Non-Linear Vibration
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Module - 3
Solution of Non-Linear Equation of Motion
Lecture - 5
Method of Harmonic Balance

So, welcome to today's class of non-linear vibration. So, this is the fifth class in the solution of non-linear equation of motions and today's class we are going to study about this method of harmonic balance. So, before going for this method of harmonic balance, let us review what we have studied in this module. So, first we have studied about the straight forward expansion method. In that method, we have seen, so due to the presence of some secular terms, so it will not yield the correct result. So, for that purpose, Lindstedt Poincare modified the method and we have studied about this Lindstedt Poincare method.

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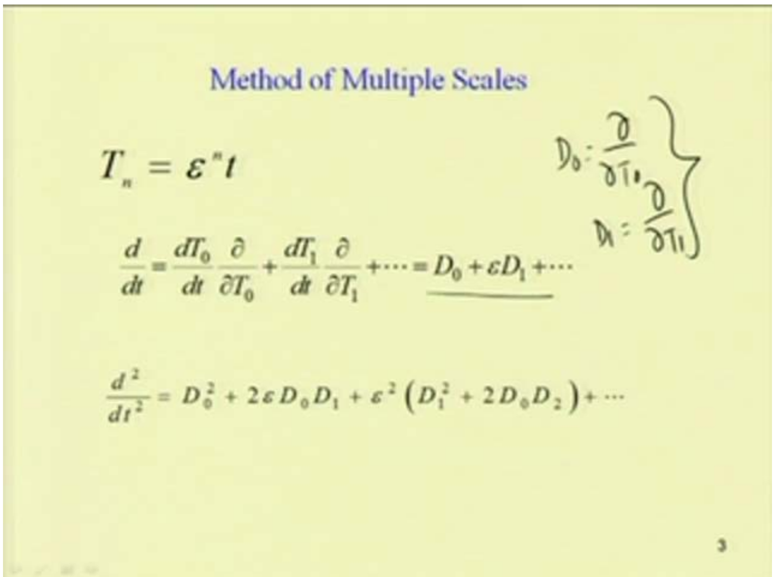

So, after Lindstedt Poincare method, we have studied about this method of multiple scales and in method of multiple scales, so we have taken different time scales and the time scales are written in this form, this T n equal to epsilon to the power n T, where epsilon is the book keeping parameter and T is the time. So, the differential, so d by d T can be written as D 0 plus epsilon D 1 and d square by d T square is written as D 0 square plus 2 epsilon D 0 D 1 plus epsilon square into D 1 square plus 2 D 0 d 2, where this D 0 equal to del by del T 0 and D 1 equal to del by del T 1. So, using this time scales, different time scales, we have found the solution of the non-linear system.

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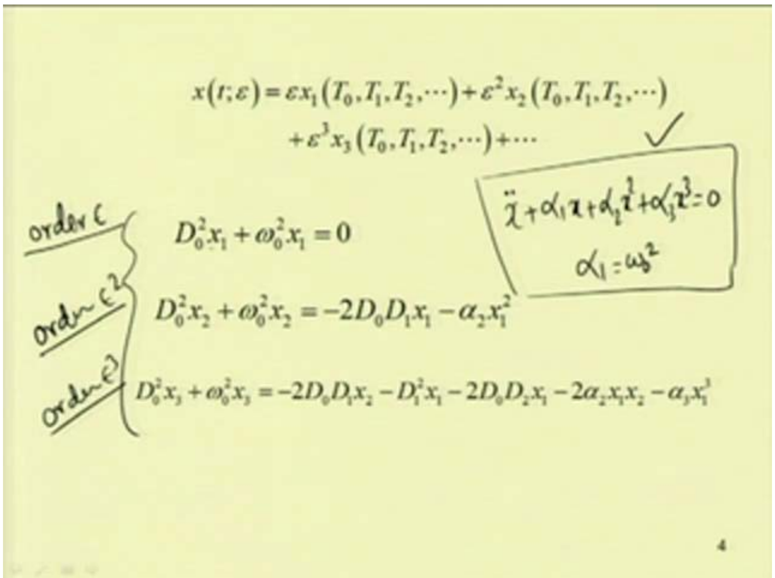

So, we have taken, like in case of Straight forward Expansion and in Lindstedt Poincare method, we have taken this variable x, which is a function of time and parameter epsilon equal to epsilon x 1 and x 1 is a function of T 0 T 1 T 2 and x 2 is also a function of T 0 T 1 T 2. So, we have taken up to 3 terms and one can take also more terms. So, x T epsilon is written to be equal to epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3. So, by substituting this in the original equation, so in this case, we have taken the equation to the, from this x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube equal to 0, so this alpha 1 equal to omega 0 square. So, taking this as the governing equation and substituting this time different time scales in this equation, so one can separate of the order of epsilon. So, this equation is order of epsilon to the power 1, order of epsilon to the power 2 and order of epsilon to the power 3, order of epsilon square and so this is order of epsilon cube. So, by separating of the order of epsilon, one can write these equations.

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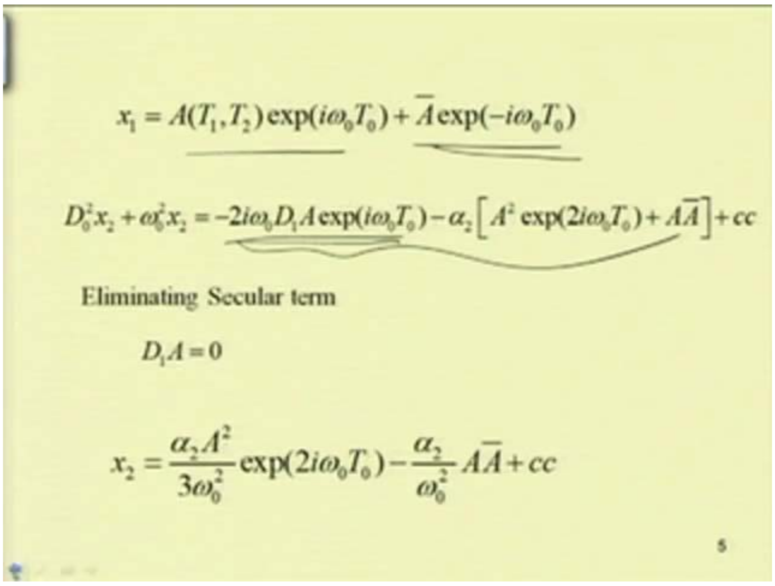

So, from these equations, one can find this D 0 square x 1 plus omega 0 square x 1 equal to 0. So one can know the solution of x 1 and this x 1 can be written as a T 1 T 2 e to the power i omega 0 T 0 plus a bar e to the power minus i omega 0 T 0. That means, this part plus its complex conjugate. So, writing this x 1 equal to a e to the power i omega 0 T 0 plus its complex conjugate and substituting it in the second equations, one can write this equation in this form. So, D 0 x 2 plus omega 0 square x 2 equal to 2 y omega 0 D 1 e to the power i omega 0 T 0 minus alpha 2 into a square e to the power 2 i omega 0 T 0 plus a a bar plus the complex conjugate of these terms. Now, due to the presence of this e to the power i omega 0 T 0 in this differential equation, so one can see that these terms will be, so this term is a secular term or this term will lead to infinity. So, T tends to infinity.

So, one has to eliminate the secular term. So, to eliminate the secular term as exponential terms, so this D 1 a should be equal to 0. So, eliminating the secular term from this one, so one has D 1 a should be equal to 0. So now, substituting D 1 a equal to 0, so one has the equation D 0 square x 2 plus omega 0 square x 2 equal to minus alpha 2 a square e to the power 2 y omega 0 T 0. So, its solution, particular solution for this equation can be written in this form, alpha 2 a square by 3 omega 0 square e to the power 2 i omega 0 T 0 minus alpha 2 by omega 0 square a bar plus its complex conjugate.

So, last class we have seen how these expressions have been derived. So, using these x 1 and x 2 in the third equation, so that is this equation D 0 square x 3 plus omega 0 square x 3 equal to minus 2 D 0 D 1 x 2 minus D 1 square x 1 minus 2 D 0 d 2 x 1 minus 2 alpha 2 x 1 x 2 minus alpha 3 x 1 cube.

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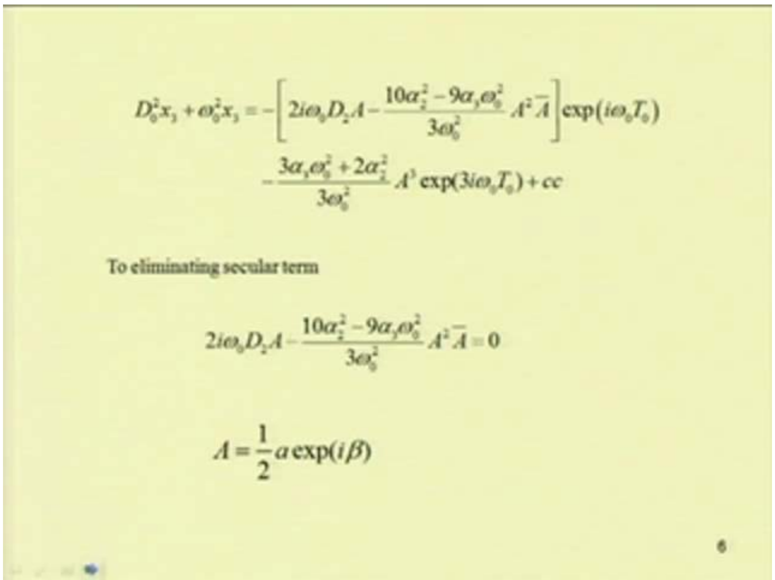

So, here in the right side, we have this x 2 term x 1 term and substituting these expressions, what we have derived from x 1 and x 2, so we can write this equation D 0 square x 3 plus omega 0 square x 3 equal to minus 2 i omega 0 d 2 a minus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 3 omega 0 square into a square a bar e to the power i omega 0 T 0 minus 3 alpha 3 omega 0 square plus 2 alpha 2 square by 3 omega 0 square into a cube e to the power 3 i omega 0 T 0. So, from this equation, it can be noted that, so this is, this term, first term is a secular term as this contains e to the power i omega 0 T 0. So, one should eliminate this term.

One should eliminate coefficient of this e to the power i omega 0 T 0 to eliminate the secular term. So, eliminating this term, so one can write 2 i omega 0 d 2 a minus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 3 omega 0 square a square a bar equal to 0. So, from this, now substituting a equal to half a e to the power i beta, that means writing this a in terms of this polar coordinate, so one can substitute this equation in this equation and separate the real and imaginary parts.

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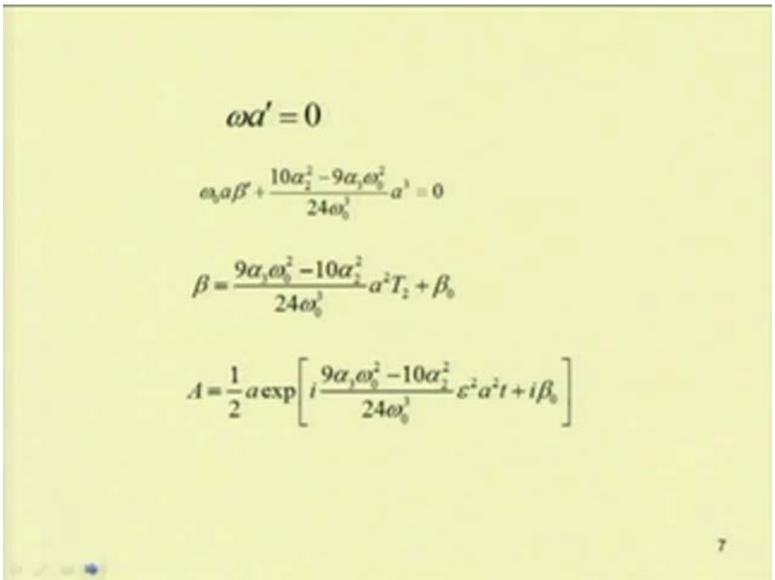

So, separating this real and imaginary part, one can write this omega a dash equal to 0 and omega 0 a beta dash plus 10 alpha 2 square minus 9 alpha 3 omega 0 square by 24 omega 0 cube a cube equal to 0. So, from this integrating one can find this a is a constant and beta will be equal to, so from this, one can write beta equal to 9 alpha 3 omega 0 square minus 10 alpha 2 square by 24 omega 0 cube alpha a square T 2 plus beta 0.

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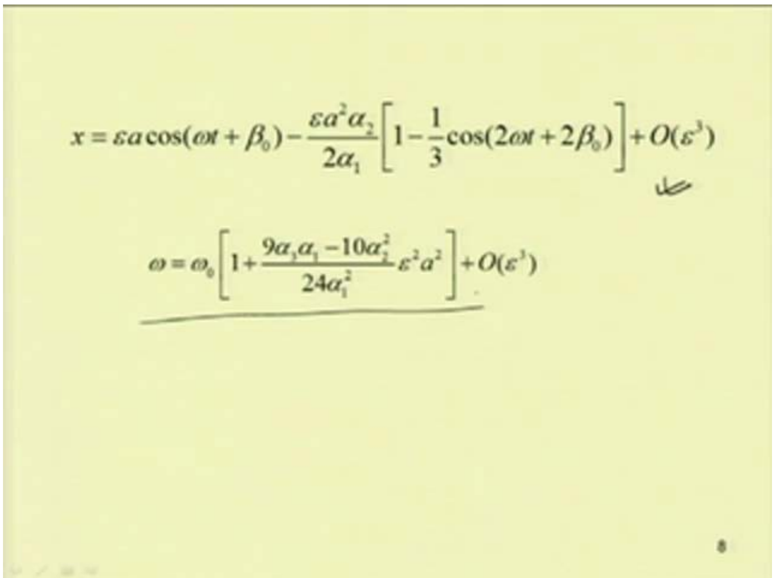

From this, one can find the solution a to be in this form and one can write this x to be this and this omega can be written equal to omega 0 plus 9 alpha 3 alpha 1 minus 10 alpha 2 square by 24 alpha 1 square plus epsilon square a square. So, these expression what we obtained using this method of multiple scales matches with that we obtained from the Lindstedt Poincare method. Now, we will see the method of Harmonic Balance. That is, another method which is used for solving this differential equation. So, this method of harmonic balance is used in a variety of non-linear equations, where the solutions are known (( )). That means, if you know that the solution is of the harmonic type, then we can use this method of harmonic balance. The demerits, the limitations of these method of multiple scale is that, so we can use this method when we have the small parameter as coefficient of the non-linear terms. For large parameter, large coefficients, so it will give this erroneous result.

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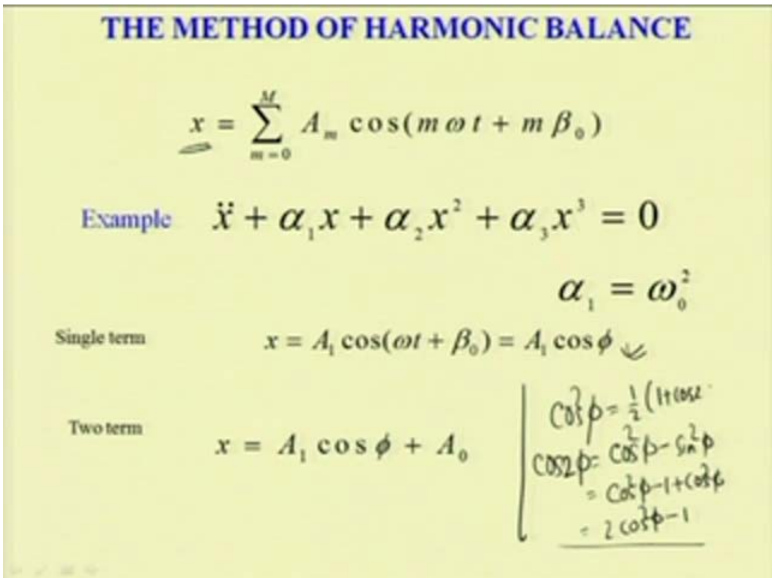

So, one can go for this harmonic balance method and find the solution, if the solutions are known to be harmonic type (( )). So, in this method, so in case of the harmonic balance method, so one can substitute this response x in this form as a Fourier series. One can write this x equal to m, summation of m equal to 0 to m a m cos m omega T plus m beta 0, where this capital m is the number of modes we are considering in the solution. So, by putting this m equal to 0, so we can have a constant term and after putting m equal to 1 2 3, so we can have the other terms. So, it can have this A 0 A 1 A 2 and similarly, we can have this beta 0 beta 1. So, we can have this beta 0 and then it will be, so for m equal to 0, so this becomes only m equal to 0, so this becomes m omega T becomes 0. So, cos 0, so this becomes 1. So, this becomes A 0 and m equal to 1. So, this expression becomes A 1 into cos omega T plus beta 0. Similarly, for m equal to 2, so this becomes A 2 cos 2 omega T plus 2 beta 0.

So, like that, one can write or expand this expression. So, one can expand this expression and depending on the number of modes, one is interested to take in this computation. But, for manual calculation, taking higher terms or taking a number of terms will be difficult to solve this equation. So for example, let us take this example what we have solved for this Lindstedt Poincare, method of multiple scales and Straight forward Expansion method. So, this is the equation, that is x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube equal to 0, where alpha 1, we have taken to be omega 0 square. So, if we take a single term, so that means, if you take m equal to 1, so then this expression can be A 1 cos omega T plus beta 0. So, let us take this omega T plus beta 0 equal to phi. So, we can write this x equal to a cos phi. So, when you write this x equal to a cos phi then or when we will take 2 term expansion, then it will be in 2 terms and then we can take either this constant or we can take, we can go up to this A 1 and A 2 by putting this A 0 equal to 0 or we can have 3 term expansion also. There we can have this A 0 A 1 and A 2.

So, we will study this harmonic balance method by substituting, taking a single term. Then, these two terms and then taking two harmonic terms, so here, we have taken a single harmonic with a constant term. So, if you are taking this x equal to a cos phi, so this equation, that means this x double dot will be equal to minus a omega square as this pi equal to omega T plus beta 0. So, x double dot will be equal to minus a omega square cos phi.

Then, this alpha 1 into A 1 cos phi plus alpha 2 into A 1 square cos square phi. Then we have to write this cos square phi in terms of, so cos square phi, we have to write in terms of this cos 2 phi, as we know this cos 2 phi equal to cos 2 phi equal to cos square phi minus. So, this is cos square phi minus sin square phi. So, as sin square phi equal to 1 minus cos square phi, so this becomes cos square phi minus 1 plus cos square phi or this becomes 2 cos square phi minus 1. So, one can write this cos square phi equal to half 1 plus cos 2 phi. So, then by substituting all these things in this equation, now by substituting this in this equation, so for cos square phi, we can write this way. Similarly, we have a term a cube. So, in this case, we have another term that is alpha 3 a cube cos cube phi. So, we have to write this cos cube phi similar to this cos square phi equal to half 1 plus cos 2 phi.

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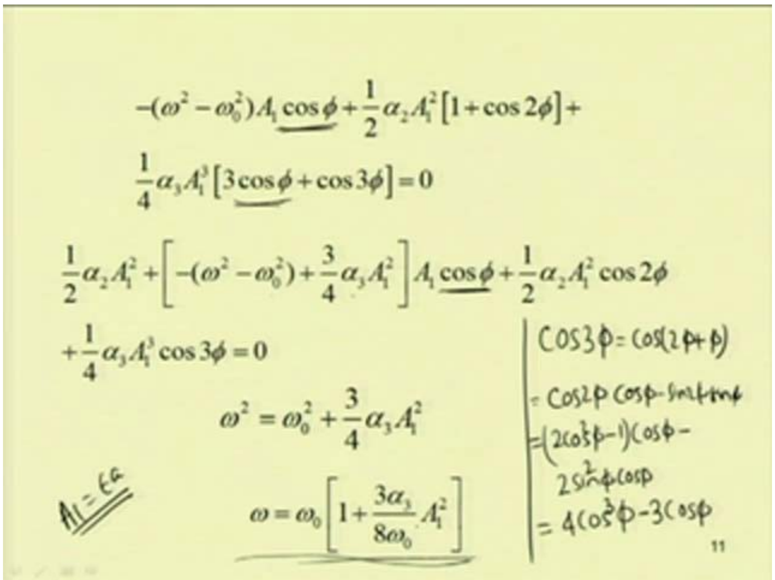

Similarly, we can write this cos cube phi in terms of cos 3 phi and we have to substitute that expression in this. So, for cos 3 phi, so we can write this cos 3 phi and as cos 3 phi equal to cos 2 phi plus cos phi cos 2 phi plus phi and by expanding that thing, one can write or one can write this way. So, it will be cos 2 phi into cos phi minus sin 2 phi into sin phi or this thing can be written as this cos 2 phi will be equal to. As we are interested to know the expression for cos q in terms of cos 3 theta, so this cos 2 phi, we can write this is equal to 2 cos square phi. Just now we have seen this minus 1 into cos phi and minus, so sin 2 phi equal to 2 sin phi into cos phi and this is multiplied by the sin phi. So, this becomes 2 sin square phi. We can substitute this sin square phi equal to 1 minus cos square phi. So, we can write this cos 3 phi equal to 4 cos cube phi minus 3 cos phi. So, for this term alpha 3 x cube, where it will be alpha 3 into a cube cos cube phi.

So, in place of cos cube phi, so we can substitute this cos 3 phi and cos phi. So, by substituting this thing expression for cos square phi and cos cube phi, so we can write this expression equal to minus omega square minus omega 0 square A 1 cos phi plus half alpha 2 A 1 square into 1 plus cos 2 phi plus 1 by 4 alpha 3 A 1 cube into 3 cos phi plus cos 3 phi. So, this will be equal to 0.

Now, equating the coefficient of cos phi to be 0, we can write, so now we have to equate this coefficient of cos to be equal to 0. So, we have, here we have a cos phi term and here also we have 1 cos phi term or first let us arrange it. So, if you arrange, then this is the constant part minus or this half alpha 2 A 1 square into 1. So, that is the constant part plus minus omega square minus omega 0 square plus this 3 by 4, this 3 into 1, that is 3 by 4 alpha 3 A 1 square into A 1 cos phi.

Similarly, by arranging the terms with cos 2 phi, we have this half alpha 2 A 1 square cos 2 phi plus 1 by 4 alpha 3 A 1 cube cos 3 phi equal to 0. So now, we have to equate the harmonics to be coefficient of this cos phi to be 0, so that we can write this minus omega square minus omega 0 square plus 3 by 4 alpha 3 a square equal to 0 or from this, we can write this omega square will be equal to omega 0 square plus 3 by 4 alpha 3 a square. Or, you can take or square root, so this omega will be equal to, so omega will be equal to omega 0 plus 3 by 4 alpha 3 A 1 square to the power half and expanding binomially, so we can write this is equal to omega 0 into 1 plus 3 by 8 alpha 3 by omega 0 A 1 square. So, by doing this 1 term expansion or by taking a single term, single harmonic, so we have seen this omega is coming to be omega 0 plus or omega 0 into 1 plus 3 alpha 3 by 8 omega 0 A 1 square.

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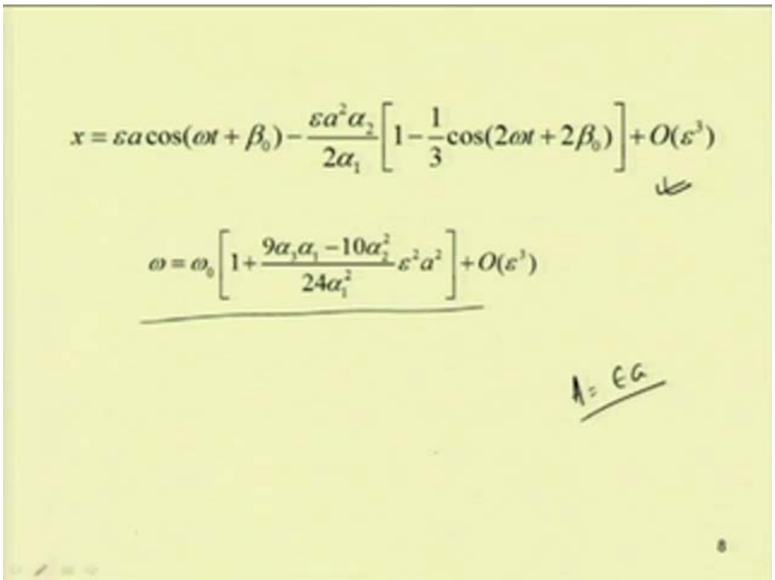

But, this approximation is not actually correct as we have seen in case of the method of multiple scale. So, this should have been omega equal to omega 0 plus 9 alpha 3 alpha 1 minus 10 alpha 2 square by 24 alpha 1 square epsilon square a square. So, in that expression, if we have substitute this a equal to epsilon a, so this a square A 1, so this A 1 square, so the expression what we obtain now, so this A 1 square, so here we can substitute this A 1 equal to epsilon a. So, this will become omega 0 into 1 plus 3 alpha 3 by 8 omega 0 into epsilon square a square. But, this term is not same as that we have obtained in case of the method of multiple of scales. So, this gives us slight erroneous result.

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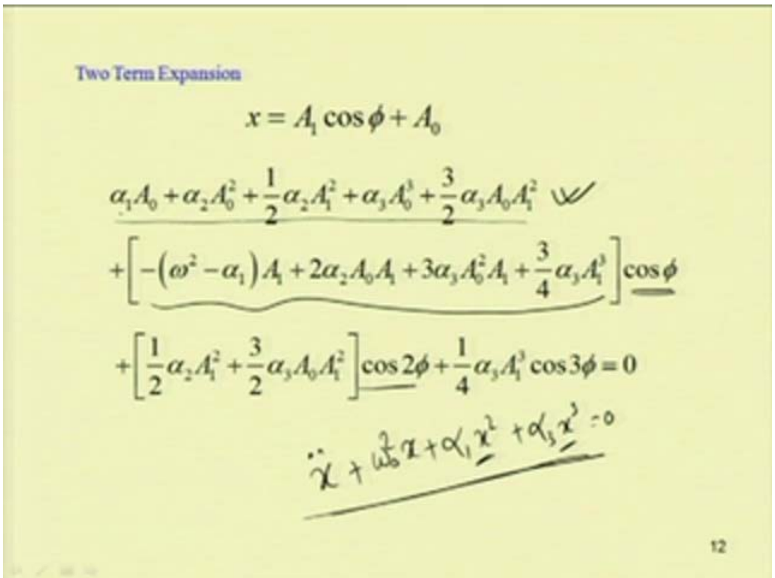

So, let us take now 2 terms in this expansion. So, if you take 2 terms that is a constant term and this harmonic, that means, we can take this x equal to A 0 plus A 1 cos phi. Then this expression will be equal to, so we have this x double dot plus, so for x double dot, it will be equal to minus A 1 omega square cos phi and we have to, so our expression, so x double dot plus alpha 1, that is omega 0 square x plus alpha 1 x square plus alpha 3 x cube equal to 0. So, here we have to expand this term x square. So, for this, it will be A 1 square cos square phi plus A 0 square plus 2 A 0 A 1 cos phi. So, for x cube also, we have to expand this thing. So, A 1 cos phi plus A 0 whole cube, so it will be A 0 cube plus A 1 cube cos cube phi plus 3 A 0 A 1 square cos phi plus 3 A 1 square cos square phi in into A 0. So, so by substituting those equations in this equation, one can and separating the order of or separating different harmonics, so one can write, so this is the constant term.

So, one can get this constant term, that is equal to alpha 1 A 0 plus alpha 2 A 0 square plus half alpha 2 A 1 square plus alpha 3 A 0 cube plus 3 by 2 alpha 3 A 0 A 1 square. So, this is the constant part and in addition to that, we have the parts, which is coefficient of cos phi. So, this part is coefficient of cos phi and this part becomes minus omega square alpha 1 where, alpha 1 is omega 0 square.

So, this into A 1 plus 2 alpha 2 into A 0 A 1 plus 3 alpha 3 A 0 square A 1 plus 3 by 4 alpha 3 A 1 cube. So, this is coefficient of cos phi. Similarly, we can have the coefficient of cos 2 phi. So, that is equal to half alpha 2 A 1 square plus 3 by 2 alpha 3 A 0 A 1 square and coefficient of cos 3 phi equal to 1 by 4 alpha 3 A 1 cube. As we have taken only 2 terms and one harmonic, so we can equate the constant term equal to 0 and the coefficient of this cos phi equal to 0 to get our frequency response curve or frequency amplitude relation. So, by substituting this alpha 1 A 0 plus alpha 2 A 0 square plus half alpha 2 A 1 square plus alpha 3 A 0 cube plus 3 by 2 alpha 3 A 0 A 1 square equal to 0 and also this one.

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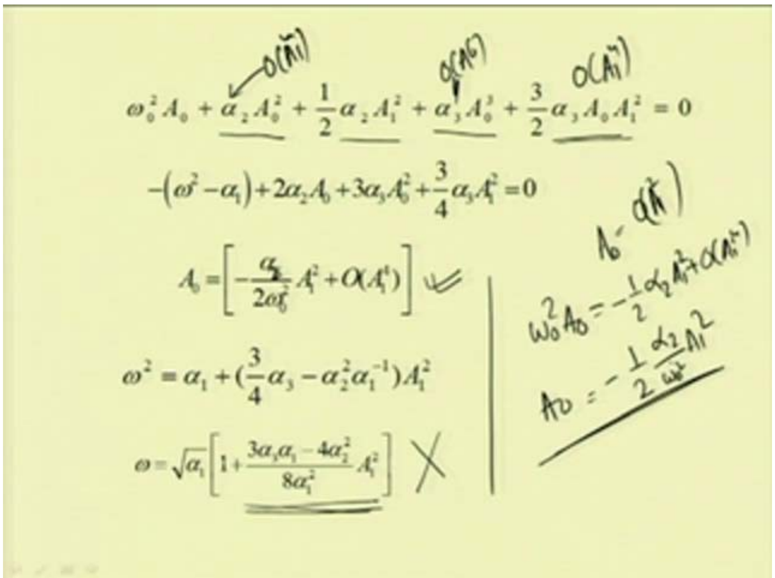

So, what you can get? So, let us see. So, from this, we can have this omega 0 square A 0 plus alpha 2 A 0 square plus half alpha 2 A 1 square plus alpha 3 A 0 cube plus 3 by 2 alpha 3 A 0 A 1 square equal to 0. So, here and the from the second expression, so we can have this minus omega square minus alpha 1 plus 2 alpha 2 A 0 plus 3 alpha 3 A 0 square plus 3 by 4 alpha 3 A 1 square equal to 0. So, from this, so we can get this A 0 equal to. So, we can write this A 0 equal to minus alpha 3 by 2 omega 0 square into A 1 square plus order of A 1 4th and from this expression, you can write this omega square equal to alpha 1 plus 3 by 4 alpha 3 minus. So, from these two, we can substitute this A 0 value, so we can get this is equal to minus alpha 2 square by alpha 1 into A 1 square.

So, we obtain this thing from this equation. So, from this expression, so we can note that we can substitute this A 0 of the order of, so A 0 of the order of a square and so assuming or we can observe from this equation, that A 0 is the order of A 1 square. So, we can write this, so we can see from this expression, so A 0 square will be of the order of A 1 4th. So, we can neglect this term and also we can neglect this A 0 6, which is of the order of A 1 to the power 6. So, we can neglect this term and also this term A 0 A 1 square, which is of the order of, so this is of the order of A 1 square is A 1 square into A 1 square, and that is A 1 to the power 4. So, this is of the order of A 1 4th. This term is of the order of A 1 4th. So, this is of the order of A 1 6 and this term is of the order of A 1 4th. So, we will take only the term A 1 square.

So, neglecting the order of 4th and 6^{th}, so we can write this A 0. That means, omega 0 square A 0. So, from this we can write this omega 0 square A 0 equal to minus half alpha 2 A 1 square plus order of A 1 4th. So, we have neglected the terms of A 1 4th and A 1 6^{th}. So, we can have this omega 0 square A 0 equal to this or A 0 equal to minus 1 by 2. So, alpha 2 by omega square, so half already we have written, so this is a square. So, this is the expression what we have written here. So, A 0 equal to minus alpha 3. So, A 0 equal to, so this will be alpha 2 A 1 square, so this is alpha 2 and not alpha 3. So, alpha 2 A alpha 2 by 2 omega 0 square A 1 square. So, this expression shows that the result what we got that is omega equal to root over alpha 1 into 1 plus 3 alpha 3 alpha 1 minus 4 alpha 2 square by 8 alpha 1 square A 1 square. So, this is also not correct as what we have seen in the method of multiple scales. So, this correction part or this additional part about this omega 0 is not so accurate when we have taken a single harmonic term plus a constant term also.

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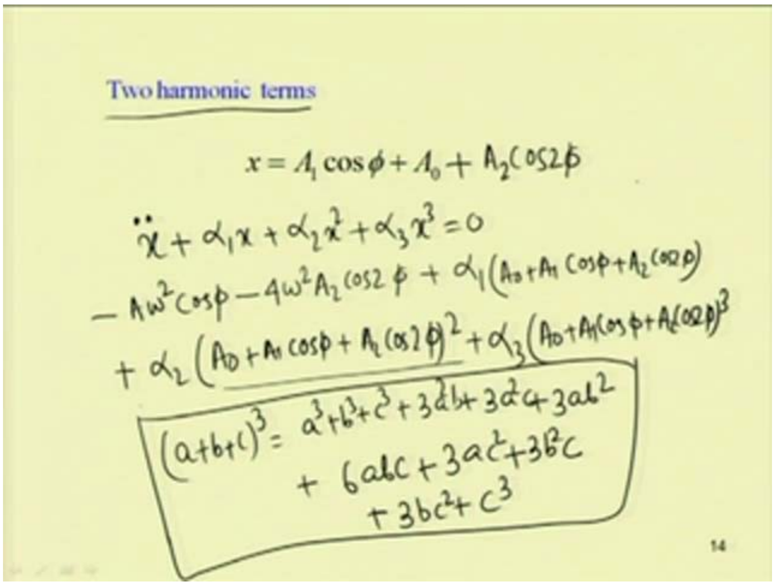

So, let us take 2 terms in this expansion and we see that it may yield some correct result. So, in these 2 terms, so you can take, so A 0 plus A 1 cos phi plus, let us take A 2 cos 2 phi. So, if you take 2 harmonic terms, then you can write x equal to A 0 plus A 1 cos phi plus A 2 cos 2 phi. So, then our expression x double dot, our equation, our equation x double dot x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube equal to 0 can be written in this form. So, for x double dot, it can written minus a omega square cos phi. So, for double differentiation of A 0 equal to 0 and double differentiation of this thing will be equal to. So, this is 2 phi, that means 2 omega. S ,while differentiating twice, so it becomes minus 4 omega square A 2 cos 2 phi.

So, this is for x double dot. Then for alpha 1, we can write this alpha 1 into A 0 plus A 1 cos phi plus A 2 cos 2 phi and then plus alpha 2 into x square. So, x square will be equal to A 0 plus A 1 cos phi plus A 2 cos 2 phi whole square and for this thing, so alpha 3 into A 0 plus A 1 cos phi plus A 2 cos 2 phi whole cube. So, we know this is similar to A 1 plus A 2 plus A 3 whole square. So, this becomes our a plus b plus c whole square. So, this is a square plus b square plus c square plus 2 a b plus 2 b c plus 2 c a. So, one can expand this in that form. Similarly, this a plus b plus c, so if you take this is a, this is b and plus c, so you can use this formula a plus b plus c whole cube equal to a cube plus b cube plus c cube plus 3 a square b plus 3 a square c plus 3 a b square plus 6 a b c plus 3 a c square plus 3 b square c plus 3 b c square plus c cube. So, using this formula for this alpha 3 into a plus A 0 plus A 1 cos phi plus A 2 cos 2 phi whole cube, so one can expand this equation and one can write equations.

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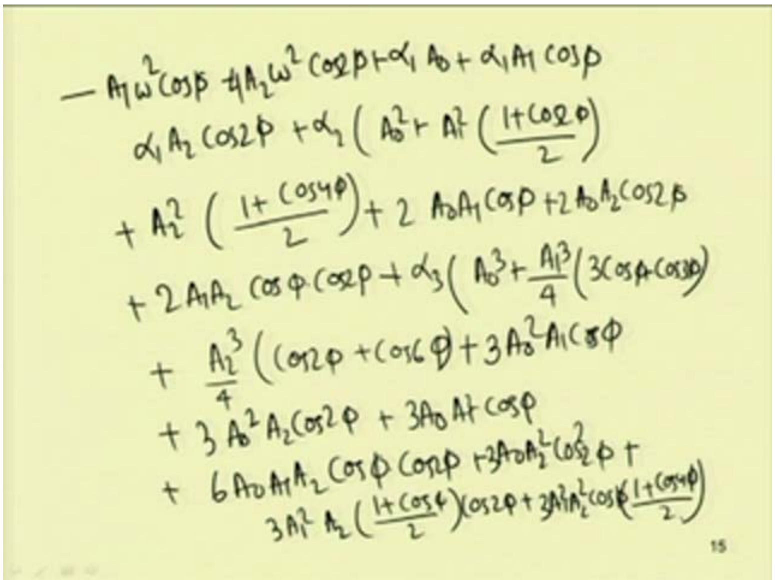

So, by expanding that thing, one can write this expression and that will be equal to minus A 1 omega square cos phi minus A 2 4 A 2 omega square cos 2 phi plus alpha 1 A 0 plus alpha 1 A 1 cos phi plus alpha 1 A 2 cos 2 phi plus alpha 2 into A 0 square plus A 1 square, into, so that cos square, theta terms one can write in this form, so 1 plus cos 2 phi by 2 plus A 2 square into 1 plus cos 4 phi by 2. So, this is cos square 2 phi. So, plus, then A 0 A 1 cos phi plus 2 A 0 A 2, so A 2 square, so this is plus 2 A 0 A 1 cos phi plus 2 A 0 A 2 cos 2 phi plus 2 A 1 A 2 cos phi into cos 2 phi plus alpha 3 into, so this thing one can write A 0 cube plus A 1 cube by, so this is A 1 cube by 4 into 3 cos phi plus cos 3 phi plus A 2 cube by 4 into cos 2 phi plus cos 6 phi plus 3 A 0 square A 1 cos phi plus 3 A 0 square A 2 cos 2 phi plus 3 A 0 A 1 square cos phi plus 6 A 0 A 1 A 2 cos phi into cos 2 phi plus 3 A 0 A 2 square cos square 2 phi plus 3 A 1 square A 2 into 1 plus cos phi by 2 into cos 2 phi plus 3 A 1 square A 2 square cos phi into 1 plus cos 4 phi by 2.

So here, one can substitute, so already I told you, for cos square theta, one can substitute cos square phi equal to 1 plus cos 2 phi by 2. Similarly, cos 3 phi can be substituted by 4 cos cube phi minus 3 cos phi.

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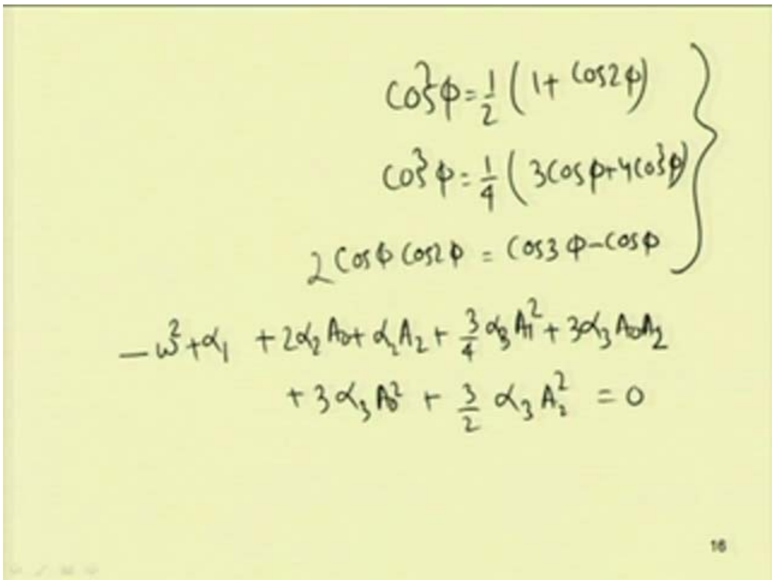

So, we can substitute this formula, that means, cos square phi equal to half 1 plus cos 2 phi. Similarly, this cos cube phi equal to 1 by 4 into 3 cos phi plus 4 cos cube phi and this cos phi into cos 2 phi. So, that thing can be substituted in this way. So, this will be equal to, so 2 into this will be equal to cos 3 phi minus cos phi. So, by substituting these in the previous equation and equating the coefficients of cos phi cos 2 phi and cos 3 phi, so one can write this expression like this. So, one can have these following 3 equations. So, one will be this omega square plus alpha 1 plus 2 alpha 2 A 0 plus alpha 2 A 2 plus 3 by 4 alpha 3 A 1 square plus 3 alpha 3 A 0 A 2 plus 3 alpha 3 A 0 square plus 3 by 2 alpha 3 A 2 square. So, this would be equal to 0.

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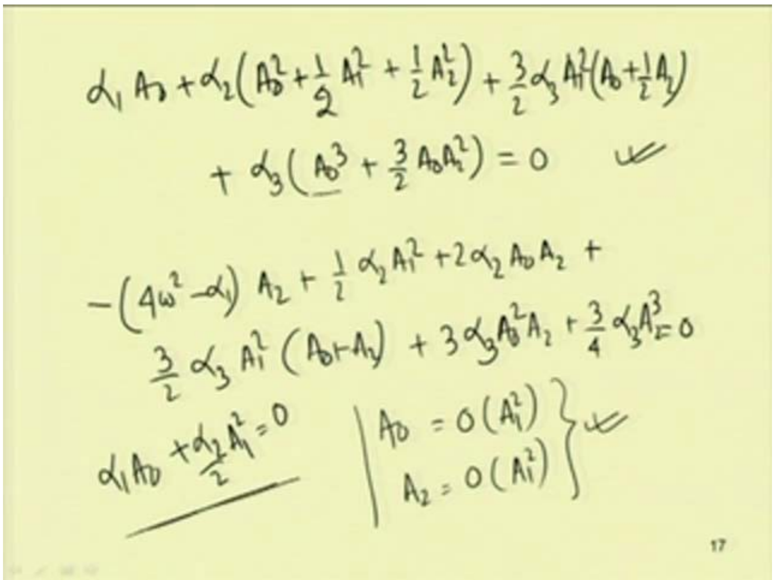

So, this is one equation. The other 2 equations will be alpha 1 A 0 plus alpha 2 into A 0 square plus half A 1 square plus half A 2 square plus 3 by 2 alpha 3 A 1 square into A 0 plus half A 2 plus alpha 3 into, so alpha 3 into A 0 cube plus 3 by 2 A 0 A 2 square. So, this will be equal to 0 and the third equation will be minus 4 omega square minus alpha 1 into A 2 plus half alpha 2 A 1 square plus 2 alpha 2 into A 0 A 2 plus 3 by 2 alpha 3 A 1 square into A 0 plus A 2 plus 3 alpha 3 A 0 square A 2 plus 3 by 4 alpha 3 A 2 cube.

So, let us see these equations. So, in this equation you can observe that the first equation, which is coefficient of cos phi can be written in this way. So, minus omega square plus alpha 1 plus 2 alpha 2 A 0 plus alpha 2 A 2 plus 3 by 4 alpha 3 A 1 square plus 3 alpha 3 A 0 A 2 plus 3 alpha 3 A 0 square plus 3 by 2 alpha 3 A 2 square. From these 3 equations we can observe that this A 0, so this A 0, so let us see the terms which are not coupled, uncoupled terms. So, if you compare with this uncoupled terms, so this is alpha 1 A 0 and this is, we have this half A 1 square. So, this is A 2 square. So, here multiplication of A 1 square A 0 and A 1 square A 2, here this is A 0 cube and here it is A 0 A 2 square. So, we can note that or one can note that this A 0 can be of the order of A 1 square.

Similarly, A 2 also can be of the order of A 1 square. So, if we can write this A 0 and A 2 in terms of A 1, then we can write this A 0 and A 2 should be of the order of A 1 square. So now, neglecting the terms which are more than A 1 square, that is neglecting the terms of order of A 1 4th and A 1 6, so we can write these expressions.

So, from the second equation, that is this equation, we can see this alpha 1 A 0, so we can write, so from this second equation, we can write this alpha 1 A 0, so this A 0 square, as this A 0 is the order of A 1 square, then A 1 square will be of the order of A 1 4th. So, we can neglect this term. Similarly, we can neglect this term as this is A 2 square. A 2 square will be of the order of A 1 4th. So, we can neglect this term also and here also we can neglect this A 1 square A 0. That means, it is also of the order of A 1 4th as A 0 is of the order of A 1 square. So, multiplied by A 1 square, so this is a to the power 4. Similarly, this A 1 square into A 2, so this is also of the order of A 1 4th and this A 0 6 is of the order of A 1 6th and A 0 A 2 square is also of the order of A 1 6th. So, neglecting the order of A 0 4th and z 0 6th, we can write this alpha 1 A 0 plus alpha 2. So, plus alpha 2 by 2 into A 1 square, so this will be equal to 0 and here we have neglected the order of epsilon to the power 4.

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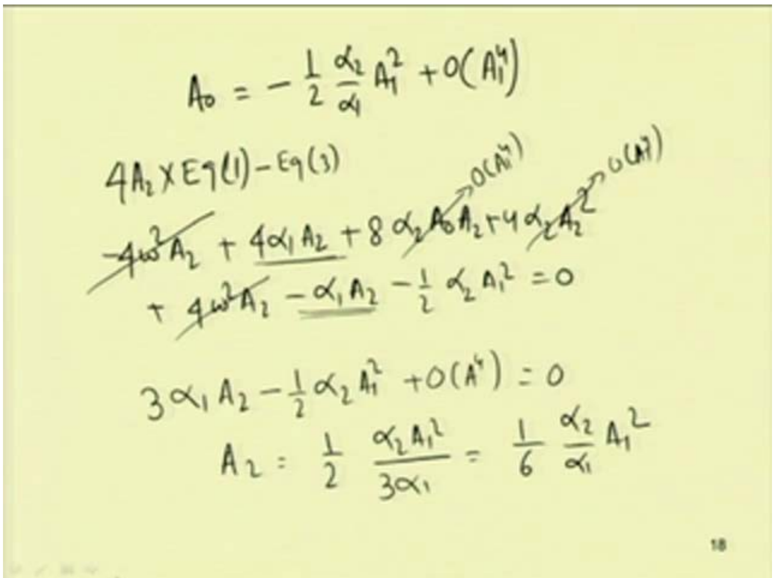

So, from this we can write A 0 equal to, so from this, we can write A 0 equal to minus half alpha 2 by alpha 1 A 1 square order of A 1 4th. Now, let us, we have to find what is A 2 or we have to write A 2 in terms of A 1. So, for that purpose, so let us check equation 1 and so let this is equation number, let me put this is equation number 1. So, this is equation 2 and this is our 3rd equation. So, in the, in this equation, 3rd equation, we have a term, this 4 minus 4 omega square minus alpha 1 into A 2 and in the first equation, we have this term minus omega square plus alpha 1. So, if I will multiply this minus, if I multiplied 4 A 2 in this equation and subtract it, so by multiplying 4 A 2 in equation 1 and subtracting that thing from equation 3, so we can write, so then we can write this way. So, this will be minus 4 A 2 we have multiplied. so this will become minus So, this becomes minus 4 omega square A 2 plus 4 alpha 1 A 2 plus 8 alpha 2 A 0 A 2 plus 4 alpha 2 A 2 square plus 4 omega square A 2 minus alpha 1 A 2 minus half alpha 2 A 1 square. So, this becomes equal to 0.

Already we have seen this A 0 A 2, this A 0 A 0 A 2 as we have seen, that A 2 should be of the order of A 1 square. So, this and already A 0 is also of the order A 1 square. So, this A 0 A 2, so this is order of A 1 4th. Similarly, A 2 square is of the order of A 1 4th and this A 1 square, so this is alpha 1 A 2 and this becomes, so this minus 4 omega square A 2 plus 4 omega square A 2 can cancel. So now, we have this 4 alpha 1 A 2 minus alpha 1 A 2, so this becomes 3 alpha 1 A 2 minus half alpha 2 A 1 square. So, plus order of epsilon 4th’ so this becomes 0 or this A 2 becomes half alpha 2 A 1 square by this 3 alpha 1. So, this becomes 1 by 6 alpha 2 by alpha 1 A 1 square.

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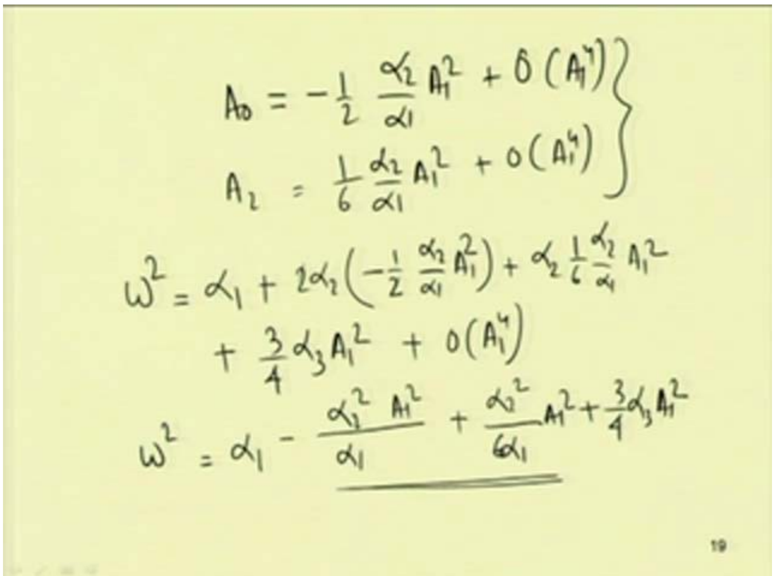

So, we got this A 0. So, already we got A 0 in terms of A 1 and that is equal to, so this A 0 equal to minus 1. So, we have written this A 0 equal to minus half alpha 2 by alpha 1 A 1 square order of A 1 4th and A 2, just now we obtained. It is equal to 1 by 6 alpha 2 by alpha 1 A 1 square. So, this is of the order of A 1 4th. So, after getting this A 0 and A 2 in terms of A 1, we can substitute it in the first equation and we can write this omega square equal to, so omega square will be equal to alpha 1 plus 2 into alpha 2 into. So for A 0, I can substitute it equal to minus half alpha 2 by alpha 1 A 1 square, then plus alpha 2 into A 2, that is 1 by 6 alpha 2 by alpha 1 A 1 square plus 3 by 4th alpha 3 A 1 square. So, then we can have of the order of about A 1 4th. So, from this we can write this omega square equal to, so this is alpha 1 and so this becomes 2 and 2 cancels, so this becomes alpha 2 square. So, minus alpha 2 square by alpha 1 into A 1 square, so plus, so this becomes alpha 2. Again, this is plus alpha 2 square by 6 alpha 1 A 1 square plus 3 by 4 alpha 3 A 1 square. So, one can write, one can combine these two because we had, we have this alpha 2 square A 1 square and here also alpha 2 square A 1 square. So, this becomes minus 5 by 6. So, minus 5 by 6 alpha 2 square A 1 square.

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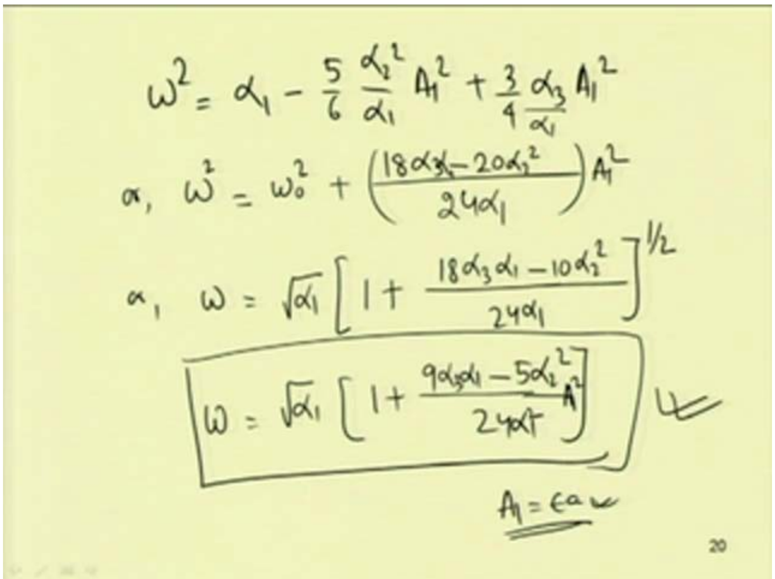

So, one can write this omega square equal to alpha 1 minus 5 by 6 alpha 2 square by alpha 1 A 1 square plus 3 by 4 alpha 3 A 1 square. Or, one can write this omega equal to or omega square equal to alpha 1, which is equal to omega square or omega 0 square omega 0 square, I can add these 2 terms. So, by adding this, one can write this equal to plus, so this becomes 24. So, by alpha 1 here, so 24 alpha 1, so this becomes 24. So, this becomes 3 into, so this 18 alpha 3 minus, so 5 into 4, so this becomes 20 alpha 2 square, so by alpha 1 here, so this is alpha 3 alpha 1. So, these into A 1 square or we can have this omega, so this will become omega 0 square or root over this thing, if I will take, so this becomes omega 0 into 1 plus.

So, this will be 18 alpha 3 into alpha 1 or I can write this equal to omega 0 or root over alpha 1. Anyway you can write. So minus, this is 10 alpha 2 square by 24 alpha 1 to the power half. So, this is or this parameter is small. So, expanding it in the binomial form, so one can write this omega equal to root over alpha 1 into 1 plus, so half of half into 18 alpha 3 alpha 1 minus 10 alpha 2 square by 24 alpha 1, so this thing can be written by 9 alpha 3 alpha 1 minus 5 alpha 2 square by 24 alpha 1.

So, this expression is same as that what we obtained in case of the method of multiple scales. So, in case of method of multiple scales, so we have, we have this expression. So, omega equal to omega 0 into 1 plus 9 alpha 3 alpha 1 minus 10 alpha 2 square by epsilon square a square and in the present case also, we obtained the same expression. So here, we have the same expression, that is alpha 1 square. So, we have taken this alpha 1 common, so 1 plus 9 alpha 3 alpha 1 minus 5 alpha 2 square by 24 alpha 1 square and the solution x, so here, so this into A 1 square. So here, we have substituted this A 1 equal to epsilon a and we obtained the same expression as that we obtained in case of the method of multiple scales.

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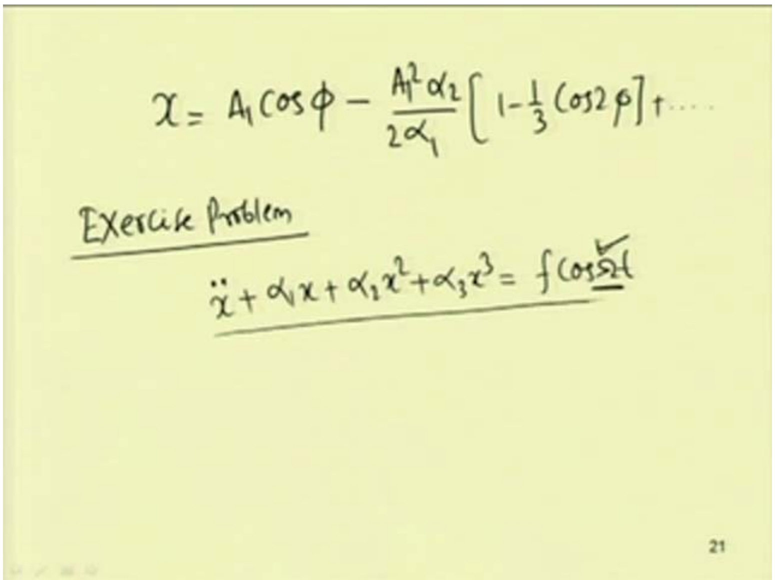

I am substituting the value of A 0 and A 1, so we can write this x equal to A 1 cos phi minus A 1 square alpha 2 by 2 alpha 1 into 1 minus one-third cos 2 phi and plus the higher order terms. So, from this, one can observe that we can use this harmonic balance method. So, if you know the solution appearing, so up to how many terms we should take in this expansion, if we know, only then we can apply this method. So, this thing, we have illustrated by using a single term, a single harmonic term, a single harmonic term with a constant term and with a single 2 harmonic terms and a constant term and we have seen. So, when we have taken these 2 harmonic terms and a constant term in the expansion, so we obtain the same expression that we obtained in case of method of multiple scales or Lindstedt Poincare method.

So, one has to know the solution (( )) to apply this method. So, this is the main disadvantage of using method of harmonic balance. Also, one can take higher order terms and by taking these higher order terms, the computational difficulty arises, which one can overcome by using this symbolic software. So, now-a-days, one can use this symbolic software to conveniently use this method of harmonic balance. But, if one wish to do this thing manually, so it is very difficult to use this method.

So, next class we will study about method of (( )) and today, in this today's class, so you can do or take some exercise problem also. So, apply this method of harmonic balance to this equation x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube. So, we have taken this is equal to 0, but let us take a term, so this is equal to f cos omega T. So, let us apply or take this as exercise problem and solve this problem. So here, take 2 term expansion and while expanding, take care of this external frequency term also. So, by taking care of this external frequency term, one can find the solution of this equation using harmonic balance method.

Thank you.