Ideal Fluid Flows Using Complex Analysis

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Lecture 9

Superposition of source and uniform flow

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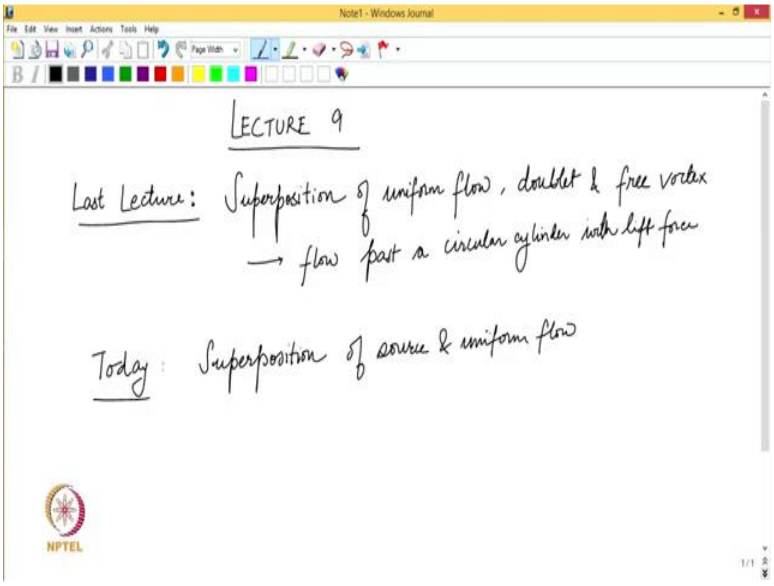

So, in the previous lecture we looked at the superposition of a uniform flow with the doublet and free vortex and what we saw was that this superposition gives us situation or scenario of flow past a circular cylinder which also brings in with itself a certain value of the left force, something which we have not quantified yet but we will do it very soon.

Now, in today's lecture I would look at another superposition problem but this time we look at superposition of a source and a uniform flow and this is quite an interesting case and also something which helps to understand how superposition typically works in recreating complex flows.

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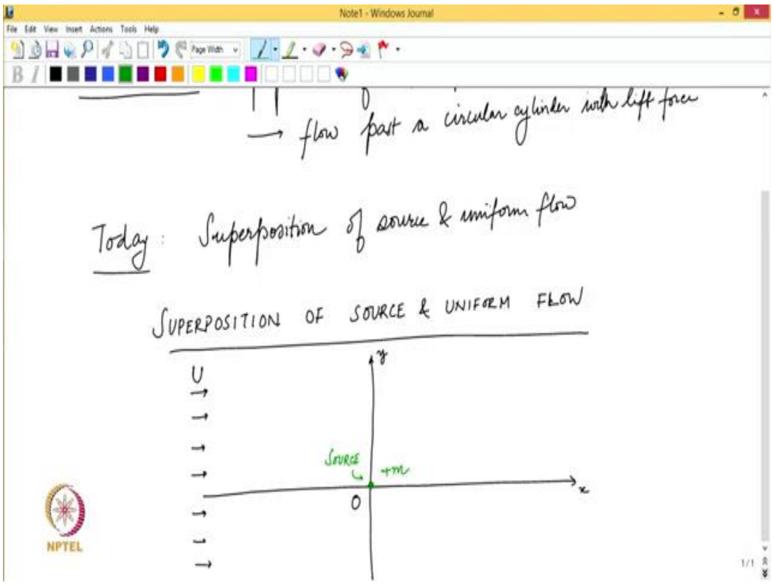

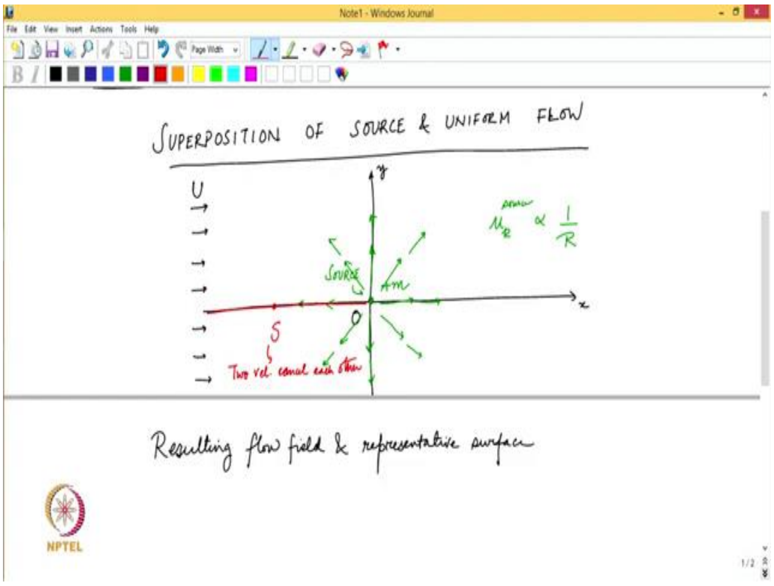

So, let us look at this situation we will start with this idea, so we look at superposition of a source, single source and a uniform flow. Now, let me first draw the schematic of what this problem would look like and then we will discuss this before we go on to actual solution. So, what I am saying is in this case consider that we have a uniform flow which has a velocity U and it is aligned along the x axis where this is x this is y and we also have a source which is located at the origin.

So, say this is a source let me represent the source by a different colour, so we have a source here which is of strength plus m, this is source which is located right at the origin. The question that I am asking now is what is the resulting flow field and also the representative surface over which the resulting flow pattern can be expected, very similar to what we did for superposition of uniform flow doublet and free vertex we saw that the resulting shape representative surface is actually a circular cylinder.

So, now recall what the definition of a source is, source is something which is giving out fluid readily in all directions. So, I could say that this source is one here which is giving out fluid in all directions and let me just draw a few arrows just to show that it is flow coming out of that location moving in all directions.

And as we go far away or further away I would say we can expect that the velocities will be decreasing where the, so the length of the arrows that I am drawing are representing the decrease in the velocity. Now, notice that we have a uniform flow coming in from the left which is going from left to right and we have a source at the origin which is radiating flow in all directions.

Since the source flow field which I am going to write as u R let us say source, this this is supposed to be inversely proportional to distance from the origin this we have already derived but just to reiterate this is what the velocity field of a source looks like and of course there is no angular component of velocity u theta is 0 for a source.

We can expect that along this line which let us draw by a red colour here along this line there would be some point where the incoming flow velocity will get equal to the source velocity such that the two components add to 0 at a given location. So, let me say that is a point S for now this is a point where the two components where the two contributions basically we have contribution from the uniform flow and we have contribution from the source these two become equal and opposite.

So, at this point S we can say the two velocities cancel each other, so two velocities cancel each other. So, this is an important point we will come back to this but for now I think what I am trying to do is to give you some physical insight on what lies ahead. As we will see this point S will be the stagnation point for this scenario.

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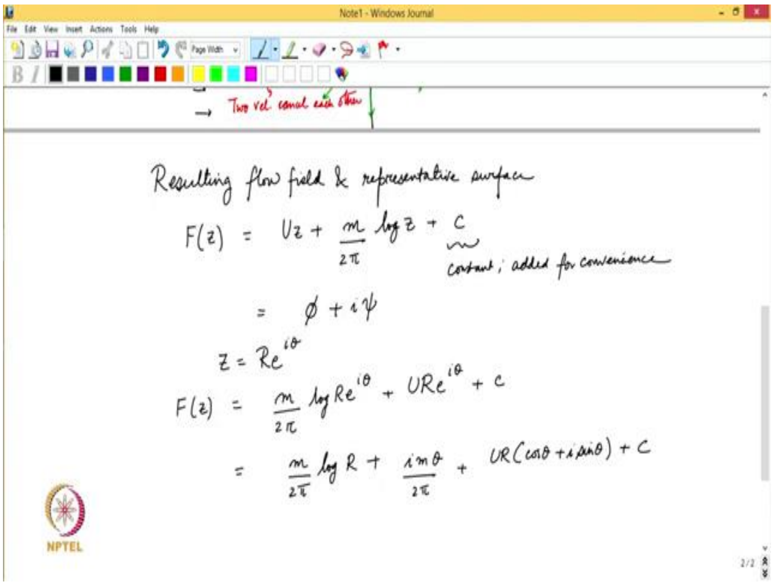

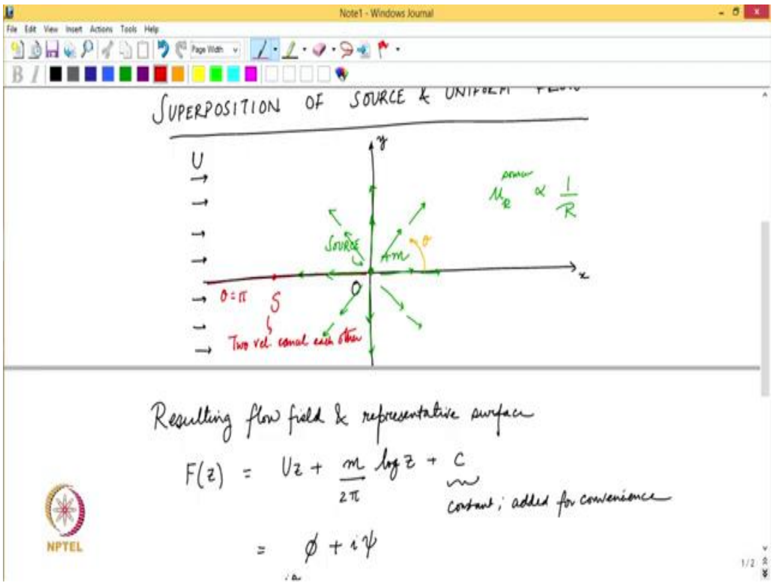

So, let us go ahead let us start our analytical analysis and so let us write the complex potential for this flow which is F of Z which will be U times Z plus m by 2 pi log of Z and let me add a constant C, this constant I am adding, this is a constant which I am adding just for convenience and I will show you what that convenience is very soon.

So, if you go to the next step which is basically we want to say this is going to be the velocity potential plus i times stream function I need to simplify my complex potential. So, let us write Z to be R times e i theta again in polar coordinates, so we will get F of Z to be m by 2 pi log of R e i theta plus U R e i theta plus C which I am sure now that you have been listening to all these lectures you now know how I proceed from here.

We will write this as m by 2 pi log R plus i m theta by 2 pi plus we can write this as U R cosine theta plus iota sin theta plus this number C. Now, as I said just based on pure intuition that there would be a point along the negative x axis where the velocities of the source and the uniform flow will cancel each other that I said is point S.

In terms of polar coordinates note that this location along the negative x axis corresponds to theta equal to pi. So, if you measure theta we typically measure theta from the positive x axis so we measure theta from this direction. So, this line that I have written here the red line is actually corresponds to theta of pi. So, let us say we want to know the particular value of stream function when theta is pi.

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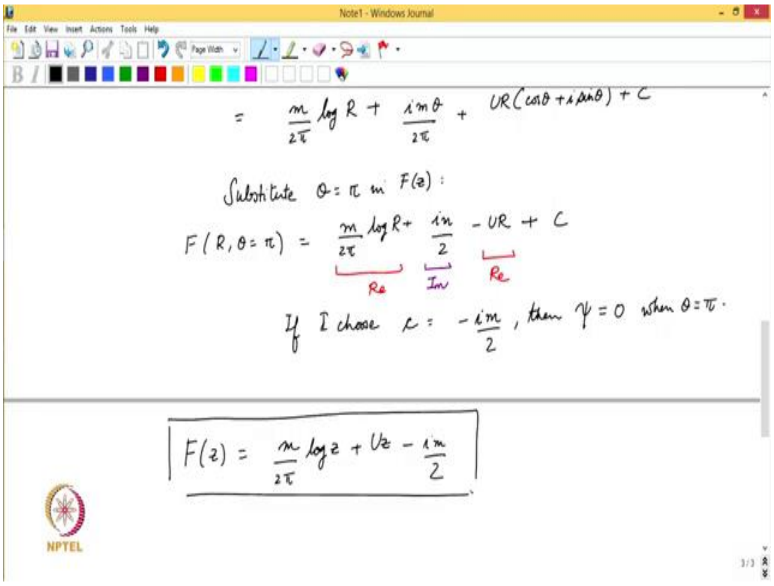

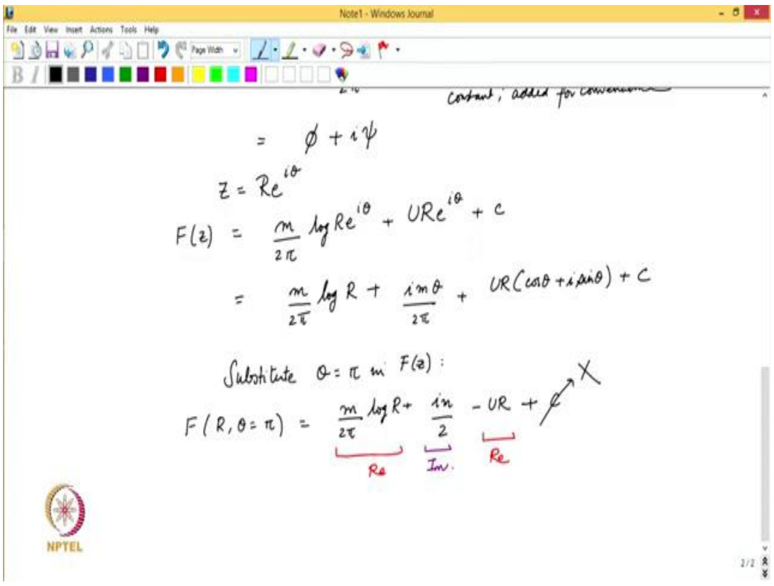

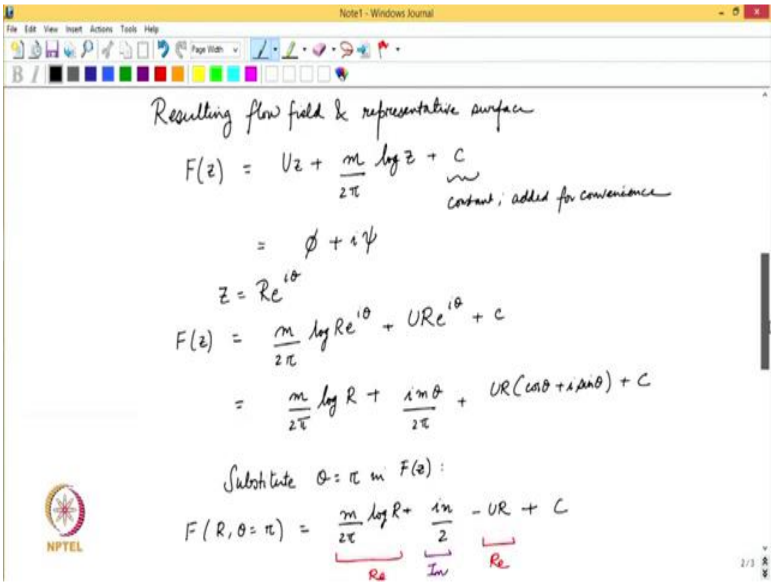

So, let us look at what happens if we substitute theta equal to pi in F of Z but the distance is any general distance. So, we could write F maybe now I can say this is R but theta equal to pi to be m by 2 pi will have log of R plus now when you put theta to be pi we will get i m by 2 plus note that when I put theta to be pi I will get cosine theta to be minus 1.

So, we will write here this as minus U R and sin theta or sin pi will actually be 0 so I would I do not get anything from there plus C. More importantly note that this is the real part this and let me say real R e means real this is the real part and this part is the imaginary part and we have not said anything about C.

But I know that the real part of this expression should be equal to phi the imaginary part should be equal to psi. As it is if there was no C, I can easily see that on theta equal to pi psi cannot be 0 if C was not there, if C was for instance not there. But since C is there the convenience that I want to wanted to show you is this that if I could choose, if I choose C to be minus i m by 2.

Then I can say psi will be 0 on theta equal to pi and then we can see that this will become sort of the genesis for a surface, we will get a surface across which there would be no fluid flow so complex potential in this case F of Z will finally be m by 2 pi log of Z plus U Z minus i m by 2 which is basically if I go back to the original equation this is what we had written U Z plus m by 2 pi log Z or log Z minus i m by 2, this is the complex potential that we would utilize.

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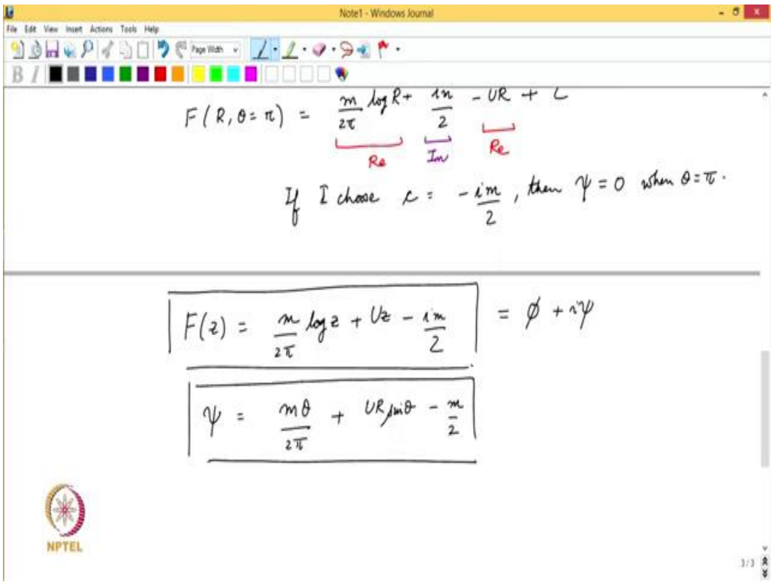

So, with this definition now what do we have is psi the stream function we have psi to be m theta by 2 pi plus note that now we will have a U R multiplied by sin theta so we will have plus U R sin theta and of course minus m by 2 this is because F of Z could be written as phi plus i psi. So, this would be our stream function for this situation. Now, what about the complex velocity?

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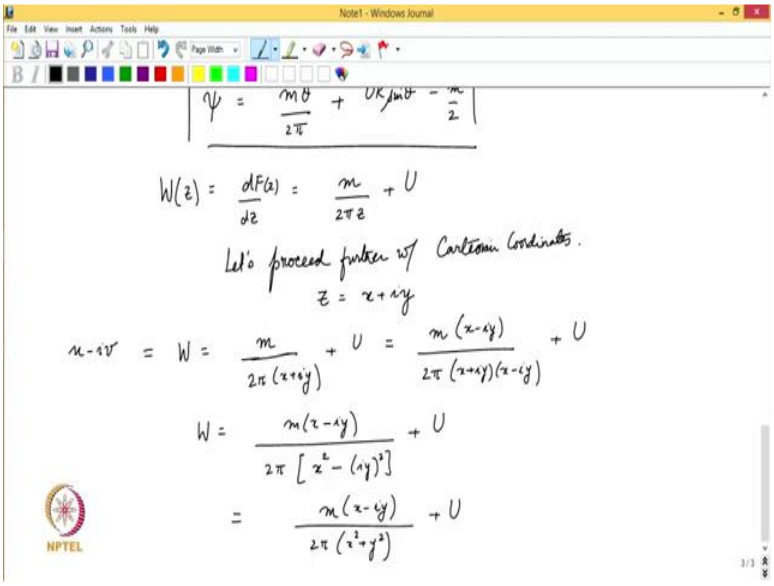

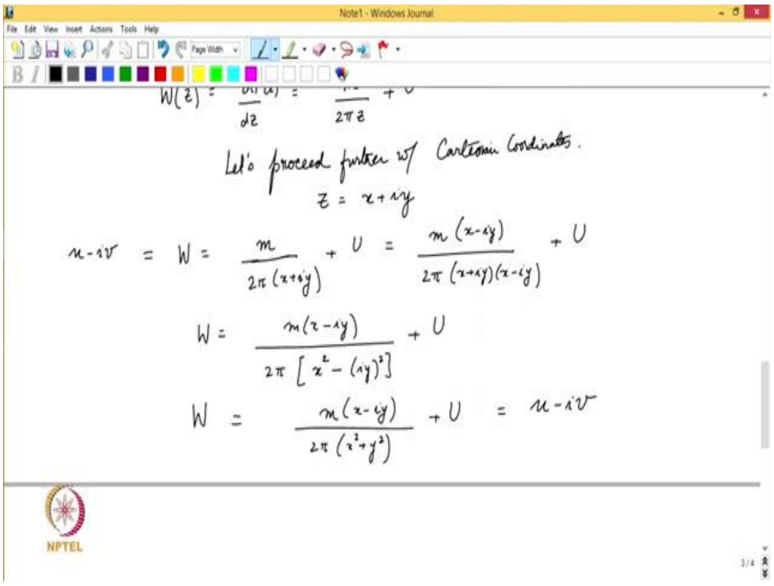

So, let us say we want to calculate W of Z which would be dF dz, so this would be m by 2 pi Z plus U anyway minus i m by 2 is a constant so its derivative with respect to Z is 0. Now, for a change I have been doing all the analysis so far using polar form of coordinates let us choose in this case Cartesian coordinates and then what I will also do is I will sort of mix and match you will see it becomes sometimes convenient to keep both coordinate systems in mind.

For now let us proceed further with Cartesian coordinates which means I would represent Z as x plus i y when you will see that it has its actually it also comes with a lot of experience but you will see that some coordinates as I have been saying some form of coordinates become more convenient to be applied in certain problem than the other type become more adaptable for some other problems, we have been looking mostly at flows which have some radial or some angular feature which enabled us to write them in polar form, but now in this case we can also write it in using Cartesian coordinates.

So, we get W to be m by 2 pi x so m by 2 pi x plus i y plus capital U and because I want to write remember I eventually I want to write W as U minus i v this is the complex velocity in Cartesian coordinates. So, I clearly want the iota to be in the numerator not in the denominator here. Here in this expression everything is a real number except what I see as a complex number in the denominator.

So, the logical thing to do is as we do in high school when we are dealing with complex numbers is to multiply the numerator denominator by the complex conjugator, so we will just say this is 2 plus, 2 pi x plus i y times x minus i y plus U. So, W would then become x, rather than m x minus i y times 2 pi.

Now, x plus i y times x minus i y is like a plus b n times a minus b so that becomes a square minus b square so here we can write, we can write this as x square minus iota y square plus u and iota square is minus 1 so this becomes m x minus i y 2 pi times x square plus y square plus u.

So, hopefully you are with me at till this point that you follow what I have done so far, this is the complex velocity which now it is in a form where I can separate out the real and imaginary parts of this complex number. So, the real part would be the x component of velocity the imaginary part will be the y component of velocity.

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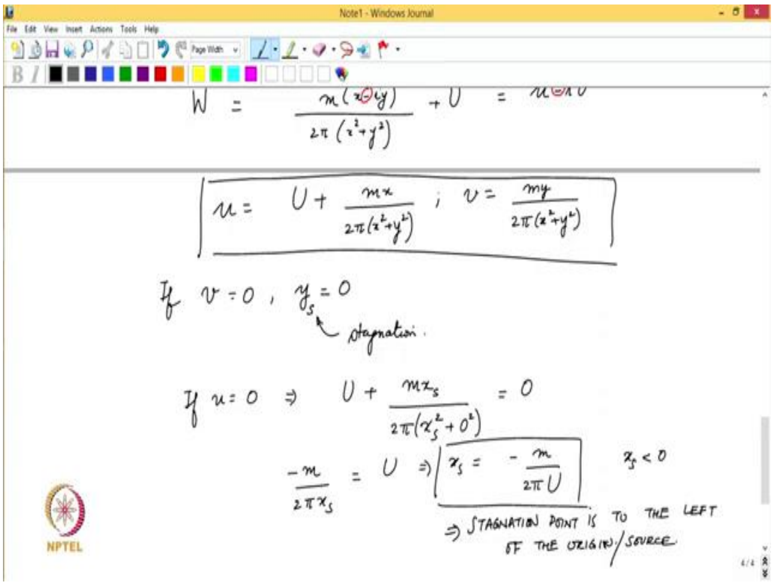

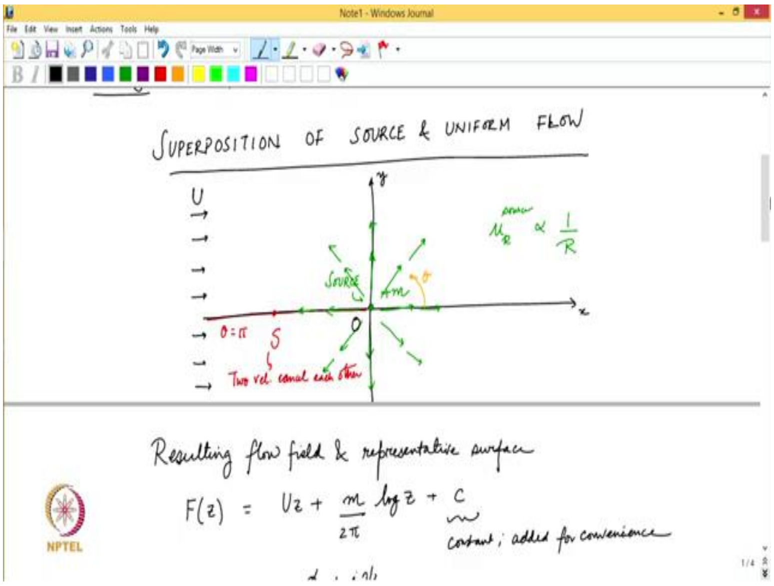

So, we can write u, small u now to be U plus m x by 2 pi x square plus y square and v to be m y times 2 pi x square plus y square. Note that there is a negative sign here and there is a negative sign here and that is the reason the y component does not have a negative sign. So, it is, it just happens to be in this form. So, we have the two components of velocity and now we have a lot more information about this flow than we had earlier.

So, now let us look at the coordinates where the stagnation point for this flow will be located. So, for the stagnation flow or the flow to reach stagnation both components of velocity should be 0 let us begin with the y component. So, let us say when is v 0, if v is 0 clearly y has to be 0. because the denominator cannot be 0 I mean it is if the denominator goes to 0 v will not be 0, so the numerator has to be 0 where y has to be 0 let us put this as y S, S the subscript S indicating it is a stagnation point or its the condition where we will reach stagnation.

So, y S is 0 so clearly the stagnation point must be somewhere along the x axis and that supports our intuition from the sketch here that the stagnation point must be somewhere along this x axis. Now when is the x component of velocity 0? So, here we will use the fact that when x component of velocity is 0 let us say it is at location x S, we should also have for stagnation the y component of velocity should also be 0 and that happens when y S is 0.

So, the denominator would just be x S square so you can maybe think of it this way that I have this as the equation y S is 0 at the stagnation point so we do not get the any contribution from the y which is in the denominator. So, we can write that minus m by 2 pi x S is U which implies that x S is minus m by 2 pi U, that is the coordinate of the stagnation point along the x axis.

Now, this is a very important result for a few reasons, one is note that we have a source, we are working with the source and uniform flow, a uniform flow which is going from left to right, so U is positive, it is a source so m must be positive. So, clearly x S is a number which is less than 0 because everything else is positive there is a negative sign in the result which means that the stagnation point is to the left of the origin because it is a point which has a negative coordinate so it should be the left of the origin or we can say to the left of the source because remember the source is at the origin. So, that is the reason now we see again going back to the original sketch that the stagnation point is to the left of the source that intuition now is helping us analyse and even sort of justify our result.

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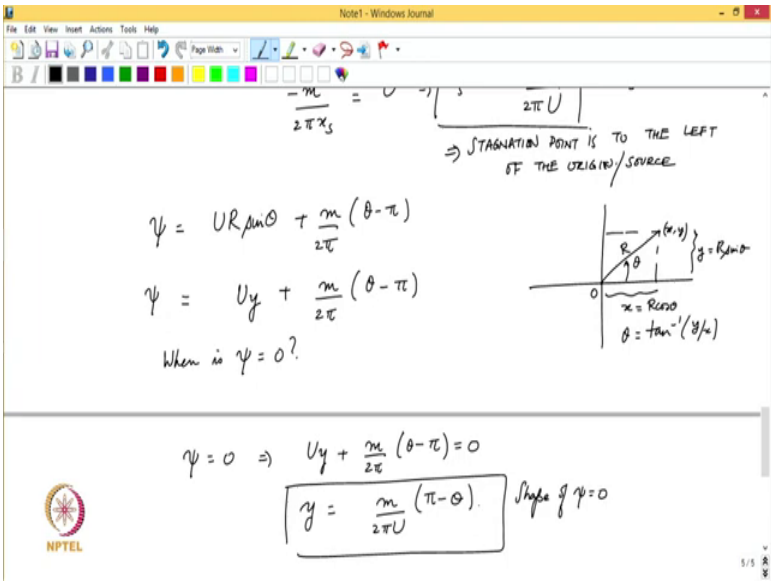

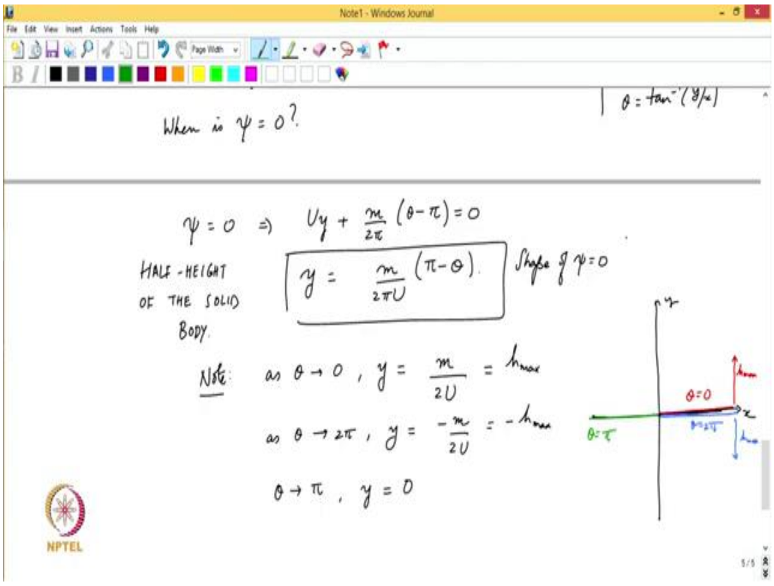

Now let us go back to psi, what was psi we said psi was u R sin theta plus m by 2 pi times theta minus pi. Now, let me do a little bit of mix and match in terms of our coordinates. Well I know that in terms of my coordinates if I have any point let us say x comma y this is say a point which is has coordinates x comma y I know that this distance from the origin is R, this angle is theta. So, clearly this distance which is x must be R cosine theta and this distance which is y must be R sin theta that is how we sort of come up with the coordinates.

So, what I am going to do is I am going to write psi as U times y noting that R sin theta is y plus m by 2 pi but instead of writing theta, theta could be written as, well it could be written as tan inverse y by x but I do not want to use an equation which I cannot solve but I would rather look at solving this equation by just pure observation.

So, I am not going to replace theta, I am just going to keep theta as it is in this equation. Now consider that I want to find this is my psi consider that I want to find psi when it is 0. So, when is psi 0 let us say this is the question that I want to answer. Well for psi to be 0 we should have u y plus m by 2 pi theta minus pi to be 0 which I would write as y to be m by 2 pi U pi minus theta.

So, if there is this relationship between y and theta or you could say y and x whichever way you want to say because remember theta is a function of y and x. If there is this relationship that is satisfied then we can say that psi is going to be 0. In a sense this gives us the shape of this curve more importantly it also tells me a lot about what the shape would be.

So, I am going to call this y what you see here as this expression as the half height of the solid body which this curve represents. Let me show you why I say this is the half height. So, let us look at some limiting cases say that as theta approaches 0 or if theta is made very close to 0 or very close to 0 we will have y to be m by 2u when theta goes to 0 will have pi n pi cancelling out from the numerator denominator so y would be m by 2 U. And this let me say is h max.

Now, note that as theta goes to 2 pi, y will now be minus m by 2 U which is minus h max. So, what it is saying is that when you go let us say very far away from the origin along theta 0 line then you get a certain height of this body and if you go from along theta equal to 2 pi line, along theta 2 pi line would mean that if this is our coordinate system, this is the x axis, this is y, theta equal to 0 limiting line is something like this which is right above the x axis and maybe theta equal to 2 pi line if I want to draw it would be right below, this is theta 2 pi, so this is theta 2 pi and we have theta 0.

So, we are if you go far away then we reach sort of like an height which is h max on this body, same thing happens here, we reach the same height above the x axis and below the x axis. And more importantly when theta goes to pi we get y to be 0, theta equal to pi is actually this line, this is theta equal to pi. So, using these facts we can now recreate the shape of this object.

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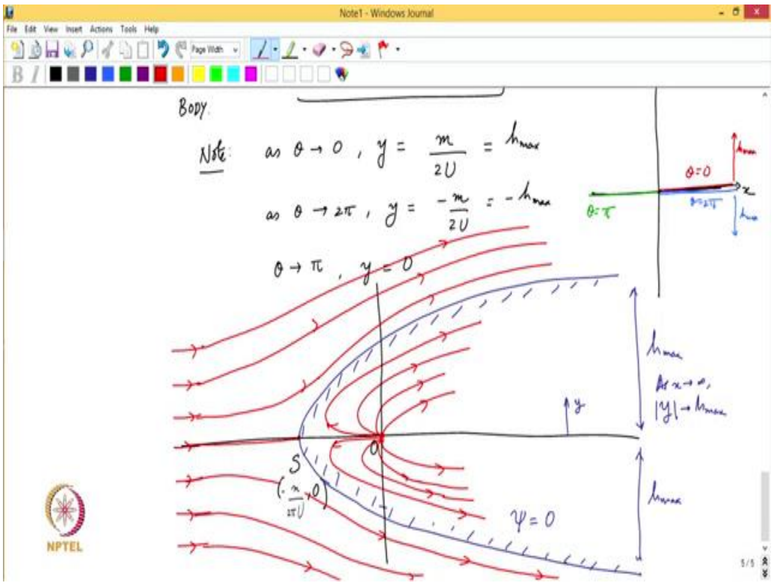

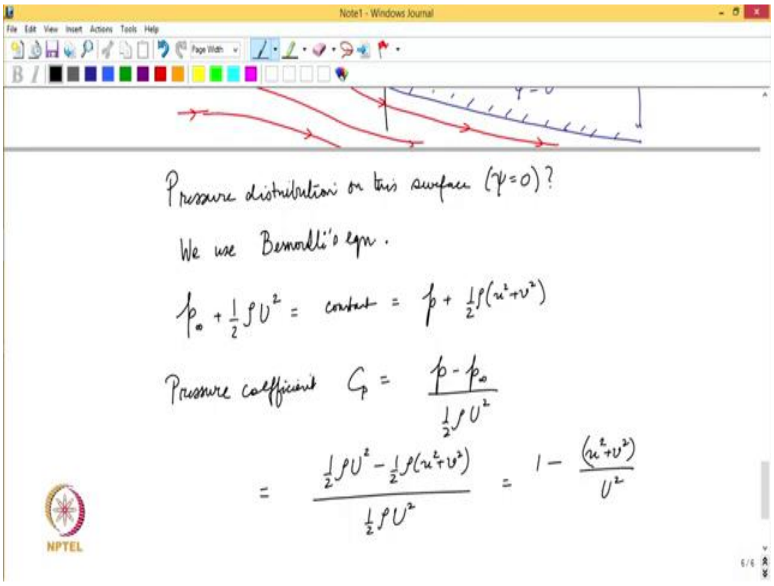

So, let us try and recreate what do we get. So, this is the origin and we have well as I said the stagnation point lies to the left of the origin so let us say the stagnation point lies somewhere here this is S whose coordinates are minus m by 2 pi U, comma 0. So, the psi 0 surface happens to be a body which would look like or a curve it would look like this, it should actually be symmetric about the x in, about the x equal to 0 axis but my sketch maybe did not do justice to it so let me try again something of that type.

And either way this would be the maximum height which happens when you go far away from the origin. So, as x goes to infinity we will have y or modulus y go to h max where y is a distance we are measuring from say the x equal to 0 axis. So, this is the psi 0 streamline, this is a stream line that corresponds to psi equal to 0.

Now, what about the flow around this object, around this solid object? So, we could draw this velocity field remember it is uniform far away so we must be having some flow coming in this fashion but as it gets closer remember it has to change it co, change its course. So, the dividing streamline if I want to put it this way it comes out straight at the stagnation point the rest of it you will see would go around something of this type, this should be the flow pattern that we would see around this object.

More importantly this is outside the free, of course outside this psi 0 streamline. What happens inside? Remember there is a source at the origin. So, internally the flow pattern is also something which is very interesting what we would see is a flow pattern which would look like the following. So, we will have a flow pattern of this type.

Now, let us take this even further so far, I have only considered recreating the shapes of the bodies. Now, let me elevate the problem a little bit and let us bring some fluid mechanics into it. Let me now say that I want to determine the pressure distribution over this body. So, from a fluid mechanics perspective I want to know what is the pressure distribution on the surface which is given by psi equal to 0, the surface corresponding to psi equal to 0.

Now, for this since we are dealing with irritation flows we can use the Bernoulli's equation and we will use the Bernoulli's equation in this fashion that we know that far upstream the velocity is U let us say the pressure there is p infinity far far away from the origin. So, I can apply the Bernoulli equation from far upstream to near the body.

So, we can say p infinity plus half rho U square and neglecting of course the potential energy contributions we can say p infinity plus half rho U square would be constant anywhere in this flow which should also be p plus half rho U square plus v square where u and v are velocities, say near the, near the solid obstacle.

Now, of course we can calculate the pressure using this equation by substituting what u and v are as a function of R or theta or x or y but from a dynamics point of view it makes more sense to not derive the pressure distribution but rather derive what we know as a non-dimensional pressure coefficient.

So, we define a pressure coefficient defined as C p or a coefficient of pressure which we write as p minus p infinity by half rho capital U square so p minus p infinity sort of becomes the difference and half rho U square is anyway the dynamic pressure. So, we use the Bernoulli equation in the C p equation we will get p minus p infinity would be just half rho U capital square minus half rho U square plus v square by half rho U square which we can write as, so half rho U square and half rho U square in the numerator denominator cancels, so we will get 1 minus u square plus v square by U square that would be the pressure coefficient using Bernoulli equation.

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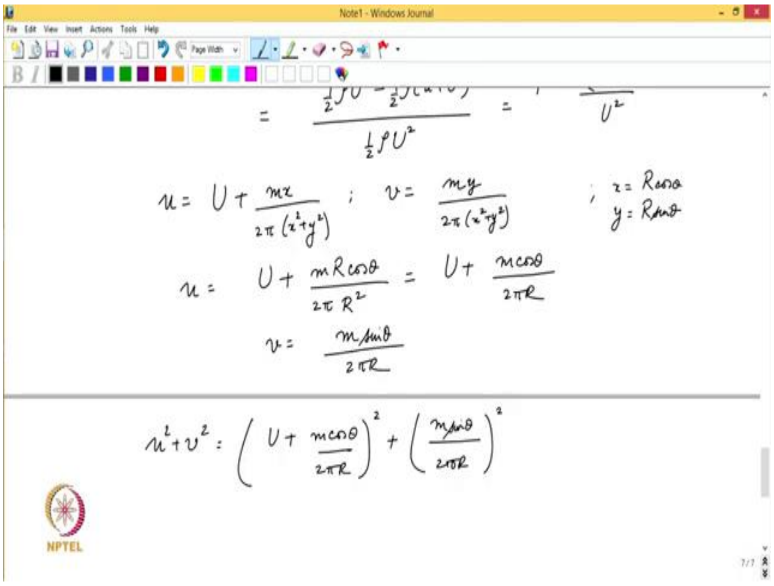

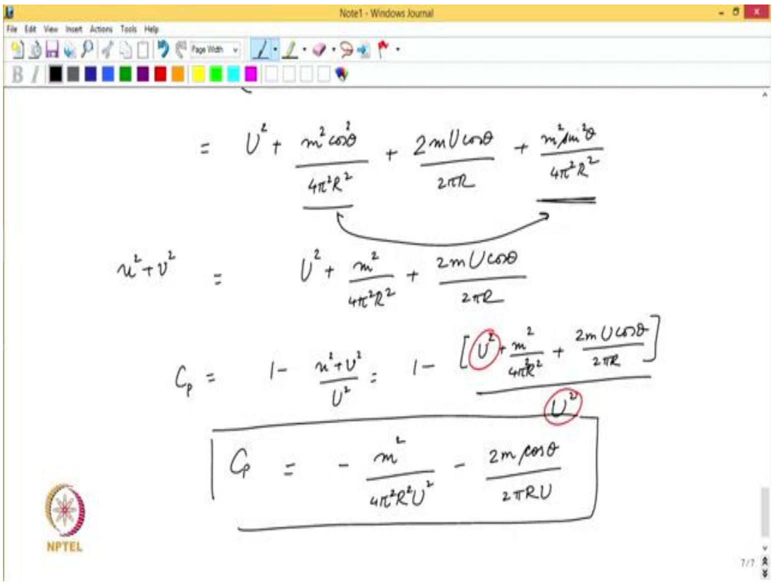

Now, we use the definition of u and v so we know that u is U plus m x by 2 pi x square plus y square and v is m remember this was actually I think m y we wrote as m y by 2 pi x square plus y square. So, I am going to change these definitions now I am going to use the idea that x is R cosine theta and y is R sin theta because of which u will become U plus m R cosine theta by can I say 2 pi R square because x square plus y square will become R square cosine square theta plus R square sin square theta, cosine square theta plus sin square theta is 1, so this is what we should get and which would be U plus m cosine theta by 2 pi R.

And v will become m times sin theta by 2 pi R. So, I am taking the liberty here of actually writing the velocity components which are in Cartesian coordinates in terms of polar coordinates but as long as we know what we are doing I think that liberty is something that we can exploit.

So, what we can write now is u square plus v square. So, u square plus v square will be U plus m cosine theta by 2 pi R whole square plus m sin theta by 2 pi R square which I could write as U square plus m square cosine square theta by 4 pi square R square plus say two times. Well, let us write at this as m u cosine theta by 2 pi R plus m square sin square theta by 4 pi square R square.

Now, this term and this, these two can actually be added up and we will get U square plus m square by 4 pi square R square plus 2 m u cosine theta by 2 pi R. So, this now brings it the velocity magnitude square to be in an easy form, so this value of C p would, which was 1 minus u square plus v square by U square can now be written as 1 minus U square plus m square 4 pi square R square plus 2 m u cosine theta by 2 pi R and this is being divided by capital U square.

Now, do you notice that if I open up the brackets and sort of divide the numerator denominator this U square and this U square will actually cancel each other it will give me 1 and that would subtract from the existing one. So, we can write this as minus m square by 4 pi square R square U square and then we will have a minus 2 m.

Well, there is a U in the numerator there is a U square in the denominator so I can just say this is 2 m cosine theta by 2 pi R U that is the definition of the pressure coefficient in this problem. So, it looks like a complicated expression but again we can do a lot more if we know what we are looking for. Well let me prove to you something first and then I will explain what I was doing. Let us first Look for, let us ask this question first that is there a location where C p could be 0.

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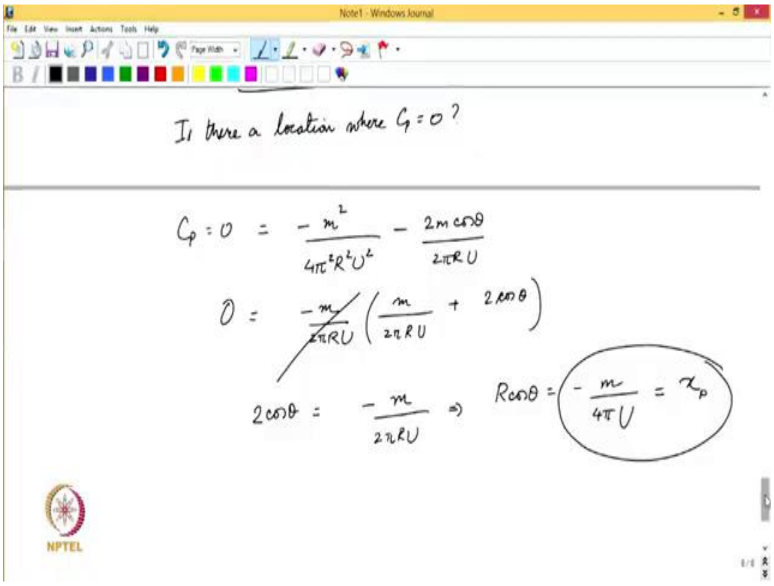

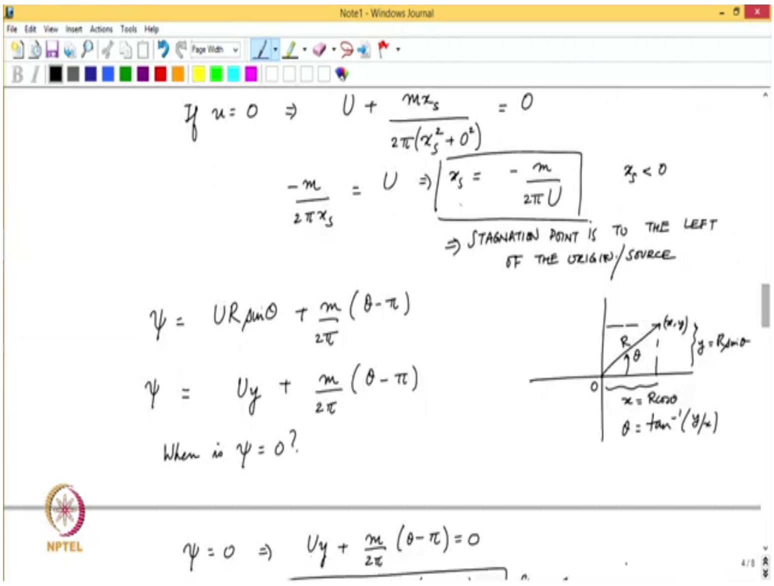

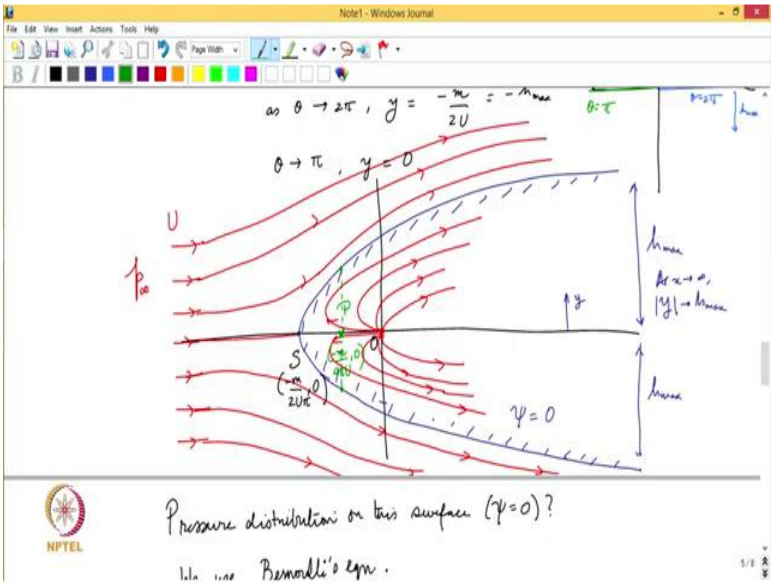

So, the question I am asking is, is there a location, location would mean physical point or particular coordinate where C p is 0. So, for C p to be 0 we should have whatever appeared on the right side of C p somehow adding to 0. So, let us try this, of course, see there is a, we could write this as minus m by 2 pi R and maybe you also if I can bring out will have m by 2 pi R U and we will have here the next one will be 2 cosine theta, I just pulled out m by 2 pi R U as a common factor from both terms. So, what we should have is whatever appears now on the screen.

So, this is a common factor as well if this had to go to 0 I am not concerned about this, this is anywhere not equal to 0. So, clearly 2 cosine theta should be minus m by 2 pi R U which means that R cosine theta is minus m by 4 pi, can I say 4 pi U, and recall that R cosine theta is x, that is a definition of x, so the location where this C p goes to 0 is some x value which let me say is x p for the point that p sort of denotes the aspect that we are looking at where C p goes to 0 so x p sort of says well it corresponds to x p and not the stagnation point.

Now, this is a very interesting finding again that in this case the pressure goes to 0 at a location x which is given as minus m by 4 pi U if you look at the location of the stagnation point and I think probably there is yeah probably this is there is a mistake here there is x S is minus m by 2 pi U, so I am sorry but I missed a factor of pi here, really sorry for this, anyway.

So, if we now look at this location which is x p which is minus m by 4 pi U can you see that it is exactly half way between these two points so this is a point which is corresponds to let us say I am going to say this is point p which has coordinates of minus m by 4 pi U comma 0. So, this is the somewhere along this line or at this point the pressure goes to 0 basically this is the coordinate at which pressure on the psi 0 boundary will go to 0. And this point p is exactly halfway between the stagnation point and the origin.

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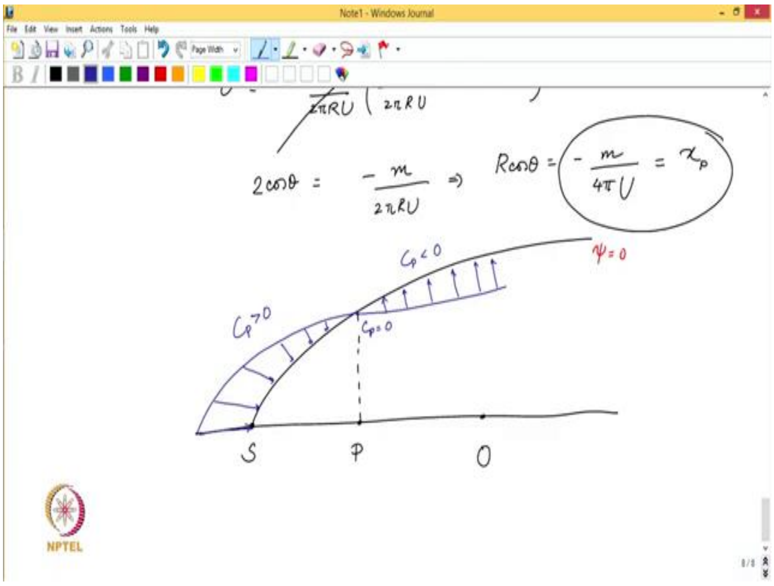

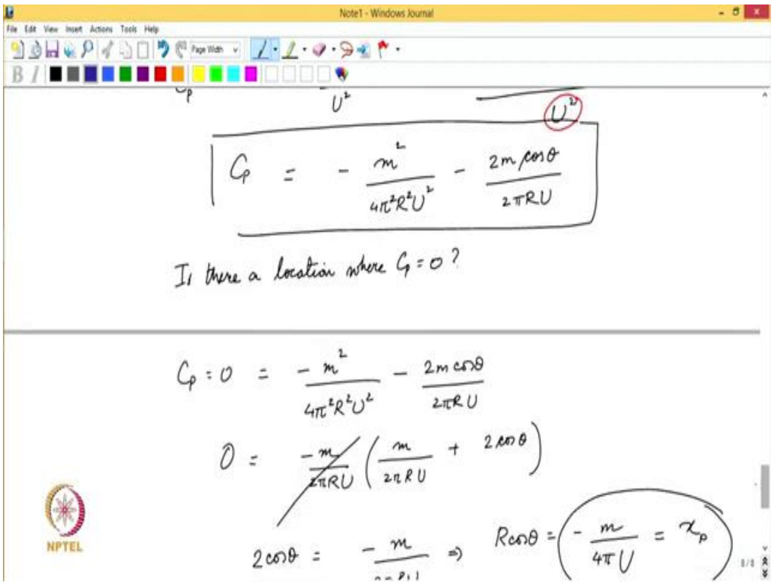

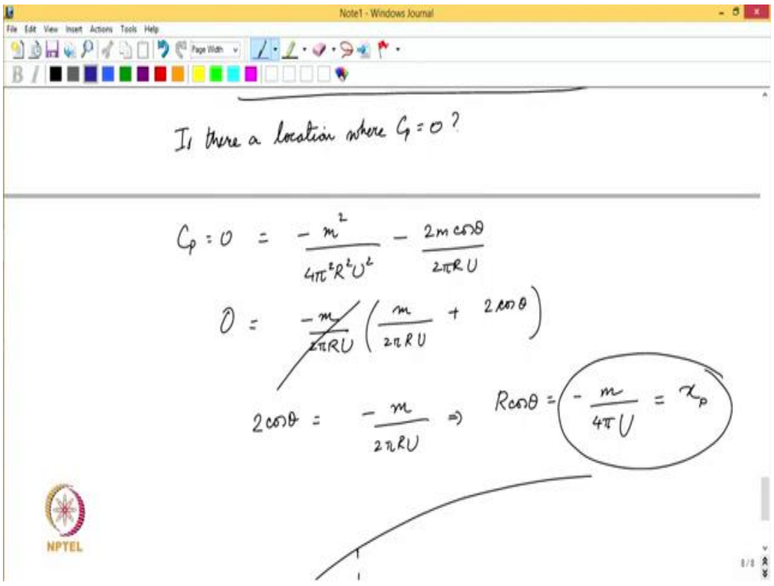

So, let me just draw this again for better understanding now so we have the x axis say this is point S, this is point say O and we have point P here exactly midway and the body shape looks something like this, let me just draw the upper half, the lower half I am sure you can figure out on your own now.

So, what I am saying is that if I look at the pressure distribution or the C p value which I am going to draw by a different colour. So, the value of C p happens to be if you now evaluate from the C p equation you will see that which I am asking you to work out on your own the value of C p will be positive as long as the x coordinate is lower than the value that we have here which is minus m by 4 pi U.

So, for all x coordinates between minus m by 2 pi U to minus m by 4 pi U the value of C p will be positive. So, the pressure coefficient will be something like this, so m this is the way the pressure would look like on this body, the pressure distribution until point p, at point p the pressure coefficient goes to 0 and then beyond that the pressure coefficient becomes negative. So, it would be of that type. So, this will be C p distribution on the psi equal to 0 line, so C p is positive here and C p is negative here and C p is exactly 0 at this point. So, that gives us a pressure coefficient or the pressure distribution on this body from the way we have been working out this case.

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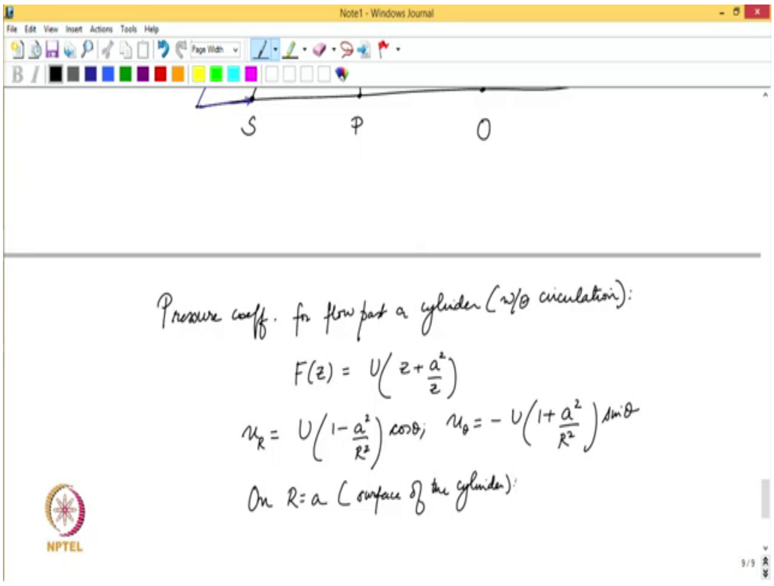

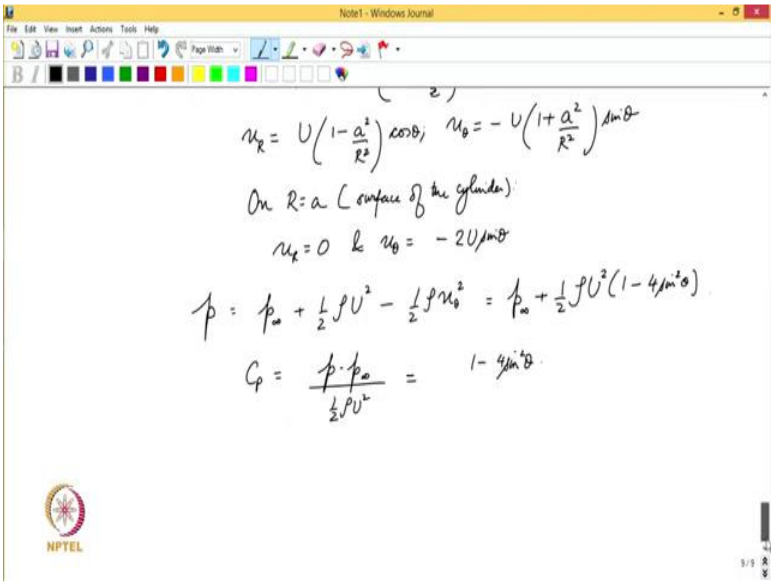

Let us also look at what was, what would have been the pressure distribution had we looked at in the previous lectures we looked at flow past cylinder. So, let us look at just to give you one more example of what the pressure coefficient should be for other problems. Say we wanted to look at pressure coefficient for flow past a cylinder and say without circulation.

In that case we had the complex potential to be U times Z plus a square by Z and we could derive from that the velocity components are U 1 minus a square by R square cosine theta and U theta is minus U 1 plus a square by R square sin theta. More importantly because on a surface of a cylinder R is a.

So, on R equal to a which will become surface of the cylinder we will have u R we can see is 0 and u theta will be minus 2 U sin theta. So, if we apply now Bernoulli equation to get the pressure coefficient on the surface of the cylinder, you would write first pressure to be p infinity plus half rho U square minus half rho u theta square which would be p infinity plus half rho U square 1 minus 4 sin square theta because there is a two sin theta in the U theta expression. So, clearly C p which is p minus p infinity by half rho U square will be 1 minus 4 sin square theta that would be the pressure coefficient for a circular cylinder. And let me now show you what it predicts for the pressure distribution on a circular cylinder.

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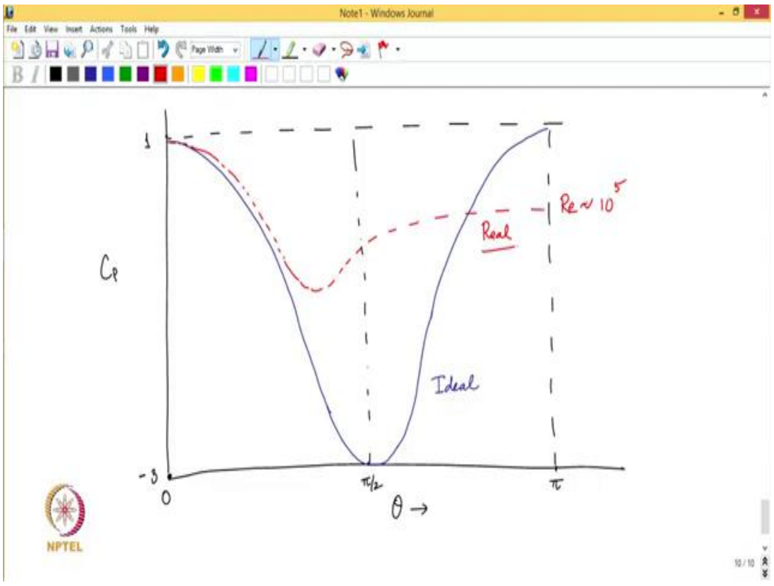

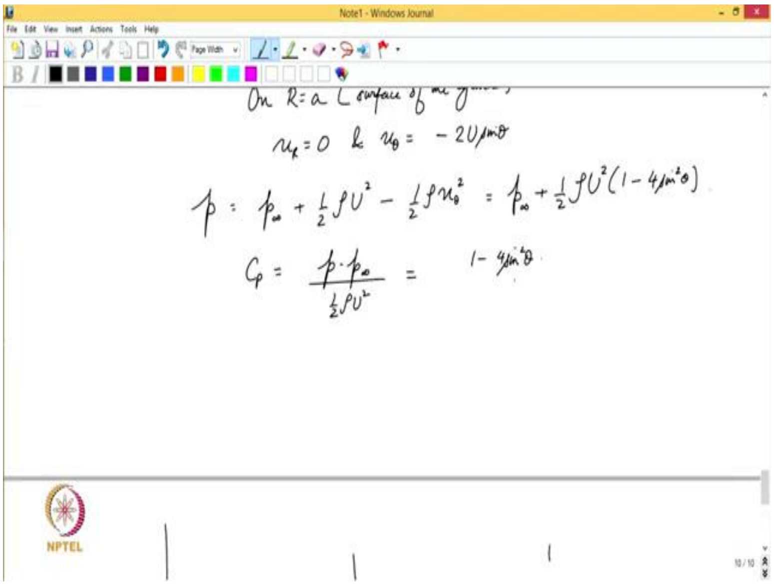

So, let us just take a new page to draw this kind of distribution, so we have C p here and we will draw theta and we will draw theta between say 0 to pi and maybe pi by 2 is somewhere in the middle. And you can see that since CP p is 1 minus 4 sin square theta what are the maximum and minimum values of C p.

So, clearly sin square theta is always a positive number, so sin square theta can only be between 0 and 1. That means C p can only be between 1 and minus 3. 1 when sin square theta is 0 and it will be minus 3 when sin square theta is 1. So, we will have this scale going from say minus 3 to 1.

So, when theta is 0 C p 1, so we will have a curve which would look like this and when theta goes to pi by 2 we will have C p going to be minus 3 so we will have this type of change and then again when theta goes to pi C p will be its maximum value, so 1, this is C p that we calculate from this ideal flow analysis.

And of course, from between pi and 2 pi it is now we see that it would be the same thing repeating itself, that is the reason I have not drawn for the entire cylinder but only the upper half, quite, this, so this kind of pressure distribution works out very well for low values of theta for real situations.

So, if I had to draw let us say the pressure coefficient in a real scenario, in a viscous fluid but at a higher Reynolds number what you would see is a behaviour of this type that the C p value will follow this curve to some point and then it starts to deviate. So, this is say R e the Reynolds number is about 10 to the power 5 or 0.1 million, this is for a real flow.

So, even though there is a large deviation between the real and ideal scenario, well, we can be quite pleased that even in a small zone up till here the pressure matches quite well and for a simple analysis of the way we are doing it analytically this is also quite a good achievement. So, we have covered quite a lot of things in today's lecture, we will see, this is one example of how we can use superposition to do much more complex analysis or much more fruitful analysis and from here on we will move on to a few more examples of superposition.

I will take up a few examples in terms of solving some problems and then we will look at how do we calculate force using the tools that we have developed, so far I have not spoken about how do we calculate the actual value of force but now we will try and address that part as well. So, thank you.