Non-Linear Vibration
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Module - 3
Solution of Nonlinear Equation of Motion
Lecture - 3
Linstedt-Poincare Method

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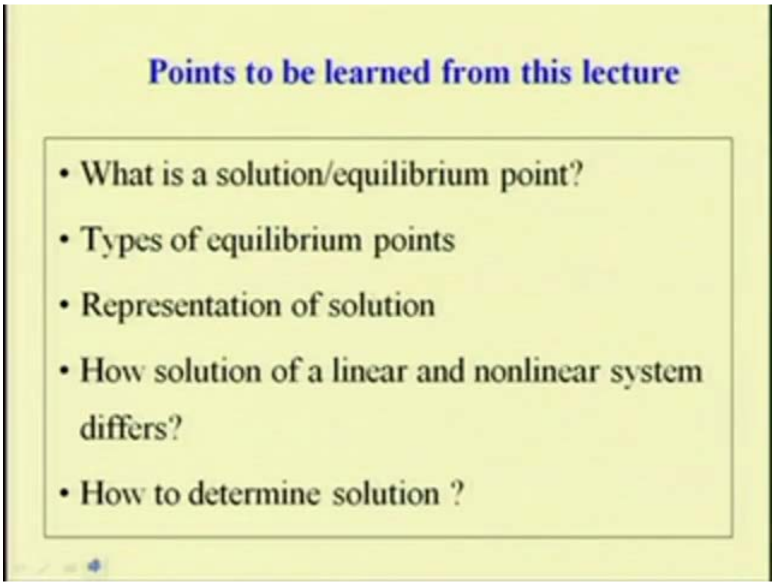

Welcome to today’s class of non-linear vibration. Today’s class we are going to study about this Lindstedt Poincare method for finding the solution of non-linear equation motion. So in the previous two classes, we have studied about this graphical methods or this qualitative analysis we did, and we know what is a solution or equilibrium point, types of equilibrium points, and representation of the solution, how solution of a linear system differs from the non-linear systems and how to determine the solutions. So, to determine the solutions, we studied some qualitative approach and quantitative approach. In this quantitative approach, so last class we have studied about the straight forward expansion method and today we are going to study about this Lindstedt Poincare technique.

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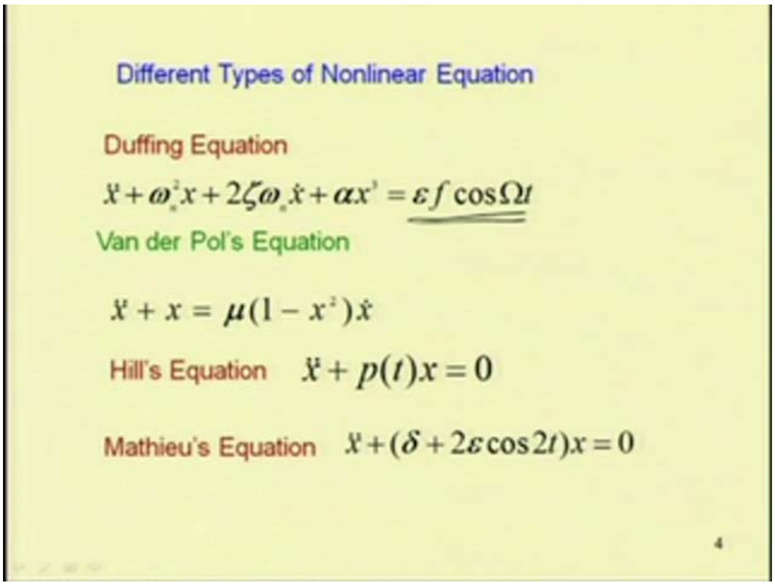

So, also I told you about these numerical methods. So, these are some of the non-linear equations we are going to study or you, by this time know about these systems. So, first one is the Duffing equation. So, in case of Duffing equation, we can take this term equal to 0 for free vibration or for force vibration, we can this term equal to epsilon f cos omega t. So, these equations may be strongly non-linear or weakly non-linear depending on the coefficients of the non-linear terms.

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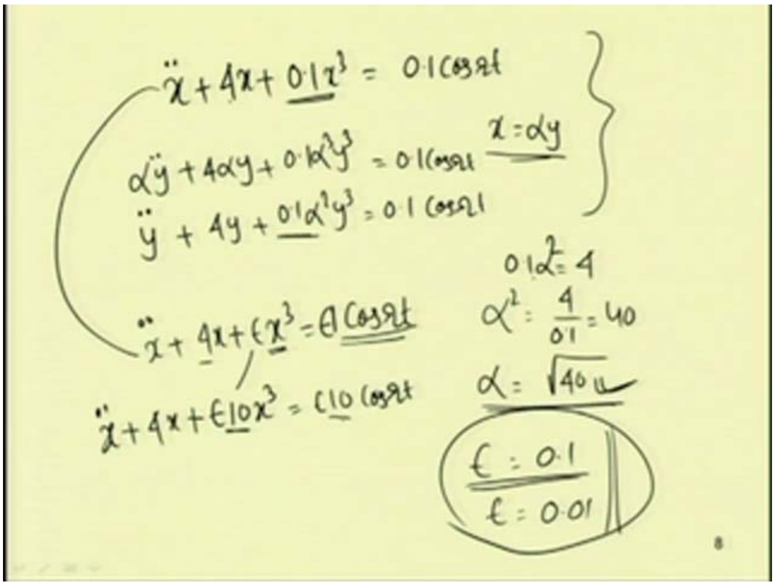

For example, if I will take an equation, so let me take an equation in this form, that is x double dot plus 4 x plus 0.1 x cube. So, this is equal to 0.1 cos omega t. So, in this case, I can use the scaling parameter and the book keeping parameter to order this equation. For example, I may use, let x equal to alpha y. So, if I will substitute this equation, in this equation, so then this will becomes alpha y double dot plus 4 alpha y plus 0.1 alpha cube y cube, so this will be equal to 0.1cos omega t. So now, I can write this equation again in this form, y double dot plus 4 y plus 0.1 alpha square y cube equal to 0.1 cos omega t. So, in this case, this if I want to make this strongly non-linear or coefficient of u cube, if I want to increase, so I can put this 0.1 alpha square equal to 4 and I can find this alpha square equal to 4 by 0.1. So, this is equal to 40 or I can find this alpha root over 40. So, which cannot be neglected.

In this equation, one can neglect this 0.1 with respect to this 4 and consider this equation as a linear system. But if I will keep this term alpha equal to root over 40, so this term cannot be neglected with respect to this 4. So, this way, one can order the system. So, by ordering the system, so one can make a weekly non-linear system to that of a strongly non-linear system and also one can use this book keeping parameter. For example, in this equation, so if I will put a book keeping parameter epsilon, which is equal to 0.1. Let me take epsilon equal to 0.1. So this equation, I can write in this form. So, x double dot plus 4 x plus, I can put this is epsilon x cube equal to, so this for this 0.1, I can write this is equal to epsilon, so epsilon cos omega t. So, here the coefficient of x cube, so will be equal to, if this epsilon I represent this is a small parameter and this coefficient will be 1, so which is comparable to this one, this 4.

Here also, the coefficient of cos omega t, so I can write this epsilon into 1 into cos omega t, so this will also be comparable to 1 or if I will take a term, which is less than this, let me take epsilon equal to 0.01. So, if I will take epsilon equal to 0.01, then the same equation I can write in this form, x double dot plus 4 x plus, so here, if I will divide this thing by 0.1 by 0.01, so then it will be epsilon into 10 x cube will be equal to, so epsilon into 10 cos omega t. So, one can change this value, that is the coefficient by substituting different type of epsilon, different book keeping parameters and one can write this equation in a different way. So, these coefficients can be varied by taking different epsilon or the book keeping parameter or different scaling parameter like this alpha. So, after getting a non-linear differential equation of motion, so we know, by using this qualitative approach, so we can solve this equation.

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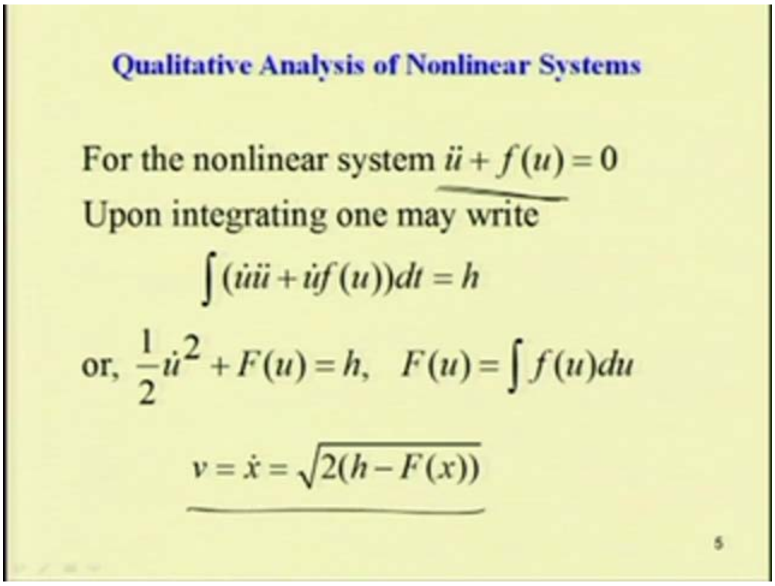

For example, here we use this qualitative approach. So, in this qualitative approach, if this equation is u double dot plus f u equal to 0, so one can write this velocity or x dot will be equal to root over 2 into h minus f x, where f x is the potential function and h is the total energy.

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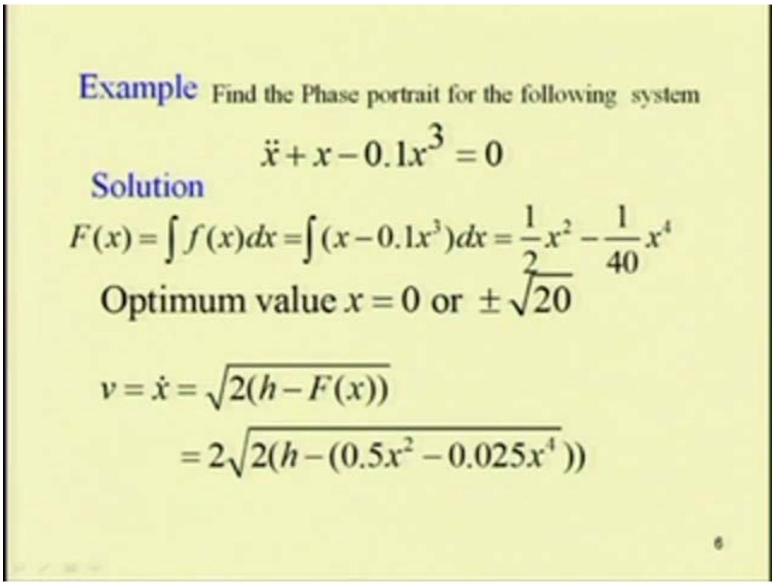

We have solved some example, that is this equation we have taken, x double dot plus x minus 0.1 x cube equal to 0 and we got this.

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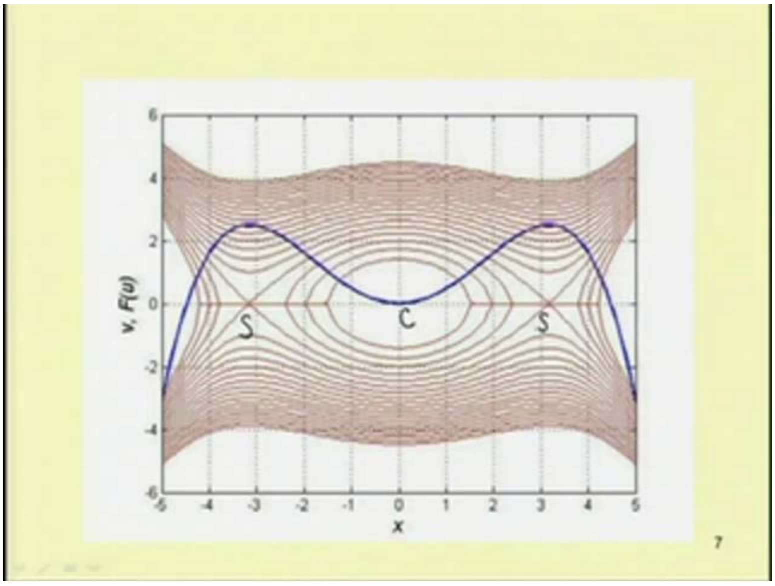

So, this is the potential function and these are the phase portraits. So, corresponding to this maximum value of the potential energy, so we have a saddle point and corresponding to the minimum value, we have a center. So, corresponding to minimum potential energy, we have a center and corresponding to this maximum potential energy, we have saddle point.

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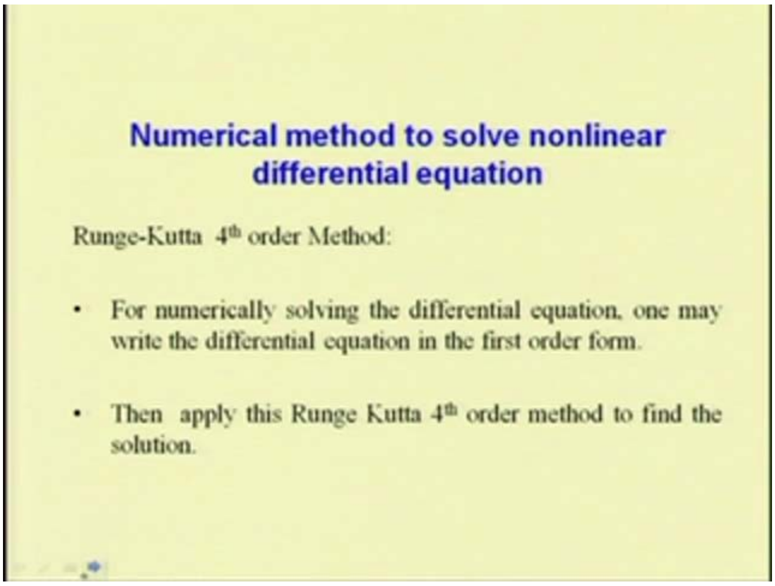

So, in this way we did this qualitative analysis. After carrying out the qualitative analysis, also we have studied in the last class by using this Runge-Kutta 4th order method.

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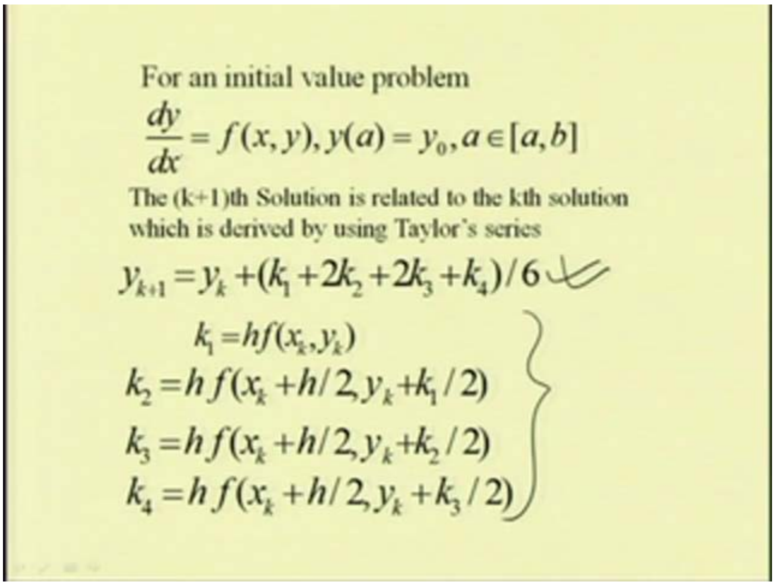

So, in which, the algorithm can be written in this form. So, given a first order differential equation d y by d x equal to f x y, where the initial condition y a equal to y 0, so a belongs to a b. So, in that case, the k plus 1th solution is related to the kth solution, which is derived by this Taylor series by this equation. So, y k plus 1 equal to y k plus k 1 plus 2 k 2 plus 2 k 3 plus k 4 by 6, where k 1 equal to h into the functional value of f at x k and y k. So, k 2 also, k 2 equal to h into the functional value of f at x k plus x by 2 plus y k plus k 1 by 2. Similarly, k 3 equal to h into functional value of x k plus h by 2 and y k plus k 2 by 2 and k 4 equal to h into f x k plus h k by 2 and y k plus k 3 by 2.

So, one can write a program taking this value k 1 k 2 k 3 k 4 and substituting in this equation and taking different time increment, so one can find this y k plus 1 in terms of y k and one can iterate it till one obtains a steady state solution.

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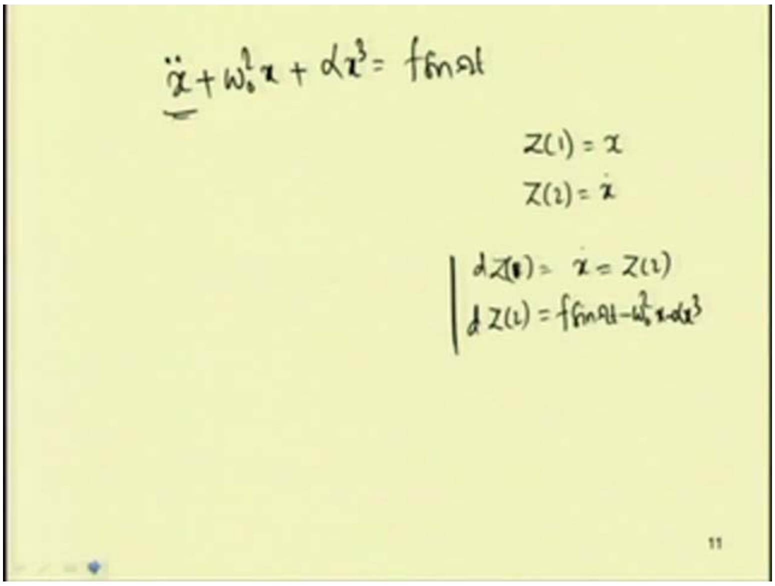

So, to use this differential equation for our purpose, so for example, in our case, our equation is, let it is x double dot plus x double dot plus omega 0 square x plus alpha x cube equal to f sin omega t. So, in this case, first we have to reduce it to that of a first order equation and after reducing this thing to a first order equation, and then only one can apply this 4th order Runge-Kutta method to find the solution. So, one can write the, to reduce it, one can write this, one can take this x dot. So, this is the first equation.

One can take x equal to, let z 1 equal to x and one take z 2 equal to x dot, so thus first equation one can write z 2 or d z 2 equal to d x dot. That is your x dot and then this x double dot d z 1, so this is d z 1, so d z 1 equal to x dot and d z 2, that is the differentiation of this x dot, that is x double dot can be written. So, this is d z 2. That is, x double dot, that thing can be written in this way, f sin omega t minus omega 0 square x minus alpha x cube. So, taking these two equations, so one can, so these are the first order equations. So, this x dot, and this is equal to z 2 d z 1, that is equal to derivative of x, that is x dot. So, that thing equal to z 2 and d z 2, that is x double dot, can be written in this form. So, this is f sin omega t minus omega 0 square x minus alpha x cube. So, after getting these two first order equations, one can use this method to find the solution. Last class, we have seen the solutions for different cases and we have plotted the time response and phase portrait for these cases.

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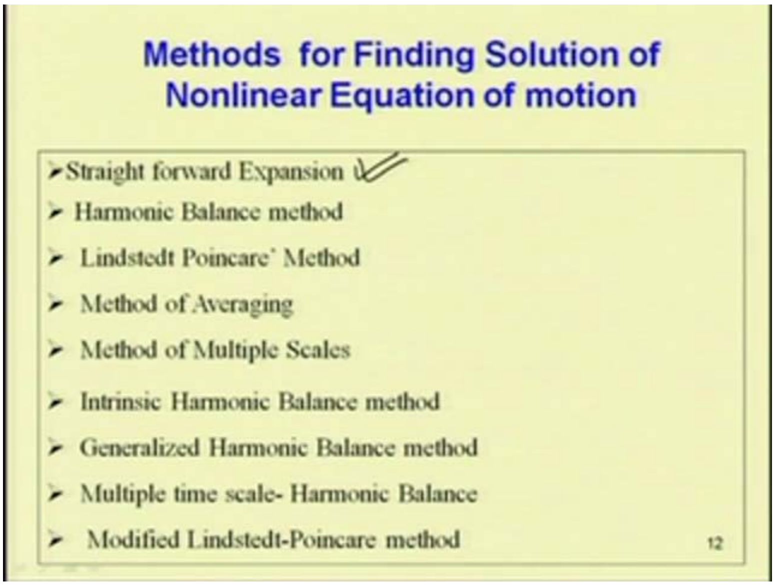

So, in case of the quantitative analysis, already we have seen the straight forward expansion method.

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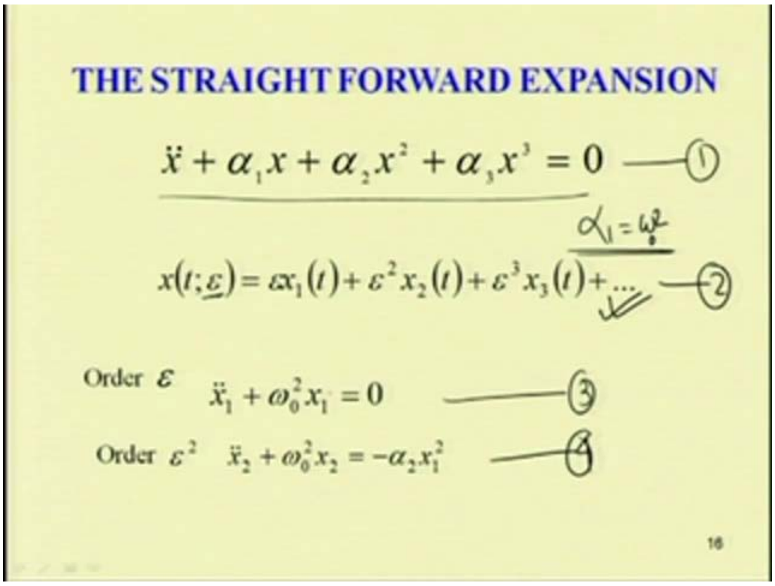

So, in case of this straight forward expansion method, so we have seen, so given a equation, let this is the original equation motion, x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube equal to 0, in which we have taken this alpha 1 equal to omega square, omega 0 square. So, in case of the straight forward expansion, we have expanded this response, that is x which is a function of t and a book keeping parameter epsilon equal to epsilon x 1 t plus epsilon square x 2 t plus epsilon cube x 3 t in the original equation. After substituting this equation, let this is equation 1, equation 2, and substituting equation 2 in equation 1 and ordering them of the order of epsilon, so one can obtain this equation, equation 3 is for order of epsilon and equation 4 is for order of epsilon square and equation 5 is for the order of epsilon cube.

So, after getting these equations, so one knows the solution of this. So, x double dot plus omega 0 square x 1 equal to 0, so the solution of this equation. So here, as the acceleration is proportional to displacement x 1, then the motion is simple harmonic.

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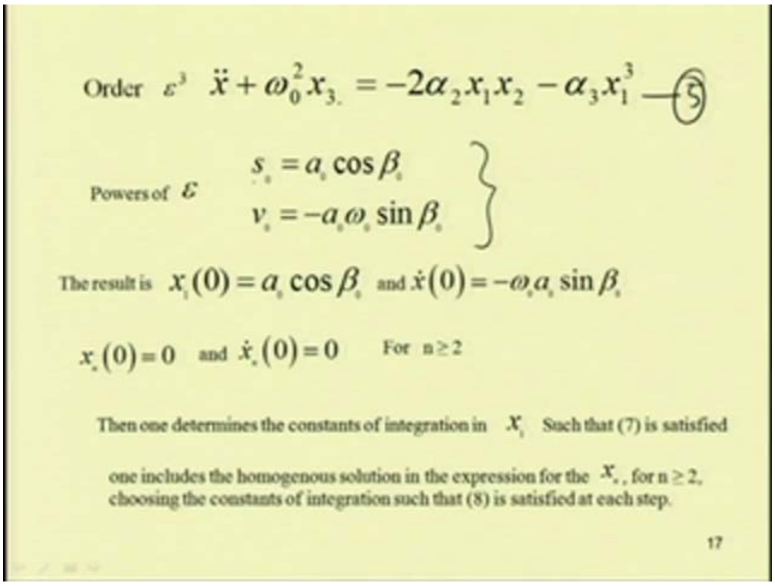

In case of the simple harmonic motion, so the solution can be written as a sin omega t. So, one can put the solution equal to a sin omega t. One can take this initial displacement and velocity in polar form and write. So, one can write s 0 equal to a 0 cos beta 0 and b 0 equal to minus a 0 omega 0 sin beta 0 and substitute it in this equation and one can find the solution.

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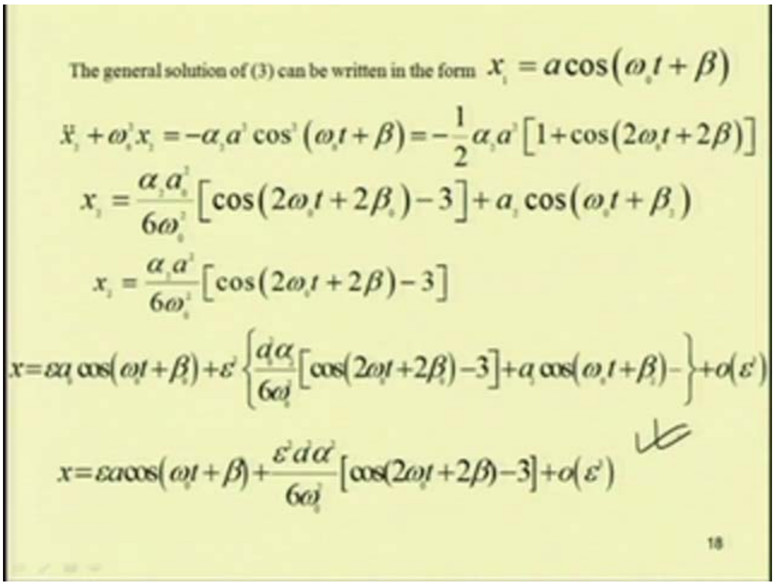

So, after getting the solution of x 1, one can substitute that thing in equation of x 2 and one can get the solution in this form.

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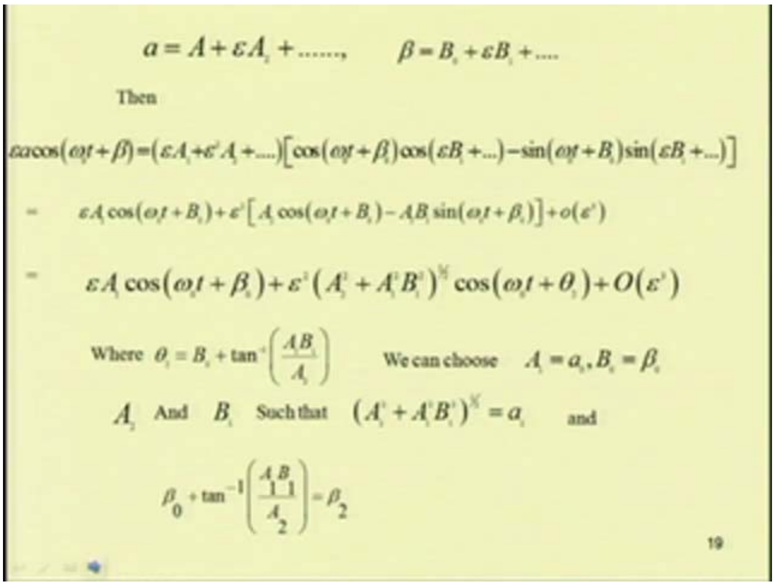

After getting the solution of x 2, one can substitute it in this equation of x 3.

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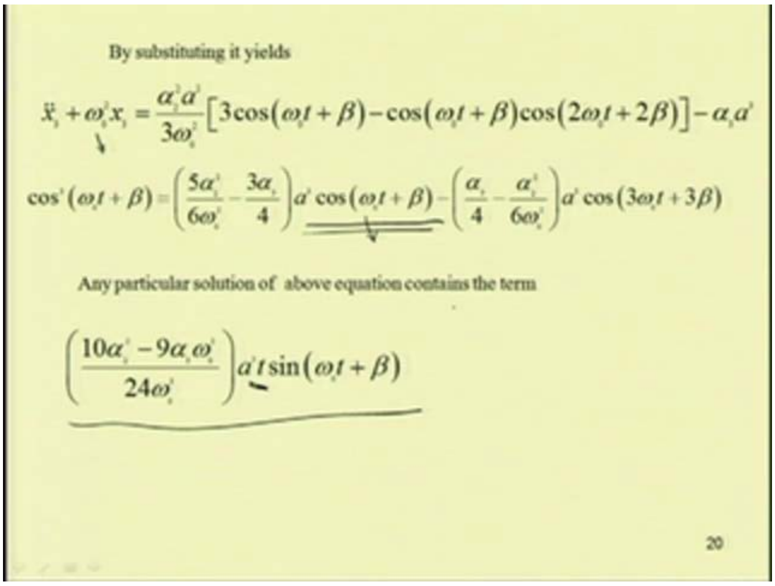

One can obtain this equation, so in which one can get this x 3 double dot plus omega 0 square x 3 equal to alpha 2 square a cube by 3 omega 0 square into 3 cos omega 0 t plus beta minus cos omega 0 t plus beta into cos 2 omega 0 t plus 2 beta minus alpha 3 a cube into cos cube 3 0 cos cube omega 0 t plus beta. So, which can be also written as, 5 alpha 2 square by 6 omega 0 square minus 3 alpha 3 by 4 into a cube into cos omega 0 t plus beta minus alpha 3 by 4 minus alpha 2 square by 6 omega square a cube cos 3 omega 0 t plus 3 beta.

So, one can observe that, due to the presence of this term cos omega 0 t plus beta, so the solution of this term, this part will contain a secular term. So, as the cos has a, so this term has a frequency omega 0 and here you have this omega 0 square x 3. So, this will lead to a secular term, which will have a solution in this form. The solution will be 10 alpha 2 square minus 9 alpha 3 omega 0 square by 24 omega 0 cube into a cube t, so a cube into t into sin omega 0 t plus beta. So, as this time term is multiplied with this term, so with increase in time or as t tends to infinity, so one can get the response to be infinity.

But, in actual case, we have observed that for minimum potential, so the response of this type of system will be periodic solution or the response or the solution is that of a center. So, in this case, the solution should be periodic. But by doing this straight forward expansion method, we are getting a solution, which shows that it will increase with time or the system becomes unstable as the time progresses. So, the solution what we obtained is not correct. So, one has to modify this procedure to correct the solution or to get a correct solution. So, for this purpose, Lindstedt and Poincare, so they have proposed a method.

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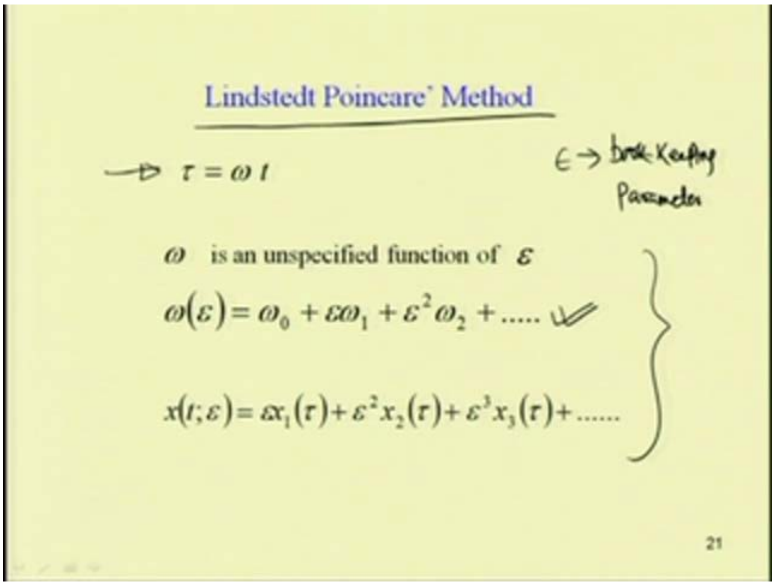

So, in that case, one can take, so this one, in the first step, one can take this time, non dimensional time tau equal to omega t and where omega is an unspecified function of epsilon and that omega can be written in this form. So, omega epsilon equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2. Like previous case, here also one can take x t epsilon equal to epsilon x 1 tau plus epsilon square x 2 tau plus epsilon cube x 3 tau plus. So, one can take as many term as possible. But it is difficult to take higher order terms. So, for that purpose, one can write a symbolic code to find the solution. So, in this Lindstedt Poincare method, the first stage step is to convert this equation by using this non dimensional time tau equal to omega t. Then substitute; write this omega in terms of epsilon. So, that is where epsilon is a book keeping parameter. So, using this book keeping parameter, so omega epsilon is written as omega 0 plus epsilon omega 1 plus epsilon square omega 2 and this. So, this is the basics of this Lindstedt Poincare method. By substituting this equation and proceeding in the previous way, one can find the solution of the system.

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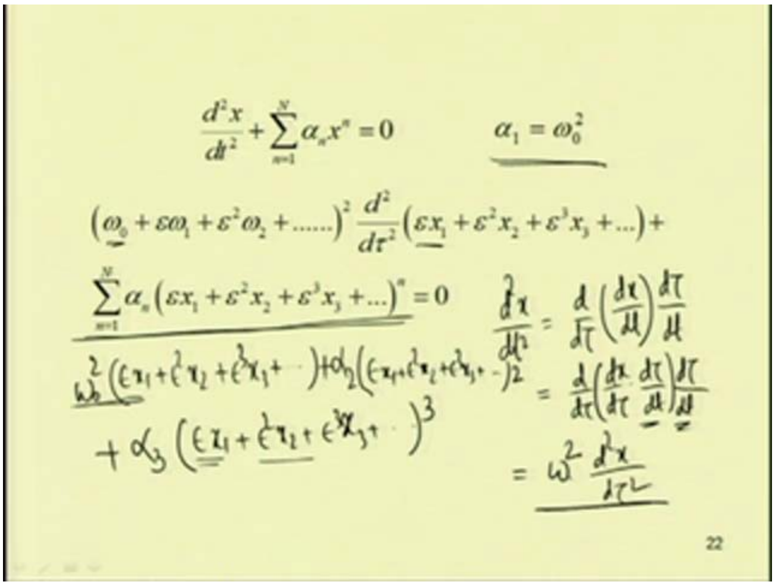

So, for example, in this case, so if we have the equation d square x by d t square plus alpha 1, so plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube, let us take up to 3 terms equal to 0. So, where one can write this alpha 1 equal to omega 0 square and then substituting this equation, so substituting the expression for omega epsilon and x epsilon in this equation, so one can write this equation. So, if you differentiate this, so d square x by d t square, so one can write this as d by d tau of d x by d t. So, into d tau by d t, so this becomes, so we have to substitute this equal to omega. So, d tau by d t equal to omega. So, one can put this equal to omega and then it can be written as d by d tau. So, inside also one can write this d x by d tau into d tau by d t into d tau by d t.

So, this is omega and this is omega. So, this will become omega square into d square x by d tau square. So, the first term can be, so with respect to this, when one differentiate, so this becomes omega square into d square x by d tau square. So, by substituting that thing, so one can have omega 0 plus epsilon omega 1 plus epsilon square omega 2 whole square into d square by d tau square. So, this is x x equal to epsilon x 1 plus epsilon square x 2 plus epsilon x cube plus higher order terms plus m equal to 1 to n alpha n. So, one can substitute these things, epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 plus higher order terms to the power n equal to 0. So, this term one can expand for n equal to 1. So for example, one can write this part for n equal to 1. So, for n equal to 1, this becomes omega 0 square into epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 and for the second term, so that is alpha 2. So, alpha 2 into, so it will be epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3 plus higher order term whole square plus; let me take up to 3 terms, so then it will be epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3, so to the power cube.

Now, expanding this one, the second one, so this is alpha 2 into, so if I will make it square, then this will be a square plus b square. So, this is a plus b plus c whole square. So, it will be a square plus b square plus c square plus 2 a b plus 2 b c plus 2 c a. So, I will have this epsilon square x 1 square plus epsilon 4 th x 2 square plus epsilon 6^{th} x 3 square plus 2 into epsilon cube into x 1 x 2 plus 1 into epsilon 4 th into x 1 into x 3 plus 2 into epsilon to the power 5 in x 2 into x 3. Similarly, one can expand this x 1 plus. So, epsilon x 1 plus epsilon square x 2 plus epsilon cube x 3. But in this case, if we are limiting our analysis to up to this 3rd order, then we can take only the first term, that is your epsilon x 1 cube. Higher order terms one can neglect because the higher order terms, this term will come, epsilon to the power 6 x to the power 6 x to the power cube and this term will come, x to the power 9 into x to the power 9 x to the x to the power x 3 to the power 3 and similarly, higher order terms will come.

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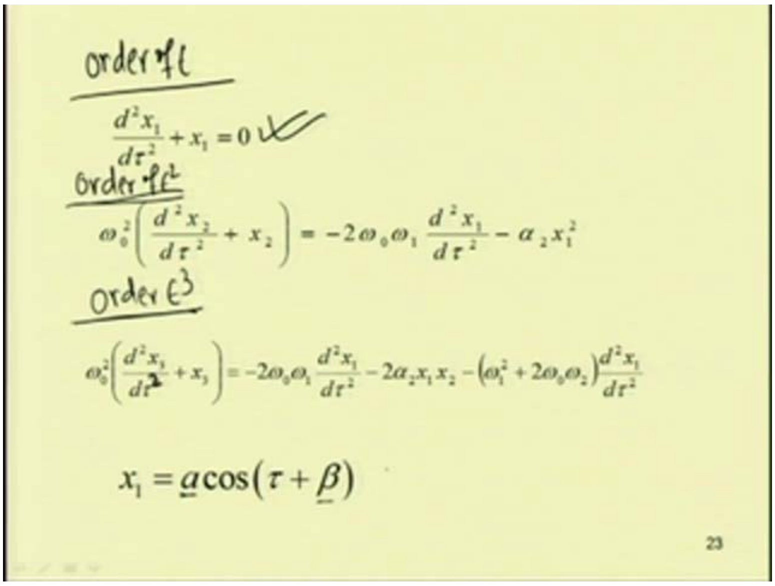

So now, arranging of the order of epsilon, so one can write this. So, this is order of epsilon. For example, for this order of epsilon one can have, so omega 0 square into, so this is d square, this is d square epsilon x 1 by d t square. So, order of epsilon, this first term will come. So, this will become, so taking that omega 0 square out, so here also we have a omega 0 square epsilon x 1. This is the term, first term and this is the first term and this corresponds to the first term. So, one can write this equation in this form, d square x 1 by d tau square plus x 1 equal to 0. Similarly, for the second term order of, so this is order of epsilon square. So, in case of the order of epsilon square, one can write omega 0 square d square x 2 by d tau square plus x 2 will be equal to minus 2 omega 0 omega 1 into d square x 1 by d tau square minus alpha 2 x 1 square.

Similarly, for order of epsilon cube, one can write this omega 0 square into d square x 3 by d tau square. So, this is d tau square plus x 3 equal to minus 2 omega 0 omega 1 d square x 1 by d tau square minus 2 alpha 2 x 1 x 2 minus omega 1 square plus 2 omega 0 omega 3 d square x 1 by d tau square. So, the solution of this equation is well known and one can write the solution of this equation. So, x 1 will be equal to a cos tau plus beta. So, by substituting this, where a and beta are constant, so by substituting this x 1 in this equation, so one can write, so omega 0 square d square x 2 by d tau square plus x 2 will be equal to minus 2 omega 0 x 1. So, differentiating this thing twice, so it will become minus a cos tau plus beta minus alpha 2 and x 1 square. So, x 1 square will give a square cos square tau plus beta.

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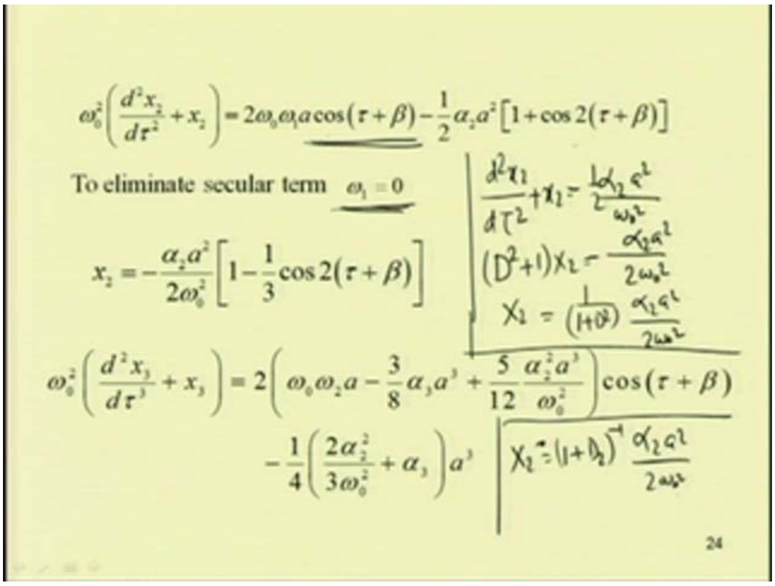

One can expand this cos square term, cos square tau plus beta equal to, so 1 plus cos 2 beta 1 plus cos 2 into tau plus beta by 2. While substituting that thing, one can get this equation. So, omega 0 square d square x 2 by d tau square plus x 2 will be equal to 2 into omega 0 omega 1 into a cos tau plus beta minus half alpha 2 a square into 1 plus cos 2 tau plus beta. Now, one can see that these terms or the coefficient of this, so this frequency equal to 1 and here also frequency equal to 1. So, this will lead to a secular term. Already we know that the secular term should not be present in the response. So, the response is bounded. So, there should not be any secular term. So, this term should be eliminated from this equation. So, to eliminate this term from this equation, which is 2 omega 0 omega 1 a cos tau plus beta. So, as this cos will have a value from minus 1 to plus 1 and this omega 0 is not equal to 0. So, the only remaining term which should be 0 is equal to omega 1. So, from this we got one condition, that this omega 1 equal to 0.

So, while initially we are assuming the solution in Lindstedt Poincare method or we have assumed this omega equal to omega 0 plus epsilon omega 1 plus epsilon square omega 2 plus epsilon cube omega 3, in this case only omega 0 is known to us, so which is equal to or omega 0 square equal to alpha 1. But the other terms we have to find by eliminating the secular terms. So, in this case, we found that this omega 1 equal to 0. So, by eliminating this secular term, so we have found this omega 1 equal to 0. So, the solution, remaining solution of this, that is omega 0 square d square x 2 by d tau square plus x 2 equal to minus half alpha 2 a square into 1 plus cos 2 tau plus beta, so it has 2 components. So, the first component, so you can write or you can have the particular integral. So, for these two components, one component equal to minus half alpha 2 a square and the other component equal to minus half alpha 2 a square into cos 2 tau plus beta.

The solution of the second component can be written or the solution of this first component will be alpha 2 a square by 2 omega 0 square. So, this thing can be obtained in this way. So, the particular integral when I am writing, so this equation can be written d square x 2 by d tau square plus x 2. So, this is equal to half minus half alpha 2 a square by omega 0 square. So, omega 0 square is divided here. So, this is the equation. So, for this, when one has to find the particular solution, so this one can write this thing as; so the auxiliary equation, so one can write this d square plus 1 x 2, so this is the, this is equal to alpha 2 a square by 2 omega 0 square.

Then, this particular solution x 2 can be written as 1 by 1 plus d square into alpha 2 a square by 2 omega. So, this negative sign is there, 2 omega 0 square. Now, by taking this term, now by taking this term to the numerator, so one can write this x 2 equal to 1 plus d 2 to the power minus 1 into alpha 2 a square by 2 omega 0 square with a negative sign. So, expanding this thing 1 plus d 2 to the power minus 1, so it will be, so for this value, so one can write 1 minus d 2 into alpha 2a square by 2 omega 0 square. As this term is constant, so d 2, that is second derivative of this or the derivative of this term equal to 0, so the remaining part equal to minus alpha 2 a square by 2 omega 0 square. So, this is the first term, minus alpha 2 a square by 2 omega 0 square.

For the cos component, so for the cos component, one can, so this is cos 2 2 tau plus beta. So, in this case, one can write this cos 2 tau plus beta, so one can, so as this, one can write d square x by d tau square plus. So, this is the equation; plus x equal to, so we can write this equation minus alpha minus half alpha 2 a square into cos.

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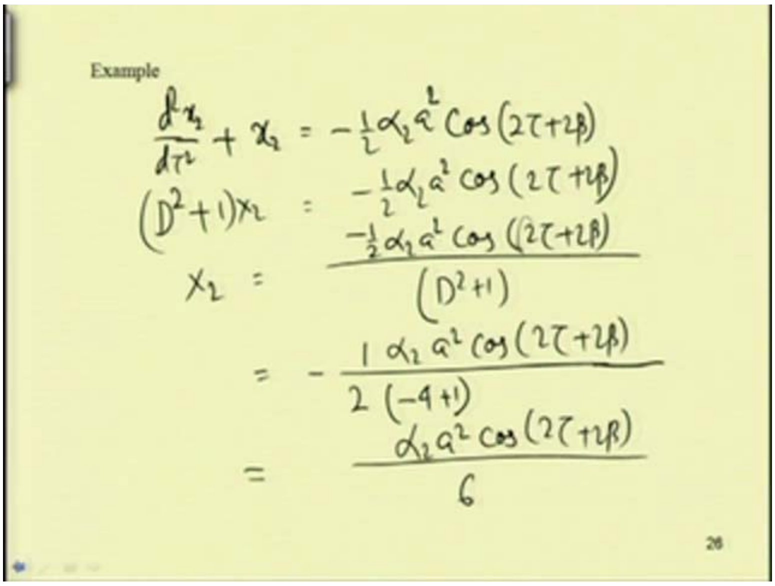

So, we have this term cos 2 tau plus 2 beta. So in this case, the particular integral, for writing the particular integral, I can write d square plus 1 into, so this is x 2, this is x 2, x 2 equal to minus half alpha 2 a square cos 2 tau plus 2 beta. So, this x 2 particular integral, you can write equal to minus half alpha 2 a square cos 2 tau plus 2 beta by d square plus 1.

So, in this case, as it is harmonic, so one can substitute this d square by square of these terms, negative, putting a negative sign, so square of 2, so that is minus 4. So, one can substitute minus 4 in place of this d square. So, the equation becomes minus 1 by 2 into alpha 2 a square cos 2 tau plus 2 beta by. So, here one can put minus 4 plus 1. So, by minus 4 plus 1 and this is minus 3 and minus minus plus. So, this becomes 6. So, this equation becomes alpha 2 a square cos 2 tau plus 2 beta by 6. So, one can add this term to the first term. So, one can get 1 minus, so 3 into 2 6, so minus minus plus. So, alpha 2 a square by 6 omega square cos 2 tau plus beta.

So, after getting this x 2 equal to this and x 1 equal to, after getting x 2 equal to this and x 1 equal to this, that is x 1 equal to a cos tau plus beta and x 2 equal to alpha 2 a square by 2 omega square into 1 minus one-third cos 2 tau plus beta. So, one can substitute these two equations, in this equation order of epsilon cube and one can write this equation and one can find that is the equation in this form, that is omega 0 square into d square x 3 by d tau cube d tau square, so this is d tau square, plus x 3 equal to omega 0 into omega 2 into a minus 3 by 8 alpha 3 into a cube plus 5 by 12 alpha 2 square a cube by omega 0 square into cos tau plus beta minus 1 by 4 2 alpha 2 square by 3 omega 0 square plus alpha 3 into a cube.

So, one can, so in this equation, one can find, so after writing this equation, so one can observe now the coefficient. So, this is cos tau plus beta. So, the frequency is 1 here and here it is 1. So, this will lead to a secular term. So, this will lead to a secular term. So, the coefficient of this should be equal to 0. So, one can substitute the coefficient of this equal to 0 to find or to eliminate the secular term. So, in this case, 2 into omega 0 into omega 2 into a minus 3 by 8 alpha 2 a cube plus 5 by 12 alpha 2 square a cube by omega 0 square equal to 0. Already we know that omega 1 equal to 0 and in this case also, we have to find this term a and beta, which depends on the initial conditions. So, to eliminate this equation, so we have to put it equal to 0 and by substituting this equal to 0, one can write this omega 2 in terms of omega 0.

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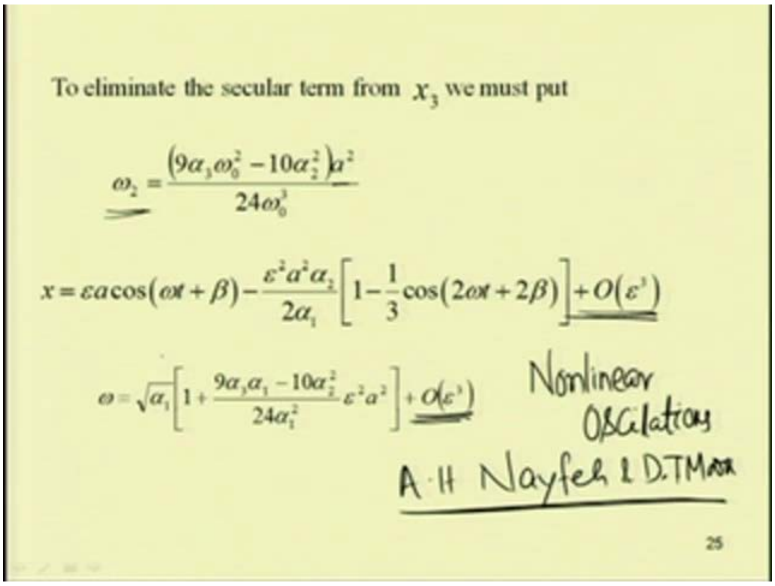

So, one can write omega 2 equal to 9 alpha 3 into omega 0 square minus 10 alpha 2 square by 24 omega 0 cube a square. So, in this case, by eliminating the secular term, that means, by putting this term equal to 0, so one can write this equation in this form, omega 2 equal to 9 alpha 3 omega 0 square minus 10 alpha 2 square into a square by 24 omega 0 cube. So here, one can see that this frequency omega 2 is a function of the response a. Unlike in case of the linear system, we have the frequency of response is independent of the amplitude of the response. So here, the frequency depends on the amplitude of the system. So, as the response amplitude changes, so here the frequency, response frequency also changes.

So, by substituting this x 2 x 1 in the first equation, that is x 0 x t epsilon equal to epsilon x 1 tau plus epsilon square x 2 tau plus epsilon cube x 3 tau, so one can find the expression for x. So, that is equal to epsilon a cos omega t plus beta minus epsilon square a square alpha 2 by 2 alpha 1 into 1 minus 3 cos 2 omega t plus 2 beta plus higher order, where omega equal to root of over alpha 1 into 1 plus 9 alpha 3 into alpha 1 minus 10 alpha 2 square by 24 alpha 1 square into epsilon square a square with higher order terms. One can neglect the higher order term. By neglecting this higher order term, so one can have this expression. So, this analysis is followed from the book non-linear oscillations by Nayfeh and Mook; Non-linear oscillations by A.H.Nayfeh and D.T.Mook. So, there are a number of variations. So in this case, it is assumed that we are taking small parameters as the coefficient of the non-linear terms.

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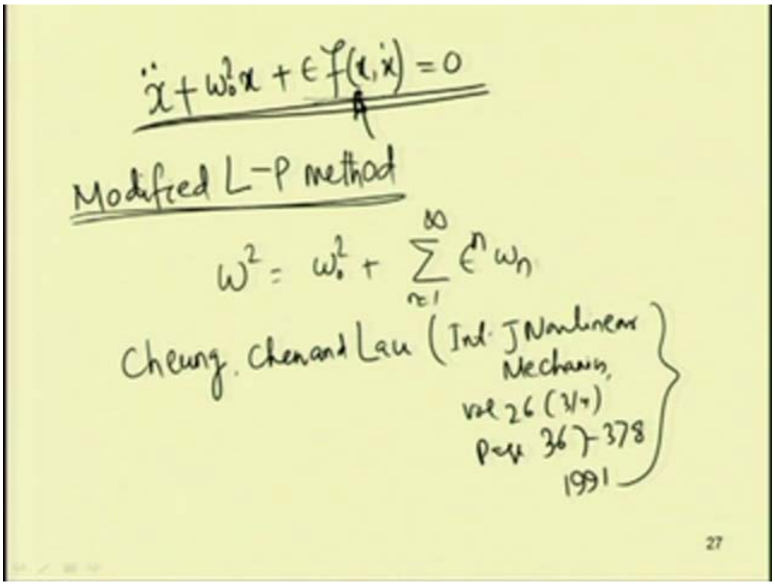

But, for large parameter variation or when the equation is highly non-linear, for example, if one has a system with very high non-linearity, that means, one can write this equation plus x double dot plus omega 0 square x plus epsilon x. So, this is a function of epsilon x and x cube x dash or x dot equal to 0. So, if this function epsilon f x x dot, which is a non-linear term is small parametory content, small parameters, then one can use this Lindstedt Poincare type of view solution method. But when this is not or the coefficients are not small parameters, that time one has to go for some other methods to find the solution or one can have the numerical solutions to find the solution of the equation.

So, there are a number of modifications to this Lindstedt Poincare method. So, one has this modified Lindstedt Poincare method for large non-linear systems. Particularly for large non-linear systems, this equation has been modified or this method has been modified. Mainly, in this modification, instead of taking this omega as a function of omega 0 plus epsilon, so one has to write this equation in terms of omega square. So, in case of the modified Lindstedt method, so one writes these equations in terms of omega square instead of omega. So, this expansion, one can write omega square equal to omega 0 square plus n equal to 1 to infinity epsilon n omega n. Also, some authors, they have introduced additional parameters like alpha, which is a function of epsilon. So for example, Cheung et al, Cheung, Chen and Lau, so this is in international journal of non-linear mechanics. So, this is volume 26 issue 3 by 4 page 367 to 378, 1991.

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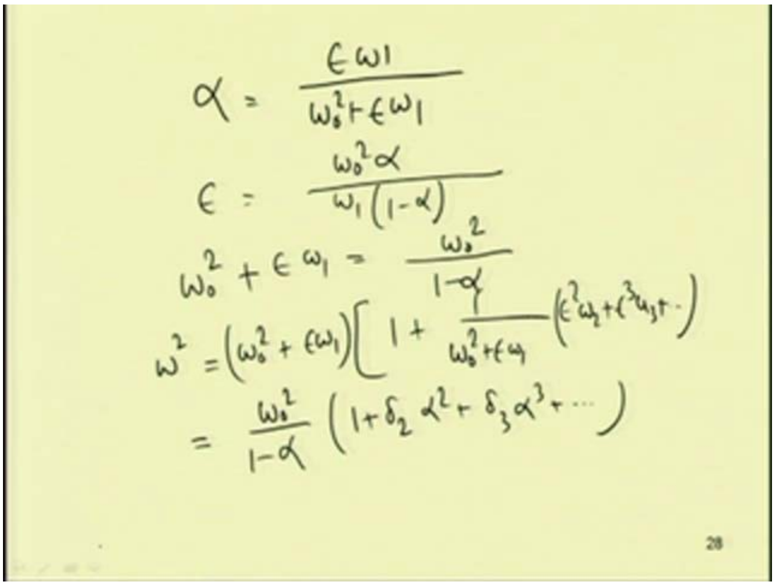

So, they have proposed a modified method, in which they have taken this omega square equal to omega 0 square plus n equal to 1 to infinity epsilon to the power n omega n. Also they have introduced some additional parameters like alpha equal to epsilon omega 1 by omega 0 square plus epsilon omega 1. Epsilon they have written in the form, that is omega 0 square alpha by omega 1 into 1 minus alpha and they have substituted this omega 0 square plus epsilon omega 1. So, they have introduced two parameters. One is alpha and other one is omega 1 and omega 0 square plus epsilon omega 1. This is equal to omega 0 square by 1minus alpha. So, one can write this omega square equal to omega 0 square plus epsilon omega 1 into 1 plus 1 by omega 0 square plus epsilon omega 1 into epsilon square omega 2 plus epsilon cube omega 3. So, this thing also they have written in this form, omega 0 square by 1minus alpha into 1 plus delta 2 alpha square plus delta 3 alpha cube.

Now, substituting this omega square and this x term in the original equation and separating the order of epsilon, one can get a set of equations, from where one can eliminate these secular terms to find the response of the system. So, this way one can modify this Lindstedt Poincare method to find the solution or find the response of the system.

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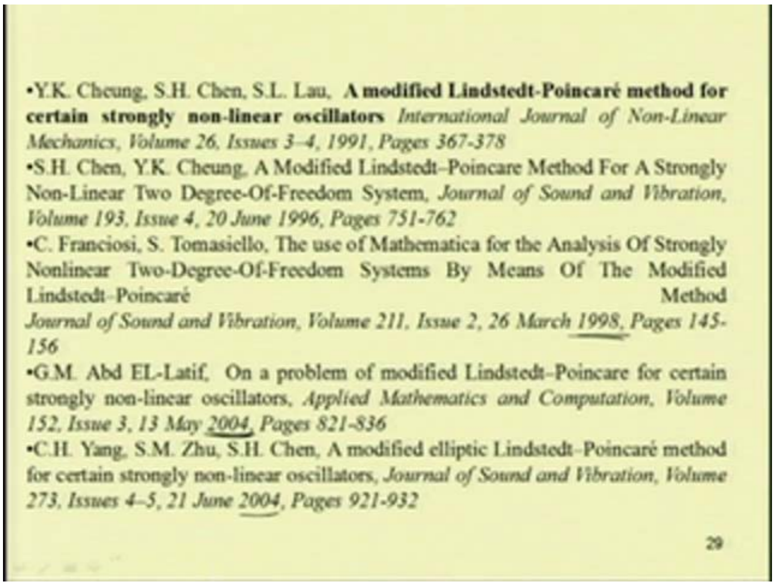

So, for further study, so one can see these papers, a modified Lindstedt Poincare method for certain strongly non-linear oscillation. So, it is in international journal of non-linear mechanics, volume 26 issues 3 4, 1991, 367 to 78 and then other prepared by Chen Cheung, a modified Lindstedt Poincare method for a strongly non-linear two degree of freedom system. So, it is published in journal of Sound and Vibration, volume 193 issue 4 and in 1996, page 751 to 762. Then one can use this paper also by Franciosi and Tomasiello, the use of mathematica for the analysis of strongly non-linear two degree of freedom system by means of the modified Lindstedt Poincare technique. So, this is published in journal of Sound and Vibration, issue volume 211, in year 1998.

Also one can study this thing on a problem of modified Lindstedt Poincare method of certain strongly non-linear system, which is published in applied mathematics and computation in 2004. Also a modified elliptic Lindstedt Poincare method for certain strongly non-linear oscillated, which is published in journal of Sound and Vibration in 2004.

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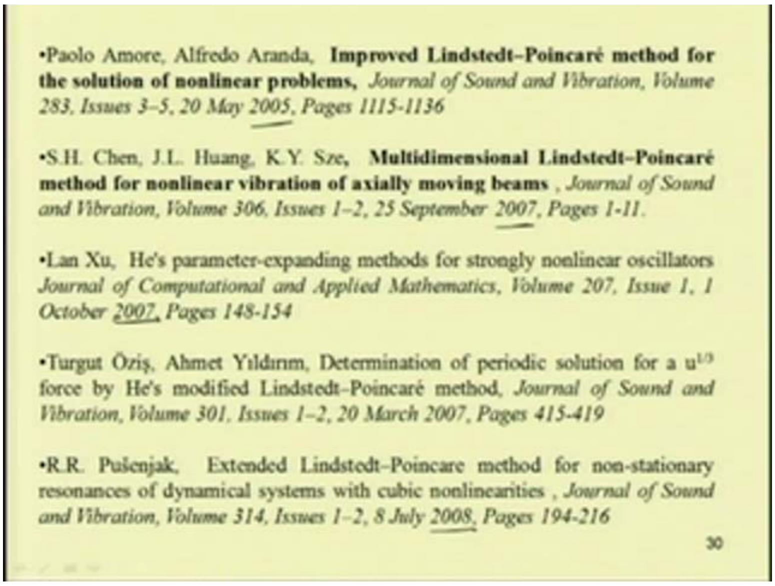

Some recent papers are also included here by Amore and Aranda, improved Lindstedt Poincare method for the solution of non-linear problems, which is published in journal of Sound and Vibration in 2005 and Chen, Huang and Sze, multidimensional Lindstedt Poincare method for non-linear vibration of axially moving beams, in 2007 it is published. Another method, so he proposed also a method for strongly non-linear system. So, one can see this paper by Xu, his parameter expanding method for strongly non-linear oscillator, so which is published in journal of computational and applied mathematics in 2007. Other papers also one can see determination of periodic solution for a, u to the power one-third u to the power 1 by 3 force by He’s method, He’s modified Linstedt Poincare method.

Another one also, extended Linstedt Poincare method for non stationary resonances by dynamical systems with cubic non-linearities, which is published in journal of Sound and Vibration in 2008. So, one can see all these recent publications for modified Linstedt Poincare technique. So, there are also some other papers where the comparison of the different methods of Linstedt Poincare technique has been carried out. So, this is a letter to editor in 2004, comparison of two Linstedt Poincare methods by Who and Cheung, which is published in journal of Sound and Vibration, volume 278 page 437 to 444. So, in that case, two different methods have been compared.

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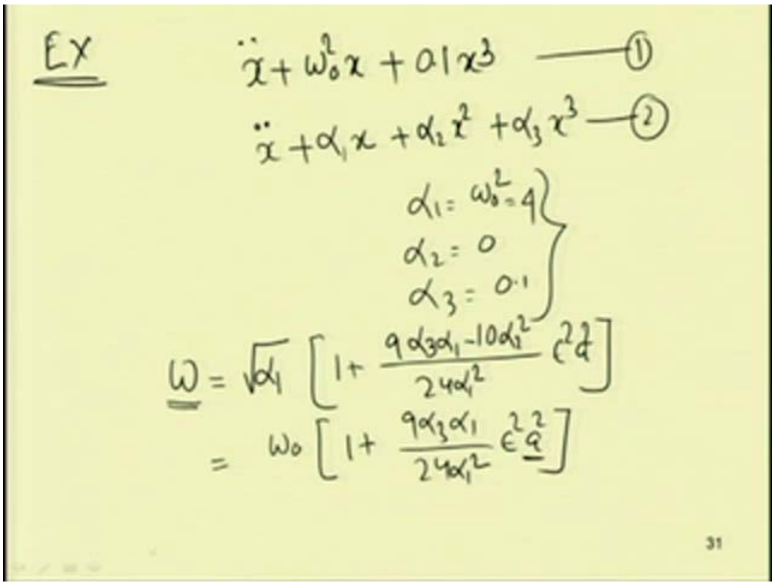

For example, one can take one example and one can solve this equation by using Linstedt Poincare technique. Let us take this example x double dot plus omega 0 square x plus, let me take this is equal to 0.1 x cube. So, in this case, one can find the solution and in this case, first one can compare this equation with our original equation, our original equation what we have solved. So, that is x double dot plus alpha 1 x plus alpha 2 x square plus alpha 3 x cube. So, by comparing this equation 1 with equation 2, so one can write this alpha 1 equal to omega 0 square and alpha 2 equal to 0 and alpha 3 equal to 0.1. So, by taking this thing, so one can find the and so by substituting this thing in our ordinary Linstedt Poincare technique method, so one can write the solution. So, the solution can be written in this form or first, one can write the frequency. So, frequency equal to root over alpha 1 into 1 plus 9 alpha 3 into alpha 1 minus 10 alpha 2 square by 24 alpha 1 square into epsilon square a square. So, in this case, as alpha 2 equal to 0, so this becomes root over alpha 1.

That means, omega 0. So, omega 0 into 1 plus 9 alpha 3 alpha 1 by 24 alpha 1 square into epsilon square a square. So, one can see this omega, the frequency of the system is related to the amplitude and one can plot this omega versus a. So, by plotting this thing, one can find, so one can write a small code for finding this omega with respect to a. Just bury a and find different value of omega. So, one can find this thing. So, one can see that when this is equal to a equal to 0, this omega equal to omega 0. So, when a equal to 0, omega equal to omega 0.

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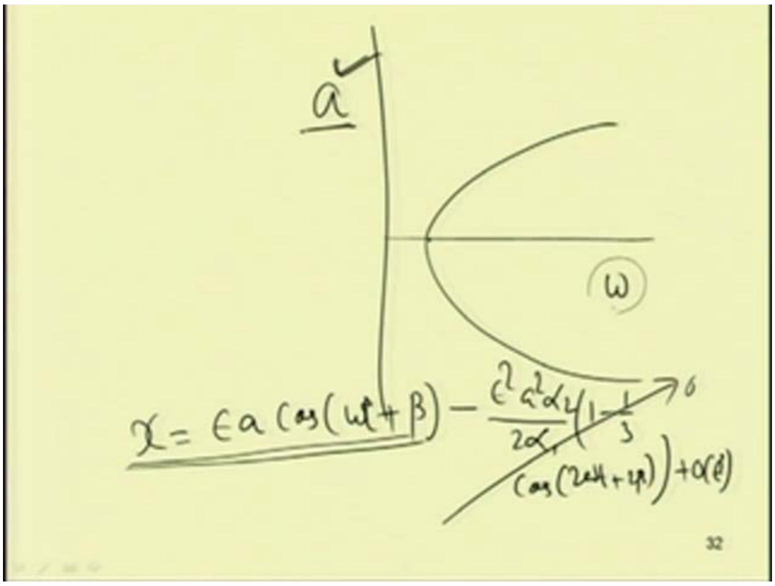

In this case, if I will take, let me take this equation omega 0 square equal to 4, so then it will start from 2. So, if I will bury this omega, so the response for, will be, so if I plot this response curve, so the response a versus omega will be like this. So, one can find with increase in a, the response or with increase in omega, the response amplitude increases. So, after taking this value of a, so one can find the solution x. So, x equal to, the expression for x equal to, one can write epsilon a cos omega t plus beta minus epsilon square a square alpha 2 by 2 alpha 1 into 1 minus one-third cos 2 omega t plus 2 beta plus higher order terms, plus order of epsilon cube. So as alpha 2 equal to 0, so this part becomes 0 and one has a solution x equal to epsilon a cos omega t plus beta. So, by taking some initial parameter, initial knowing, initial displacement and velocity, so one can find this parameter and one can see this a is burying with omega and one can find the response of the system.

So, in this way, by using Linstedt Poincare technique, one can find the response of the system. But the method what we know now is, depends on whether the system is strongly or weakly non-linear system. So, if it is weakly non-linear system, one can use this Linstedt Poincare method. But if it is strongly non-linear, one can go for the modified Linstedt Poincare method, which I told very briefly. In the next class, I may tell you about this modified Linstedt Poincare method and another method, and that is method of multiple scales. Thank you.