting this expression for this d square u by d t square and this d u by d t in this equation that is d square u by d t square plus omega 0 square u equal to epsilon f u u dot.

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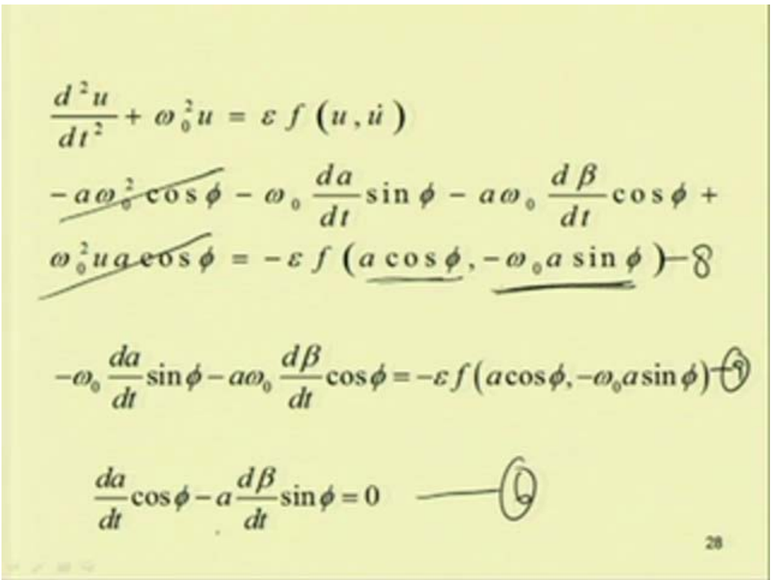

So, one can write this equation; that means, d square u by d t square equal to minus a omega 0 square cos phi minus omega 0 d a by d t sin phi minus a omega 0 d b by d t d beta by d t cos phi plus omega 0 square u a cos phi equal to minus epsilon f. So, for u we have substituted a cos phi and for u dot we have substituted minus omega 0 a sin phi. So, this is the point to be noted that this u is substituted by a cos phi and u dot equal to minus omega 0 a sin phi, so now taking this equation, so we can write this by separating the coefficient of cos and sin, so we can write, so from this equation, so we have two equation. So, this is one equation and previously we have this equation d square u by d t square plus omega 0 square u equal to epsilon f, so in this equation we have substituted the expression for u square, but this is one equation d a by d a by d t cos phi. So, let us put the number, so this becomes equation 6 and substituting equation 7 in equation 1.

So, we can obtain this equation, so that is this is equation number 8, so in this equation number 8. We can write or we can rearrange this thing and we can write this equation in this form that is minus omega 0 d a by d t sin phi, so this the term minus a omega 0. So, in this case minus a omega 0 square cos phi, so this is plus and this is minus they cancel. So, this equation reduce to this form that is minus omega 0 d a by d t sin phi minus a omega 0 d beta by d t cos phi, so this will be equal to minus epsilon f a cos phi minus omega 0 a sin phi, so this is equation 9 and again we have written this equation 6 here, so this is equation 6, so from this equation 9 and 6 we can find the equation with d a by d t and d beta by d t to obtain that thing.

Now, by multiplying this first equation by cos phi and the second equation by minus omega 0 sin phi and adding these two equation will get the expression for d beta by d t. Similarly by eliminating this two terms; that means, by multiplying this equation by sin phi and the second equation by omega 0 cos phi, so we can eliminate these two terms and we can get the expression in terms of d a by d t.

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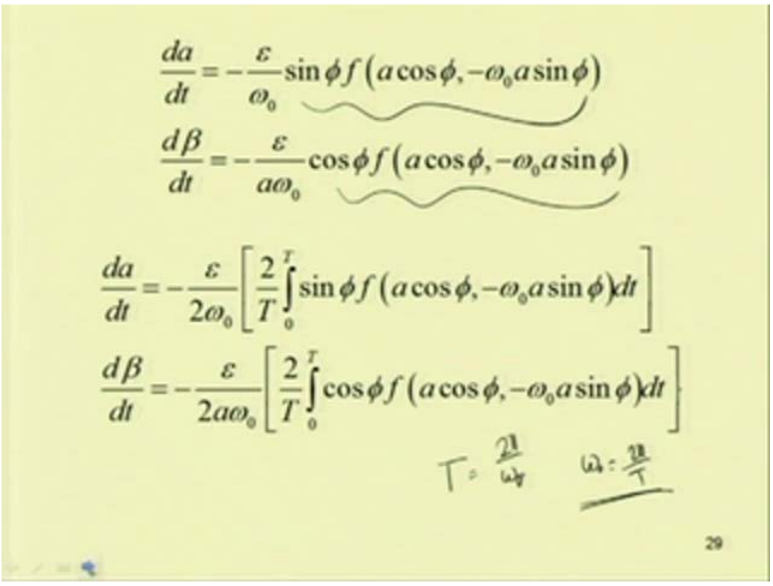

So, we can write this equation now d a by d t equal to minus epsilon by omega 0 sin phi, so function of a cos phi and minus omega 0 a sin phi. Similarly this d beta by d t equal to minus epsilon by a omega 0 into cos phi into function of a cos phi minus omega 0 a sin phi as already we have pointed out this a and beta are slowly varying function of time and this right side one can observe these are periodic as we have the sin phi and cos phi terms, so both these are periodic terms, so these are periodic with a period of 2 phi, so one can write this T or this T equal to 2 pi by omega 0, so 2 pi by omega 0 one know this omega 0 equal to 2 pi by T, so as these are slowly varying function of slowly varying function this a and beta, so one can obtain or one can see or one can find that this value will not vary mass in a cycle.

So, averaging this value over a cycle, so one can write this equation d a by d t equal to minus epsilon by 2 omega 0 2 by T integration 0 to T sin phi into f cos a cos phi minus omega 0 a sin phi d t or the second equation by averaging over the cycle one can write this d beta by d t equal to minus epsilon by a omega 0, so this becomes 2 by T integration 0 to T cos phi f a cos phi minus omega 0 a sin phi. So, it can be noted that it is integrated over 0 to T and just divided by the time t, so this 2 2 cancels, so this is just written by integrating this equation and dividing this by the term t, similarly also this equation is written by integrating the right hand side and dividing that thing by t. So, now, one can express this time in terms of this frequency omega 0 and one can write this equation.

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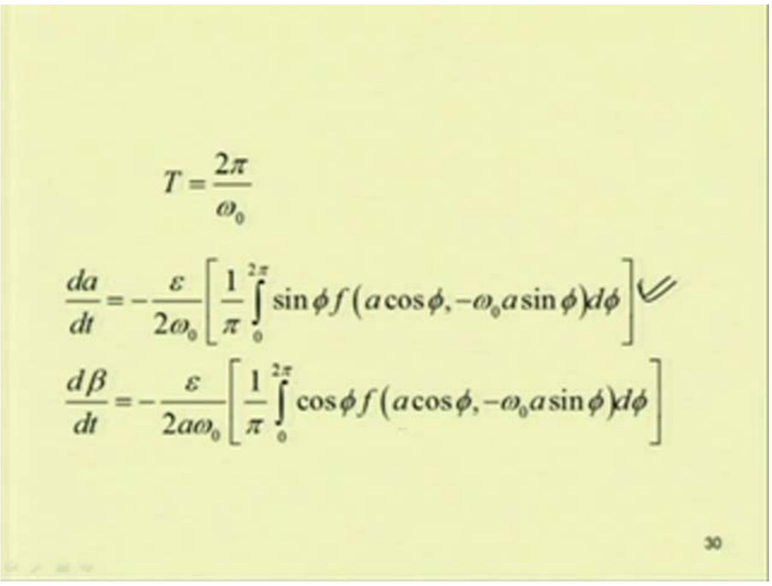

This d a by d t equal to minus epsilon 2 by omega 0, so by substituting this T equal 2 pi by omega 0, so one can write this equation, so this d a by d t will be equal to minus epsilon by 2 omega 0 into 1 by pi 0 to 2 pi sin phi f a cos phi minus omega 0 a sin phi d phi. Similarly this d beta by d t will be equal to minus epsilon by 2 a omega 0 into 1 by pi integration 0 to 2 pi cos phi into f a cos phi minus omega 0 a sin phi d phi.

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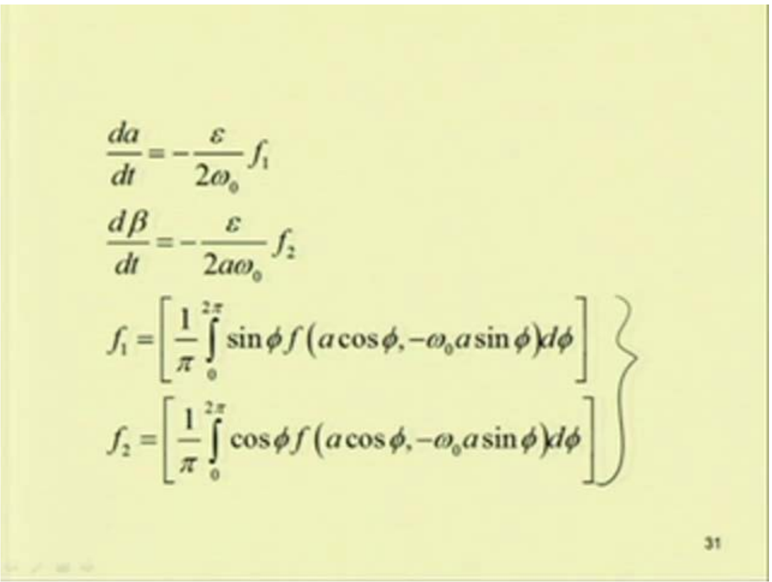

So, one can write this equation again in this form that is d a by d t equal to minus epsilon by 2 omega 0 into f 1 and this d beta by d t equal to minus epsilon by 2 a omega 0 into f

2, where this f 1 equal to 1 by pi integration 0 to 2 pi sin phi f a cos phi minus omega 0 a sin phi d phi, and this f 2 equal to 1 by pi 0 to 2 pi cos phi f a cos phi minus omega 0 a sin phi d phi, so by taking these functions are by evaluating these function, one can find this d a by d t and d beta by d t, and for steady state one can substitute this d a by d t equal to 0 and d beta by d t equal to 0 and one can obtain the frequency equation. So, one can take a simple example for example, let us take the simple linear system.

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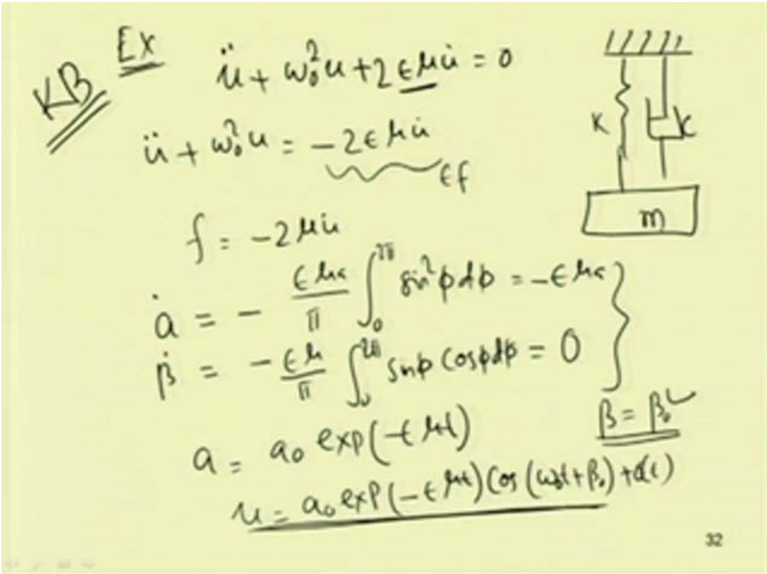

So, in this linear system linear damping, let us take u double dot plus omega 0 square u equal to or plus omega 0 square u plus 2 epsilon mu u dot equal to 0, so this is the equation for simple harmonic for a simple spring mass damper system, so this is not an non-linear system. But let us see how we can use this method of averaging for this case, so one can write the equation of motion in this form that is u double dot plus omega 0 square u plus 2 epsilon mu u dot equal to 0, where this is equal to epsilon mu equal 2 omega 0 into zeta u dot. So, now one can write this equation in this form u, so it will be equal to minus 2 epsilon mu u dot, so our, so this becomes our f. So, here f equal to minus 2 mu u dot, so this is epsilon f, so f equal to this, so this by substituting this f equal in that form, so one can write this a dot equal to minus epsilon mu a by pi integration 0 to 2 pi sin square phi d phi, so this becomes minus epsilon mu a.

Similarly, beta dot one can write equal to minus epsilon mu epsilon mu by pi integration 0 to 2 pi sin phi into cos phi d phi, so this becomes as sin phi integration 0 to 2 pi becomes 0, so one can find this thing, so as a dot equal to minus epsilon mu a, so one can write this a equal to a 0 e to the power, so e to the power minus epsilon mu t and this beta dot equal to 0, so beta will be a constant, so beta will be equal to beta 0, so beta will be beta 0 as beta dot is a constant. Similarly a dot one can obtain, so the original the solution becomes a 0 e to the power minus epsilon mu t into cos omega 0 t plus beta 0, so plus order of epsilon. So, this way one can use this method of averaging or this K B method. So, here this example shows this K B method how this Krylov-Bogoliubov method can be use to apply for this equation. So, next class we will study or we will take few more examples to study this K B method and also I will tell you about this generalized method of averaging and other methods of averaging also.

Thank you.