Ideal Fluid Flows Using Complex Analysis
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Lecture 8
Superposition of Uniform Flow, Doublet and Free Vortex

So, in the previous lecture, we looked at the complex potential for a flow past a circular cylinder. And we saw that, this flow could be derived by superposition of a uniform flow and a doublet.

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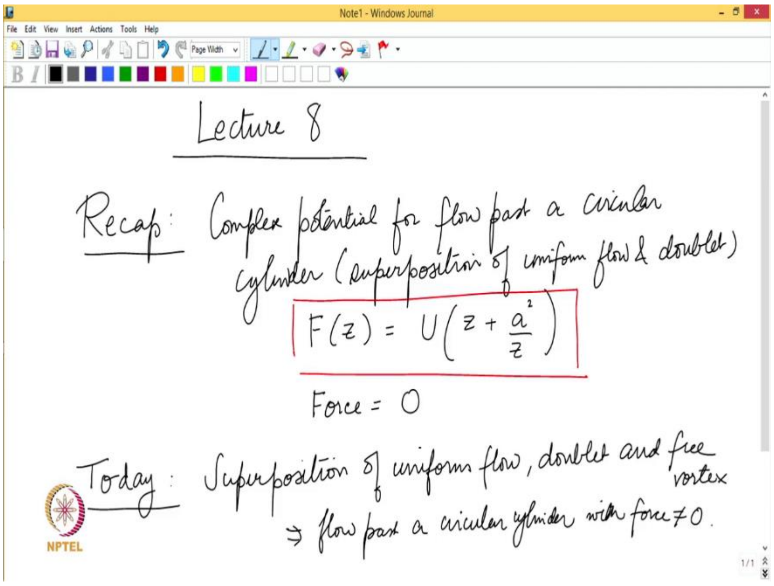

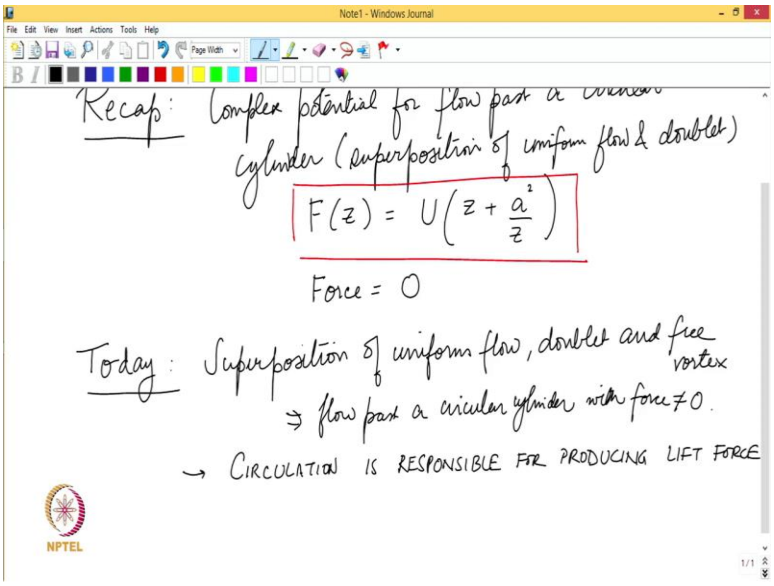

And in that case, the complex potential we showed could be written as F of z in terms of, the superposition as I said of a uniform flow, which is U times z and the doublet which is U square by z. And by doing this, we could show that on our surface, which is at a distance a from the origin, the streamline corresponds to psi 0. And that was a surface where we could replace that streamline with a circular cylinder. And in that case, we also showed that the flow around the cylinder is symmetric about both x and y axis. So, the net force is 0. So, clearly that is a hypothetical case.

And what we will do today is to look at a more general case, which is to look at flow paths a cylinder with some finite force. And we will see that, that force that we will quantify would be basically lift force, or force in a direction which is transverse to the direction of the uniform flow. So, the agenda today is to look at superposition of uniform flow, doublet, and free vortex. And what we will show is that this will give us flow past a circular cylinder with the force now not being equal to 0, or having some finite value.

More importantly, what will show is that it is a circulation which causes a force, or it produces a lift force. And eventually we will prove why the circulation, determines the lift force? So, we will see in this case, that circulation is responsible for producing a lift force, eventually we will prove it.

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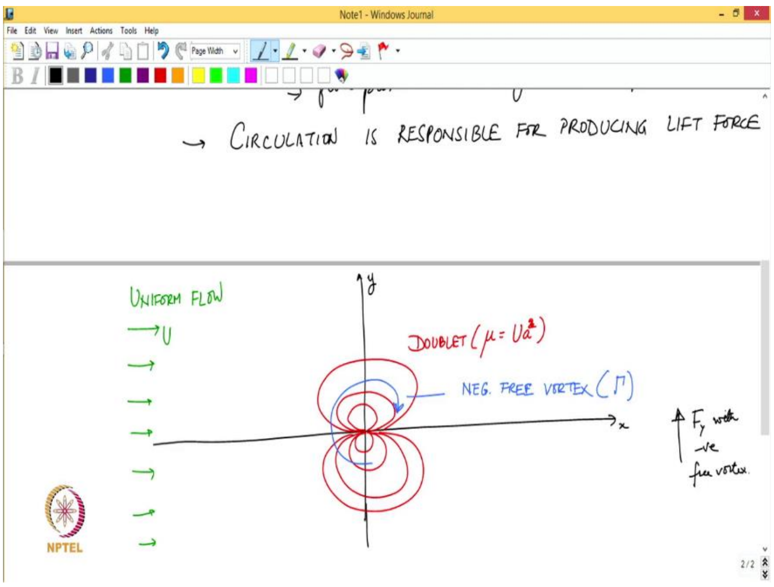

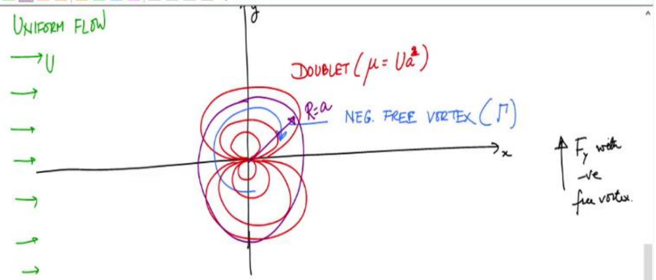

$$F(z) = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\log z + C$$

constant

Surface $R=a \rightarrow \psi = 0$ streamline.

$$W(z) = \frac{dF}{dz}$$

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$\underline{C?}$ $\quad z = ae^{i\theta}$

$$F(z) = U\left(ae^{i\theta} + \frac{a^{\cancel{2}}}{\cancel{a}e^{i\theta}}\right) + \frac{i\Gamma}{2\pi}\log ae^{i\theta} + C$$

$$= aU\underbrace{\left(e^{i\theta} + e^{-i\theta}\right)} + \frac{i\Gamma}{2\pi}\log a + \frac{i\Gamma}{2\pi}i\theta + C$$

$$= 2aU\cos\theta + \frac{i\Gamma}{2\pi}\log a - \frac{\Gamma\theta}{2\pi} + C$$

$$= \phi + i\psi$$

Assume $C = C_r + iC_i$

$$F(z) = 2aU\cos\theta - \frac{\Gamma\theta}{2\pi} + C_r + i\left(\frac{\Gamma}{2\pi}\log a + C_i\right.$$

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$= \phi + i\psi$

Assume $C = C_r + iC_i$

$F(z) = 2aU\cos\theta - \frac{\Gamma\theta}{2\pi} + C_r + i\left(\frac{\Gamma}{2\pi}\log a + C_i\right)$

$$= \phi + i\psi$$

$$\phi = 2aU\cos\theta - \frac{\Gamma\theta}{2\pi} + \cancel{C_r} \rightarrow 0$$

$$\psi = \frac{\Gamma}{2\pi}\log a + C_i = 0 \Rightarrow C_i = -\frac{\Gamma}{2\pi}\log a$$

Then, $\psi = 0$ at $R = a$.

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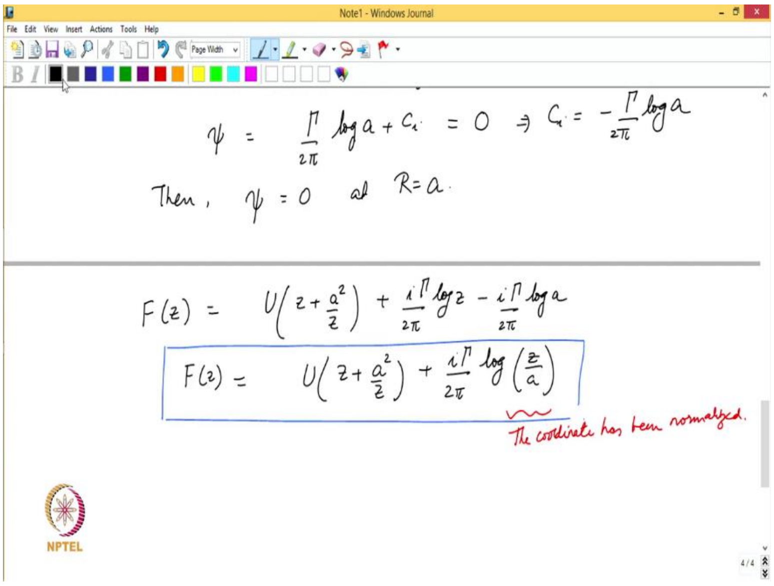

So, let us look at the situation that we have at our disposal. We are trying to now superpose, three types of flows. And let us represent them each by a different colour. So, we have a uniform flow. This is uniform flow, which has a potential of U times z. So, I will just represent that by velocity U, we have a doublet which is formed by superposition of a source and sink. So, this is a say a doublet.

We saw in the previous lecture that when we add these two, we get flow past a circular cylinder, but without any force. Now, what I am proposing is to include on top of this problem, a free vortex, and we will include a negative vortex, a negative vortex will be 1 which has a circulation in the clockwise direction. So, we will have a negative free vortex. So, its strength I am going to write as gamma.

So, it will produce a flow in the clockwise direction, and that is centred at the origin, that is very important. So, the singularity of the free vortex is at the origin itself. The doublet would have a strength which we are writing as mu, but for all reasons we will say would be mu a square. Now, why did I choose a negative vortex? Well, as I will show you, in this derivation, or in the mathematics that will develop today, that the free vortex of a negative circulation is one that produces a positive lift.

So, if we put our coordinate system as x and y, so we will get a positive lift, which is a positive force say F y, with negative free vortex, or a free vortex with the clockwise rotation. So, let us begin this and see how we will develop the mathematics for this. So, we can write by superposition that F of z would be U z plus a square by z that is, as usual, from the previous lecture. And now we are going to add a negative vortex. So, recall that for positive vortex, for instance, the potential is minus i gamma by 2pi log of z. Since, this is a negative rotation vortex, we will have a plus i gamma by 2pi log of z as a complex potential corresponding to this clockwise rotation vortex.

Now, what I will do is? I will add a constant to this function, which is, which as you will see is a complex number. But I will add this to ensure that the surface corresponding to R equal to a, gives us a size 0 streamline. In other words, this number C will also ensure that I have normalised my free vortex coordinates. So now, an obvious question could be why are we adding this constant? Does it change the way we do our mathematics? And the answer that we get at the end of this? Now, it does not change the flow field. And this I want you to be able to verify on your own.

But in case you are not able to think about this, or you are not, you are having some issues, trying to come up with a logical explanation, what I would say is that if you look at the velocity field, which is W of z, which is dF, dz. So, adding the constant does not change the derivative. So, the velocity field will still be the same, irrespective of whether you have a constant or not. So that is why we can justify adding a constant to the complex potential. Now, let us derive what this number C would be? So, my first question is what is C?

So, what I will do is, I will go on the surface of the cylinder, so cylinder would be of radius a, any coordinate on this cylinder should be ae i theta. So, on this figure, if we can maybe take the liberty of drawing a circle with radius a. So, we have a radius circle of radius a, and this is R equal to a, so I want this to be a streamline. So, what we can do is, we can say F of z or F at this point, so F of z corresponding to this number would be U ae i theta, plus a square by ae i theta plus i gamma by 2pi log of ae i theta plus C. So, that is what the complex potential would be on this surface, or on the surface of this circle, which is at a distance a.

So, let us simplify this. So we have one a will cancel from here, so we can write this as U, a will be common, so we will have ei theta plus e minus i theta plus i gamma by 2pi, we will have a log of a plus i gamma by 2pi, log of e i theta will just be i theta, plus C. e i theta plus e i minus theta will just give me 2 cosine theta. So, I can write this as 2aU cosine of theta plus i gamma by 2pi log of a. This will become minus gamma theta by 2pi, because i square is minus 1, plus C.

Now, we have not reached the conclusion yet, which is that I want to write this as phi plus i psi. So, I want to associate the real part to phi, and the imaginary part to psi. As of now, it is not going to serve my purpose, because if I was to assume C to be real, then C would go to phi, and then I would still not be able to get psi equal to 0, at on this surface.

So, I want psi to be 0. So, clearly, if I was to assume C to be some real part plus iota times some imaginary part, then F of z would be 2au cosine theta, and let us put the real parts together minus gamma theta by 2pi, and we will have a real part coming in from C, plus will have iota times gamma by 2pi log of a plus C i.

Now, this I can say is phi plus i psi. So, phi which is a real part would just be 2aU cosine theta minus gamma theta by 2pi plus C r, at which point I am going to assume that this constant is such that C r is 0, because it is not going to have any influence on phi either way, and I will take psi which is gamma by 2pi log of a plus C i, which I want to be 0 at the surface. So, C i should be minus gamma by 2pi log of a. So, that should be the imaginary part of this number. So, if you choose it in this way, then psi becomes 0 at a distance of R equal to a, and in such a case then R complex potential, F of z could then be written as U z plus a square by z, plus i gamma by 2pi log of z.

Now, remember we had a plus C, C is just minus iota times C i which is gamma by 2pi log of a which we can write as. So, this becomes a complex potential for this problem. So, the purpose of adding C as I was saying earlier is to normalise this coordinate. So, a becomes a distance which is normalising this coordinate. So, I hope till here this is pretty standard, this is clear. So, now let us get into some rigorous math with this complex potential because now things will get interesting how we use this complex potential to derive different scenarios of flows?

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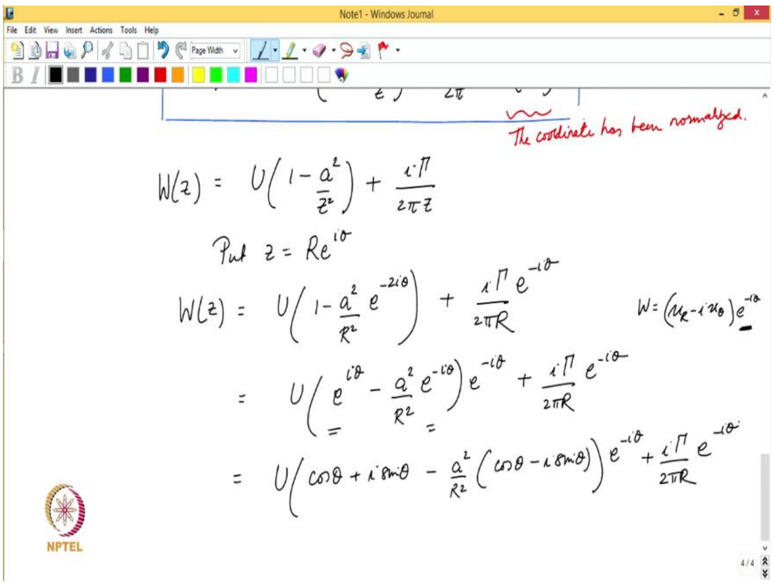

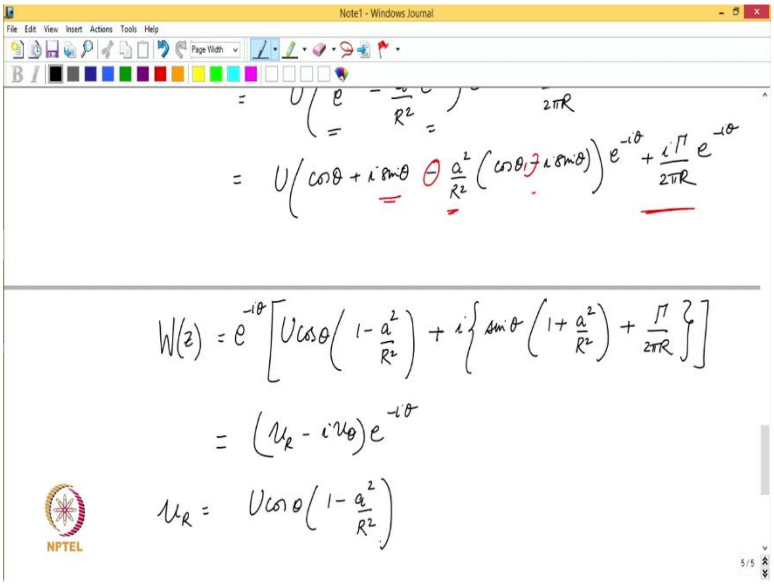

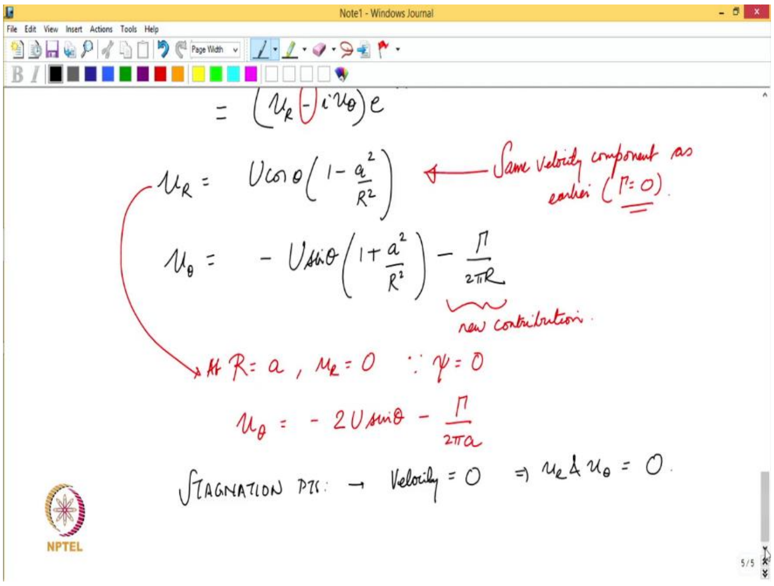

So, the way we do it is, we first derive what would be the complex velocity of this flow? So, that would be just U times 1 minus a square by z square plus i gamma by 2pi times z in the denominator, because when you take the derivative d by dz of log z, you will get a z. Now, in for obvious reasons this will be a problem which can be easily solved using cylindrical coordinates.

So, we put z to be Re i theta. So, W of z would be U 1 minus a square by R square, we will have e to the power minus 2i pi theta plus i gamma by 2pi R e to the power minus i theta. So, as usual I will try and take an e to the power minus i theta outside of all whatever we have in the velocity expression, because I want to write this eventually as u R minus i u theta e minus theta, I want to get to this point.

So, I need an e to the power minus i theta outside. So, I will try and rearrange numbers here in such a way that I get this factor out. So, I can write this as U e i theta minus a square by R square e minus i theta times e to the power minus i theta, plus i gamma by 2pi R e to the power minus i theta. What we can also do is look at or expand e to the power i theta, and e to the power minus i theta in terms of cosine and sine theta. So, this will become U cosine theta plus iota sine theta, minus a square by R square cosine theta minus iota sine theta times e to the power minus i theta, plus iota gamma by 2pi R e to the power minus i theta.

The whole purpose of doing this is to separate thing, is to separate the real parts and the complex parts. So, let us start collecting the real parts together. So, we will get, now this, you have to do it carefully, we will have U cosine theta 1 minus a square by R square, the imaginary part will be plus say iota times let me keep an e to the power minus i theta outside here, plus iota times we will have a sin theta times 1 plus a square by R square, which comes from, 1 comes from here, a square by R square, plus comes because there is a negative sign here and that multiplies by this, so it gives us a positive sign.

So, we will get sin theta 1 plus a square by R square. And then we will also have plus this number from the side, this contribution, so we can say this would be gamma by 2pi R, we can close this bracket and that is it, that is what we should get, this is W the velocity, complex velocity for this situation. So, this should be u R minus i u theta e to the power minus i theta. So, now let us separate out the real and imaginary parts.

So, u r would just be U cosine theta times 1 minus a square by R square. And u theta will be, this will be minus, and I am sorry, that I am missing a U here, there is a U that is missing there. So, it would be minus U sin theta 1 plus a square by R square minus gamma by 2pi R, and the negative sign appearing because we have a negative sign here, in front of U theta. So, that is the reason we get a negative sign with all quantities on the side of U theta.

Now, some things you should notice are that first of all, this is the same velocity component as earlier, which was when we had no circulation like which when gamma was 0, or gamma was not there, we got this as the velocity component. In u theta now, which is we have this as a new contribution because of rotation, because of a free vortex, which adds circulation to the flow. Now, a thing that we can easily verify is that when we take this, this component at R equal to a, we have u R to be 0. So, there is no radial component. That is also expected because psi would be 0, and it is a constant stream function line R equal to a. So, that is logical that u R is 0.

But what about u theta at R equal to a, u theta will become minus 2U sin theta minus gamma by 2pi a. So, it is a function of theta, but it has this additional term. In the previous case we saw this was just minus 2U sin theta. So, this is the flow field. Now, a way to draw the flow field in this situation is to first identify where are the stagnation points for this new situation. Remember, the stagnation points are points where when we say that the velocity is 0, in our case, that would mean that u R and u theta are both 0. Now, we have seen that u R is 0 at R equal to a which becomes logical to think that can u theta also be 0 at R equal to a?

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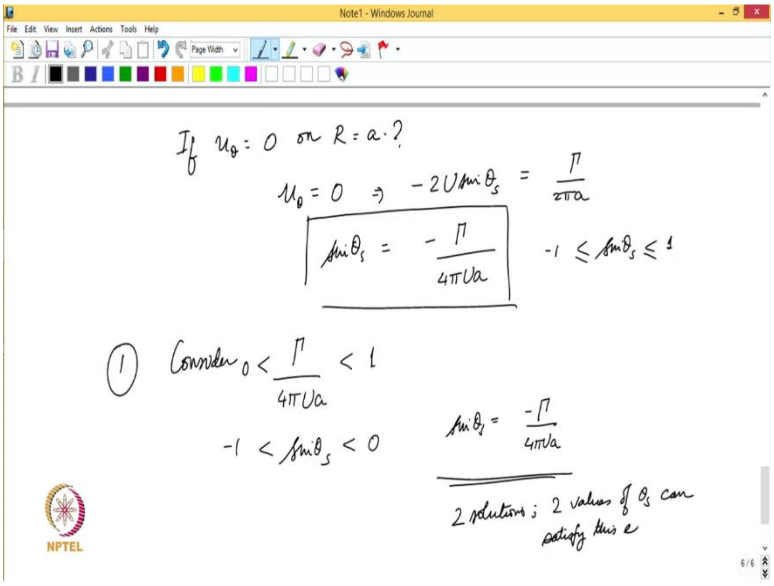

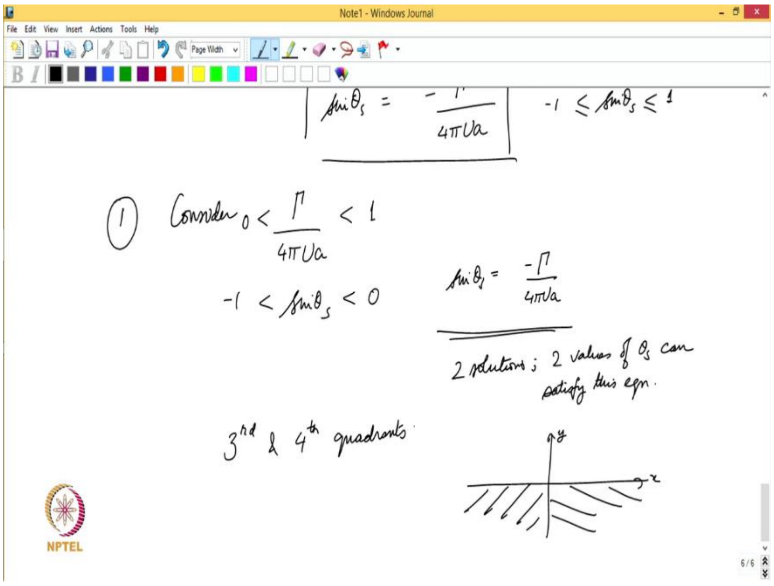

So, let us consider the situation where we try and identify if u theta can be 0? Let us say on R equal to a. So, what should be the values of theta corresponding to this case? In the previous lecture, we saw that u theta could be 0, because the gamma contribution was not there. So, u theta could be 0 if sin theta was 0, which means theta was either 0 or pi.

In this case, now, for u theta to be 0, we need minus 2U sin theta to be gamma by 2pi a. Which means let me say theta s is the location of the stagnation point, which is, how I am going to represent that value. So, sin theta s correspond to the stagnation point would be minus gamma by 4pi U a, that would be the location where the stagnation point will be found. So, at that theta value.

Now, this equation can have quite a few solutions. And it also depends on the right-hand side, I know that sin theta s can only be between minus 1 and 1. So, the only way this can be valid is that, of course, if the right side is also a number which follows this criteria that it is between minus 1 and 1. So, let us first consider this possibility that gamma by 4pi U a, where all these numbers are positive by the way, gamma is positive, U is positive, a is positive. So, let us consider this scenario where this is less than 1. It is of course greater than 0, needless to say because everything is positive, but say it is less than 1. So, if gamma is let us say small value as compared to 4pi U a, what do we get?

In this case, sin theta s would be between minus 1 and 0, sin theta s would definitely be negative, it will be more than minus 1. So, for such a case where we are trying to calculate sin theta s to be minus gamma by 4pi U a, it should be clear that this right side is less than 0 but more than minus 1. So, there could be two roots to this equation, or two solutions. Which I am, by what I mean is two values of theta s can satisfy this equation.

And these values will be, think about this because sin theta is negative. So, these values should be in the third and the fourth quadrant. So, the two roots will be in the 3rd and 4th quadrants. By that, what I mean is we had our x y axis, so the solution would be either here in this, or it would be here. Those are the only two places where the solution can be found. It cannot be in the upper half, or anything above y equal to 0, it would be definitely in the lower half of the plane.

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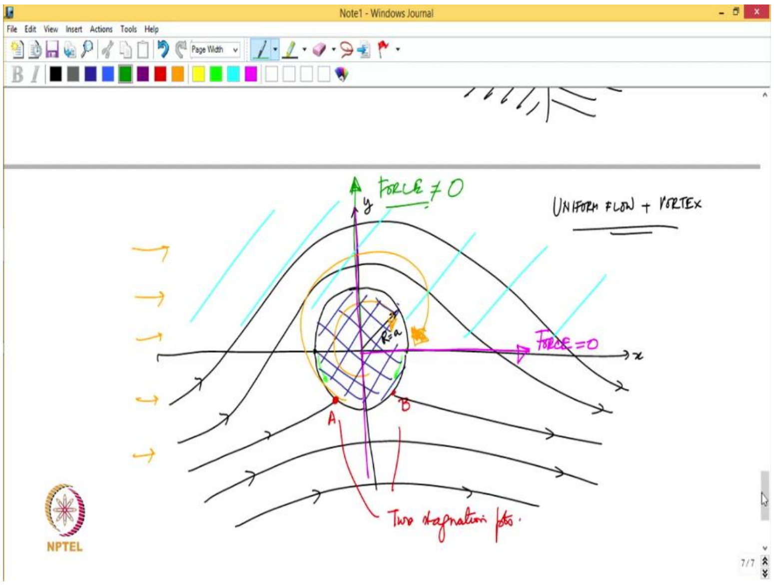

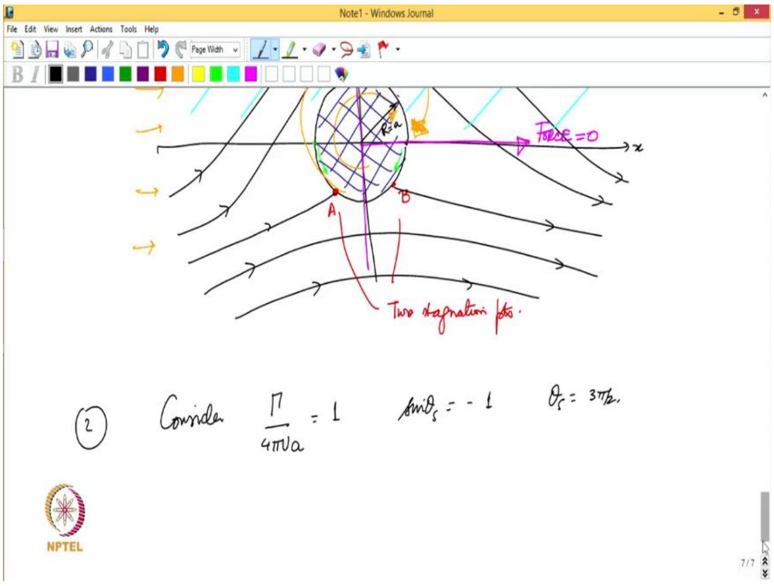

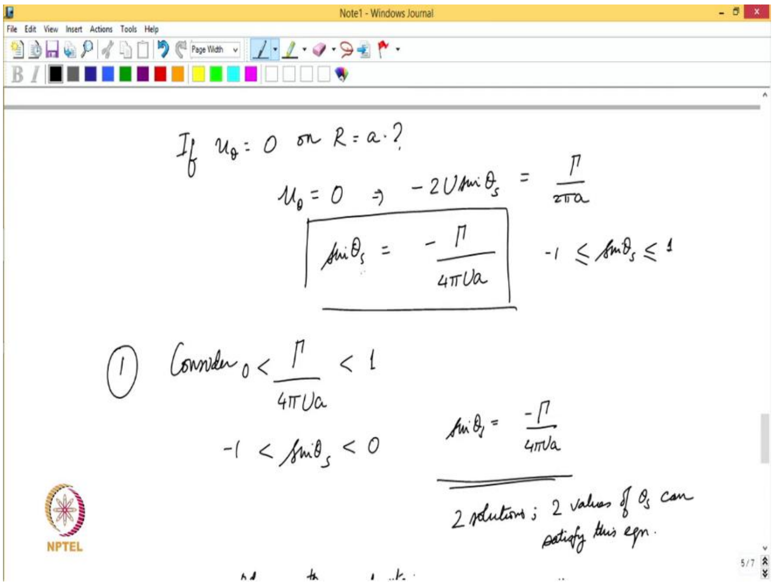

So, it so happens in this case now, that depending on the value of gamma the roots, would look something like this. So, we can draw this cylinder, surface which is a circle. So, this is a circle at R equal to a, and let me say this is solid so, let me denote that this is a solid cylinder. So, the stagnation points on the lower half, and they will also be located symmetrically about the y axis, that is a property of the sin function that is going to make it happen. So, we can have the stagnation points say one stagnation point is here A, the other one would be diametrically opposite at B. I should not say diametrically but it should be opposite to A. So, these will be the stagnation points where the flow comes to rest.

So, our stream lines would look in a certain way there will be some symmetry about y equal to, about x equal to 0 rather. So, the flow that you would see would be of this type. So, that would be the flow profile that we would see for this case. So, these are the two stagnation points again, and these are going to be located symmetrically about x equal to 0. More importantly, I want you to think about what is the role of gamma here?

If we put gamma to be 0 in this equation itself here? If we look at this equation and we put gamma to be 0, we get sin theta s to be 0, that clearly means the stagnation points must go back to theta 0 and theta pi. So, they must be these locations, but because gamma is nonzero, the stagnation points start to move along the lower half of the cylinder.

So, they are actually moving in this direction depending on what is the magnitude of gamma, for very small values of gamma they would be nearer to the equator, but as you increase the circulation, or if the circulation is higher, relatively it is higher, the stagnation point start moving towards the south pole if I can say, the south pole of the cylinder.

And what does the circulation do? That is also very important here, the circulation is actually recall that it is a circulation in the clockwise direction something like this in the clockwise direction, what is it doing? We had a uniform flow coming towards the cylinder. So, the circulation is actually adding up with the uniform flow on the upper half. So, in the top half of our problem. So, in this part of the domain, the uniform flow and the vortex are sort of adding. So, the velocities are higher in that part. So, the fluid is going to move faster on the upper half.

On the lower half of the cylinder, the vortex is actually opposing the incoming flow, the vortex is inducing velocities, which would be opposite to the incoming flow, and that is the reason we get stagnation points on the lower half. Because these are the locations where the velocities are opposed in a manner of saying equally.

So, we get stagnation points on the bottom half. More importantly the stagnation points being on the lower half causes the pressure on the lower half to be higher than the pressure on the upper half. So, pressure on the lower half is going to be high, pressure on the upper half is low. So, we get a net force in the vertical direction. So, the cylinder experiences a force in the vertical direction because the pressure on lower half of the cylinder is higher, the pressure on the upper half is lower, and that is what is going to cause this lift force to be produced.

However, if you look at flow about the x equal to 0 line, which is this line itself, which is this line, flow if you look at flow about this line, the flow is symmetric. And since there is no viscous force, or there is no viscous shear, the net force along the horizontal direction is 0. So, force along this direction is still 0, but the force along the vertical direction is not 0.

So, that is the role of circulation in inducing a vertical force, we will also quantify that force exactly how that much force is, and that would be very in the next, in the upcoming lectures, but for now, I think we should at least make some sense to visualise what has happened here? So, we looked at the case where gamma by 4pi U a was less than 1.

Now, let us look at the special case where gamma by 4pi U a is exactly equal to 1. So, what has happened here is that as I said the circulation increases, the stagnation points start moving downwards, A and B move away from the equator. So, when gamma by 4pi U a is equal to 1. This is a critical case, in this case sin theta s should then be minus 1, which means theta s should be 3pi by 2, which is the point on the negative y axis.

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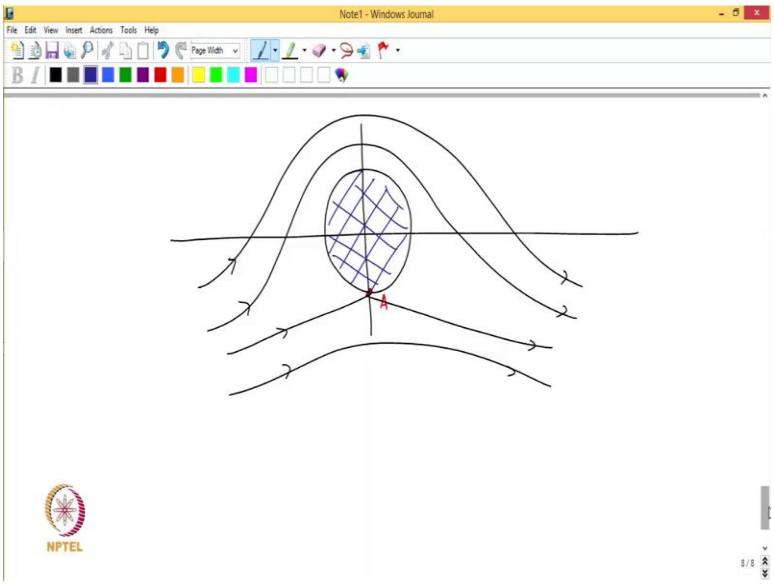

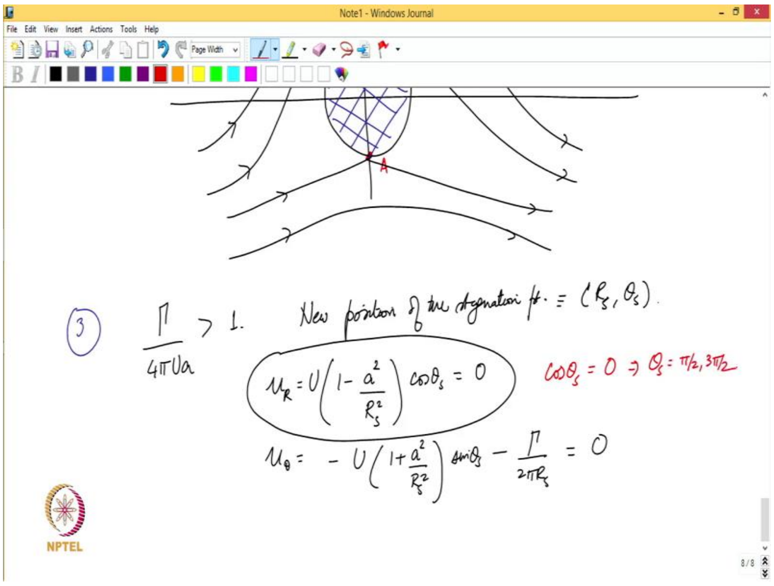

So, I do not think I need to explain what is going to happen in this case, but at least graphically we can draw this that now we have a cylinder, and we will have the stagnation point being located at this point. So, this is the only solution, and the flow field would look like the following, in this solid cylinder. So, that would be the profile and only one stagnation point could be observed and that too at the lower most point or the south pole of this cylinder. Now, the last aspect that I want to consider is that vortex could have a higher strength, than what we get by this situation?

So, the third scenario could be, what if gamma by 4pi U a is more than 1. Then, clearly, we cannot have a stagnation point on the surface of the cylinder, because the conditions cannot be satisfied. So, we will calculate the new position of the stagnation point in this case, and so let us say the newest position of the stagnation point is given by some location R s theta s in polar coordinates. To find out what this point is? We need to go back to the definition of the radial component of the velocity and the tangential component to the velocity.

So, we can say u R, which was given as U times 1 minus a square by in this case now let us say R square cosine theta s. This should be 0 for an, for a stagnation point and u theta which was minus U 1 plus a square by R square sin theta s minus gamma by 2pi R s. This should also be 0, both should be 0 to get a stagnation point. And so, we have two equations and two variables R s and theta s.

So this is not very challenging to solve, if I look at the first equation here, it can have either R s equal to a, which sort of gives us U R equal to 0, but that is a trivial solution, we have already seen that, more importantly, R s equal to a will again bring us back to the cylinder and you can verify that it is not going to work out because gamma is more than 4pi U a. So, clearly only cosine theta s can be 0, that is the only feasible solution, which means theta s could either be pi by 2 or 3pi by 2, one of those two values to give us a solution, Now, which of those values? And what would be the value of R s corresponding to that value theta? Let us work it out.

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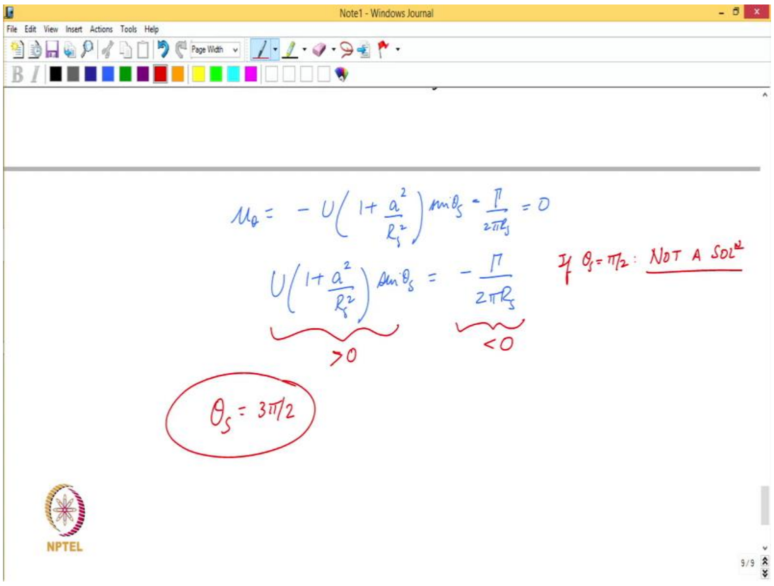

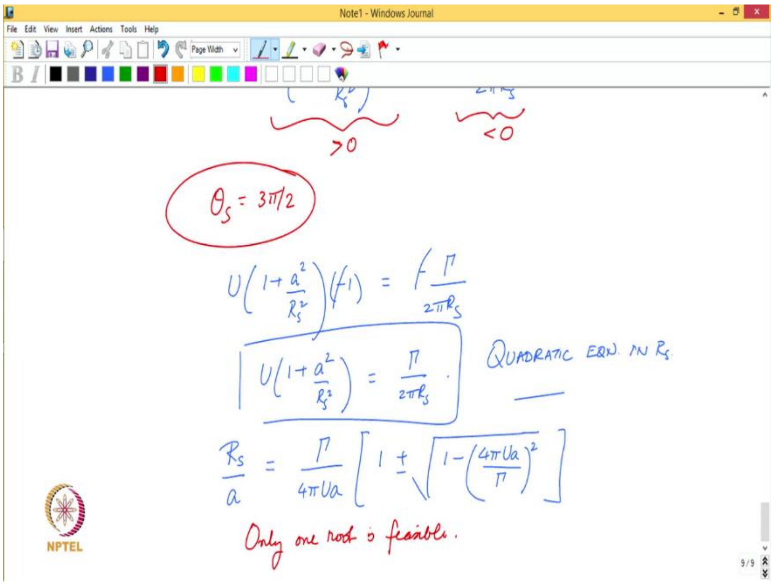

So, if I look at u theta, which is minus U 1 plus a square by R square sin theta s minus gamma by 2pi R s, which is 0. So, we should have U 1 plus a square by R square sin theta s to be minus gamma by 2pi R s, for u theta to be 0. Now, just by observation, we can see that if theta s was pi by 2, then the right side would be a number which would be more than 1, or which will be more than 0, rather. It would be a positive number, because every quantity on the on the left-hand side is positive. So, if theta is more than pi by 2, every quantity on the left-hand side would be more than 0, it would be positive number.

But if you consider the left-hand side, then this is actually a negative number. So, clearly, this equation cannot be satisfied, if theta s is pi by 2, a positive number cannot be made equal to a negative number. So, this is not a solution to a problem. So, the only possible solution is theta s to be 3pi by 2. So, the stagnation point would again be somewhere along the negative y axis.

Now, we can see if we use this and try and now look at the U theta equation, so we will have U 1 plus a square by R square sin 3pi by 2, sin 3pi by 2 is actually minus 1. So, this is minus 1, this becomes equal to minus gamma by 2pi R s. So, these negative cancels, and what we now have is a quadratic equation in R s, we have U 1 plus a square by R s square to be gamma by 2 by R s.

So, we will have two roots to this equation. So, you can work this out the two roots, I would write them the two solutions in terms of a normalised coordinates. So, R s by a normalised distance, this would be gamma by 4pi U a times 1 plus minus square root of 1 minus 4pi U a by gamma square, that would be the solution to this problem. These will be two roots to this problem. However, only one root is feasible. So, you can take a moment here and try and think about which would be a feasible root?

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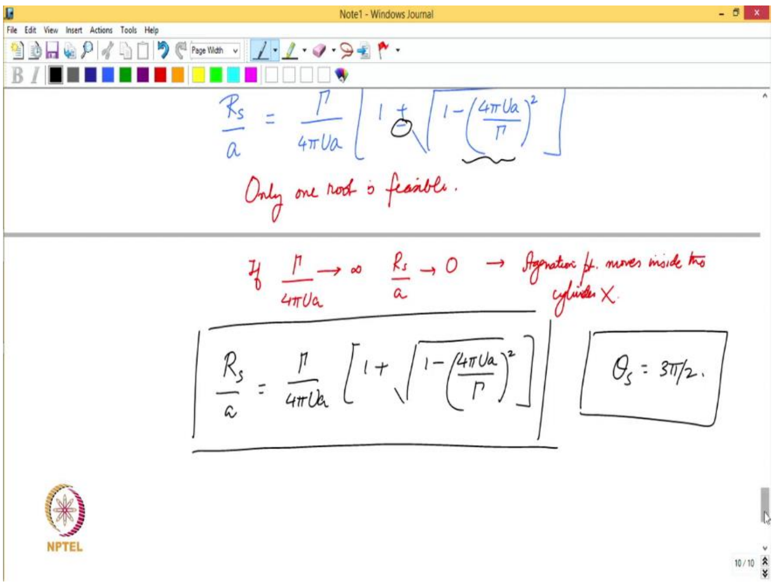

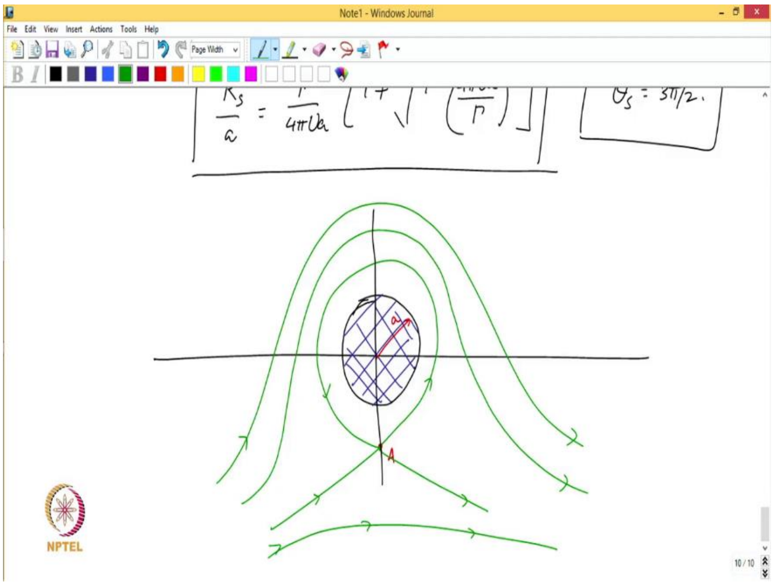

The way I would prove to you is that, if for instance, gamma, it was a very strong vortex such that gamma by 4pi U a, is maybe going to infinity, it is a very strong vortex. In this case, the value of R s by a would actually approach 0, which indicates that the stagnation point moves inside the cylinder which is not possible, which is not physically possible.

So, this happens when you have a negative value inside, where when this number goes to 0, you will have 1 minus 1 which is 0. So, the value of R s comes inside the cylinder. So, the only possible solution is R s by a to be gamma by 4pi U a 1 plus 1 minus 4pi U a gamma square, that is the only feasible solution. So, the coordinates of a stagnation point are R s given by this equation, and theta s to be 3pi by 2 that is where the new stagnation point would be located.

Physically, if we draw this again, this is a circular cylinder which is of radius a. Now, the stagnation point has moved downwards, and it would be located somewhere here, it is theta s corresponding to pi by 2 is on this line. So, we will have a solution, which would look very interesting. So, we will have a flow pattern which would look like the following.

So, we will sort of have an envelope of fluid which encircles this cylinder, and everywhere else, the flow would look like that. That would be the flow pattern that we would see right at the stagnation point is where this would be an envelope of fluid which would be completely encapsulated in that streamline. So, this would be the case when you have a very strong vortex at the singularity, which is z equal to 0.

So, we come to an end to superposition of this type. Now, deriving specially flow past a cylinder with and without circulation. So, we will take a few more examples in the upcoming lectures. And then we will also look at deriving the idea that force on any structure in this case, using potential flows is proportional to the circulation gamma.

So that I have not proved yet. But needless to say, that the pressure field or the velocity field, and the idea that the pressure would be higher with the velocity is lower tells us that in all these cases that I have taken up today, the force would be in the vertical direction. So, we will use that, and we will also try and prove it, algebraically, which I have not done, but we will do it in the next lecture. So, thank you, and see you in the next class.