Ideal Fluid Flows Using Complex Analysis
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Lecture 7
Superposition of Uniform Flow & Doublet

So, in the previous lecture, we looked at an elementary flow which is called as a doublet. And you may recall that the doublet was actually constructed by a linear combination of a source and a sink, which were of equal strengths and located very close to each other near the origin.

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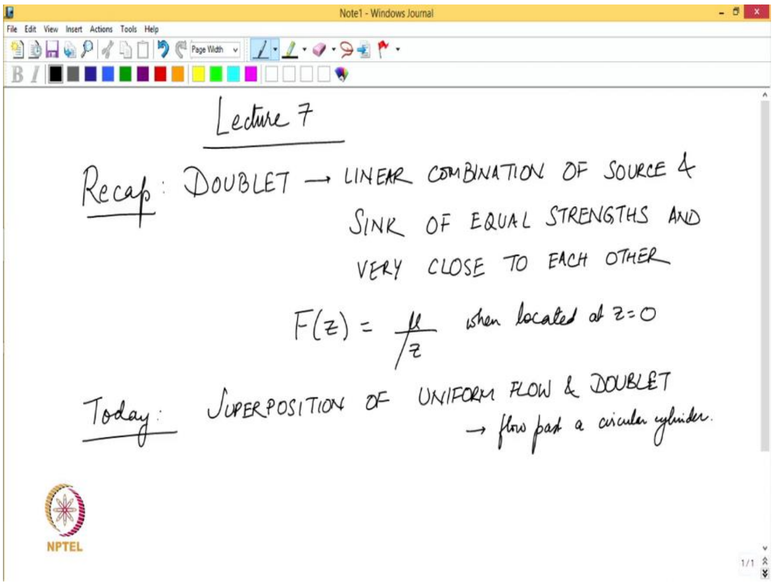

So, in that case, we showed that the complex potential for a doublet is given as mu by z, when we say that the doublet is located at the origin or at z equal to 0. So, what I will do today in this lecture is to extend that idea and show you what we get when we do superposition of a uniform flow and a doublet. And this is a very interesting problem, because as I will show you, this corresponds to flow past a cylinder, or circular cylinder. So, we will try and prove this today using the arguments that we have been working on so far.

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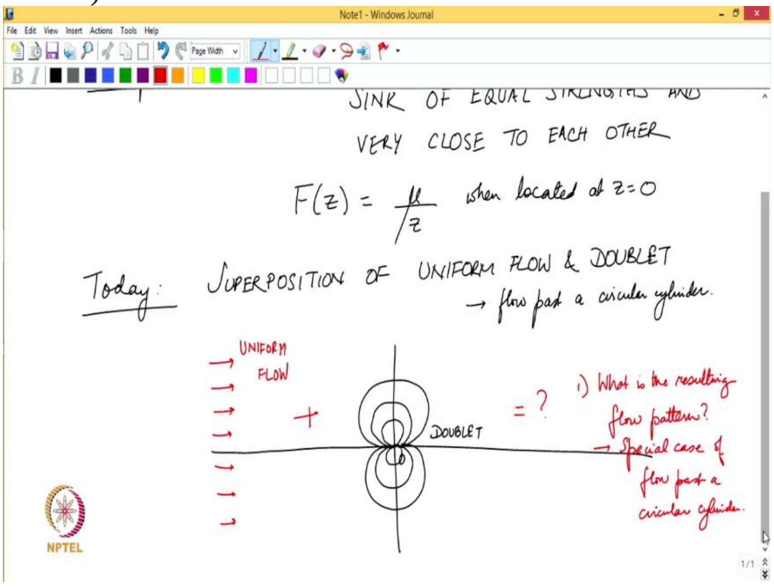

So, what is the problem saying let me show you in terms of schematic. What we are saying is that if we have a doublet which is located at the origin. So, this is the origin, and we have a doublet. So, we have some flow patterns here, this is what we derived in the last lecture. So, we have some flow pattern near the origin which is because of doublet, and I superpose, or I include a uniform flow over this doublet. So, this is uniform flow. So, when you add these two, we are going to answer what is the resulting flow pattern, that is one. And we will show that this happens to be the special case of flow past a cylinder. So, let us get started, let us see how this works out. And we will again do it by first principles, which is to go with the complex potentials which have been constructed using complex variables.

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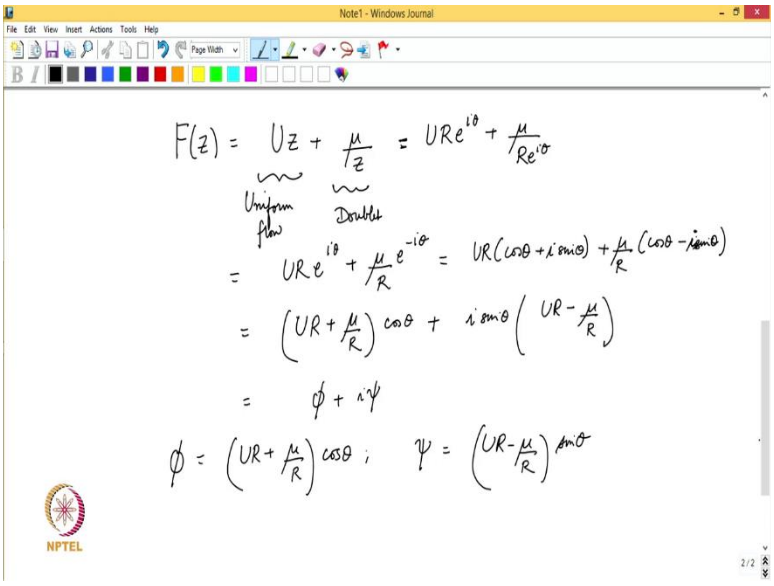

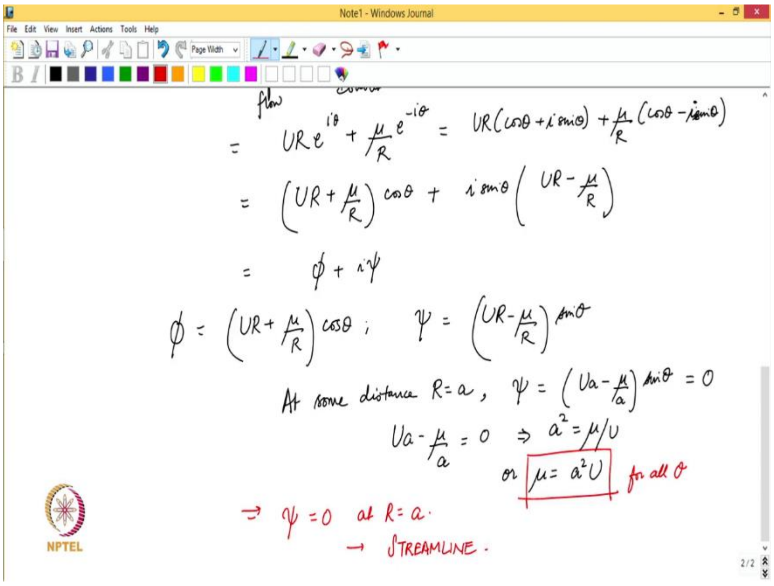

So, we can write that for this problem of the uniform flow, let us say of velocity U, and oriented along the x direction that is very important here. So, what happens to when you add uniform flow with a doublet, doublet maybe of strength say mu. So, what do we get? So, we can write that the, when we put these two together, the new complex potential should be U times z plus mu by z, where the doublet is cantered at the origin. So, this is the doublet, the complex potentially corresponding to a doublet, and this is the complex potential corresponding to a uniform flow, which is oriented along the x axis.

So, I want to now derive what do we get when we add these two potentials? So, as usual, let us again start with the polar form of our coordinates. So, we can write this as U R e i theta plus mu by R e i theta. So, we can write this as mu R e i theta plus mu by R e minus i theta, and then we use the definition of the exponential of a complex number which is e i theta here.

So, we can write this as in terms of cosine and sin functions I can write this as U R cosine theta plus iota sin theta plus mu by R cosine theta minus iota sin theta, and we can combine the real terms in 1 expression in the imaginary terms, so, we can separate them out. So, we can write this as say U in this case R plus mu by R cosine theta plus iota sin theta of U R minus mu R. So, I hope that so far, this is very routine stuff for you now.

So, I can say this is going to be the sum of the velocity potential plus iota times the stream function. So, clearly, the velocity potential is U R plus mu by R cosine theta, and the stream function is U R minus mu by R sin theta. Now, note that I did not say anything about what is the magnitude of velocity or how does the velocity compare against the doublet strength.

Now, what I am going to present to you now is sort of a you can consider this as one of those thought experiments that what if I could say that at some distance say R equal to a, at some radial distance a away from the origin, the number a is such that I could write for instance an R equal to psi which becomes U a minus mu by a sin theta, what if at some distance a this becomes 0, for all values of theta.

So, clearly sin theta will not be 0, but rather we will say that U a minus mu by A is 0, which implies that this distance a square or by the square of the distance is mu by U, or I could say mu is a square U. So, we have some distance a real number which when square and multiplied by the strength of the velocity field gives us the doublet strength.

Now, that is a fairly, easy thing to see it we need not even talk about, assuming some distance a but what we are saying is that at some distance a this must be satisfied, a could be either small or large depending on what is the strength of the doublet, and what is the velocity field that we are trying to impose, but this must happen somewhere because a can go from 0 to infinity. So, this equation must be satisfied at some location R which is equal to a. Now, if this is a case, which happens, let me say for all theta then at this distance clearly psi would be 0. So, psi would be 0 at R equal to a, since psi is 0 at this distance, it is clearly a streamline because the stream function value is a constant on this circular line.

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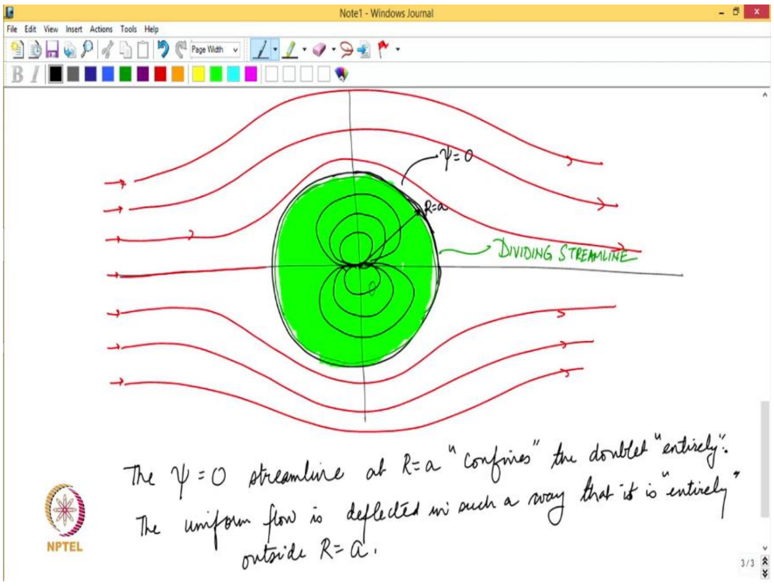

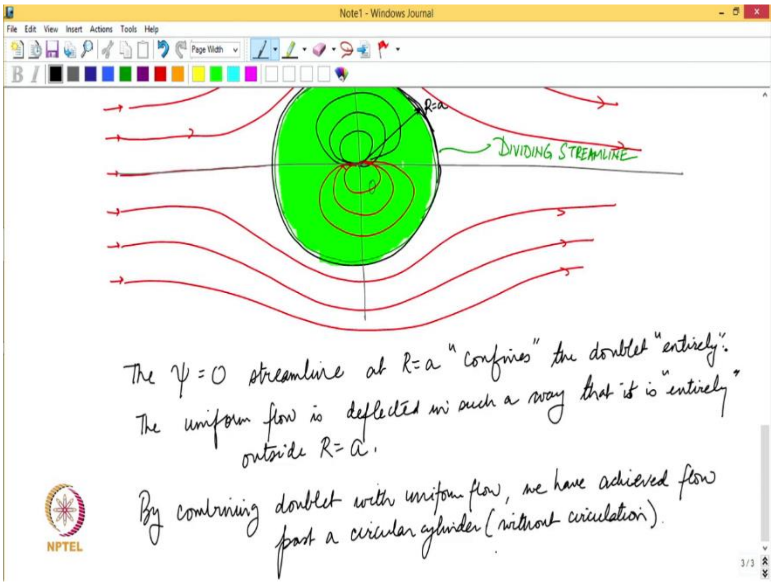

So, recall what R equal to a means, if we draw the complex plane in this case, this is the origin. So, R equal to a would mean that we are at some distance, R equal to a and the circle passing through the distance, that is a circle on this circle here. All points on this are in equal distance from the origin. Psi is 0 for all theta. So, recall now that we had a doublet at the origin. So, what we had was a doublet flow which was right here.

So, we had a doublet at the origin, and we had a uniform flow outside. Since psi equal to 0 is a streamline think about what I am going to say now, that can you make out that this surface is a line which separates the doublet from the uniform flow entirely. What I mean by that is that uniform flow that starts from outside this cylinder remains confined outside the streamline, at the streamline which starts outside this surface psi equal to 0 remains confined outside, and the doublet remains confined to this region only within the cylinder. So, I am trying to draw green line here, which is very close to the cylinder surface just to make sure that the two lines do not coincide. But basically, it is the same R equal to a line.

So, the effect of the doublet is confined to this region only so, the effect of the doublet is confined if I could probably try that, it is confined to this region only. And outside this streamline, which is now what I am calling as a dividing streamline, the uniform flow prevails or the effects of the uniform flow prevail.

So, note that now, we can say that the psi equal to 0 streamline which is at a distance R equal to a confines the doublet entirely within the circle, and the uniform flow is deflected in such a way that it is entirely outside R equal a. The point that I am making here is that the dividing streamline has ensured effect to the doublet remains within the cylinder. And the uniform flow sees the dividing streamline as an obstacle. So, to say.

And so, the uniform flow gets deflected around this, just because R equal to a line happens to be 1 across which mass transfer cannot happen, or there cannot be no fluid flow, just because the way the problem has been set up. So, now, let us begin a simple thought experiment. So, so far, we have only looked at two things, we have superposed a doublet with a uniform flow. Now, let me talk about a thought experiment with you, and this would require you to just go along with, or think about what I am saying and we will do it very slowly. So, now, we see that what happens when you have a doublet with a say in this case, the uniform flow.

Now, I know that R equal to a is a streamline and across this there is no normal component of velocity. In this case, now, if I was to slide in a cylinder, a hollow cylinder, which is of radius a, if I was to gently slide it in, into this plane, such that it is cantered at the origin, and it coincides exactly with this stream nine, which is at R equal to a.

So, say let me do this first. So, think about this in this thought experiment, I slide in a cylinder which is hollow, and which has a radius a, and it sits exactly where the streamline psi equal to 0 is located. Now, at this point, when you slide it in, there will be no change to the flow, because the surface or the cylinder would actually become a streamline, because there is no normal component of velocity.

Now, note that, at this point, since the flow remains the same, flow still remain the same, it will be frozen. Now, what I am going to do is in my thought experiment, I will pull out the doublet. So, I pull out the doublet from within the cylinder somewhere, this is the thought experiment. So, I can just go in pull the doublet out, what would happen to the flow now?

I am sure you can imagine even when I pulled out the doublet the flow will not change, because the uniform flow would be deflected because of the shape of the dividing stream line, which is the surface of the hollow cylinder. Now, finally after I pulled out the doublet I pour in the same let us say concrete maybe the cylinder is made of concrete or some steel let us say, I pour in molten steel and I allow it to solidify within the hollow cylinder, or the hollow cavity. And it becomes a solid cylinder, a solid object would the flow change, no again not.

So, the flow will again not change. And so, what we would have achieved as part of this exercise of this thought experiment that we gradually put first the hollow cylinder, a shell. Then you pull out the doublet, and then you pour something the metal or the concrete into this hollow cavity to fill it up completely, there would be no change to the flow pattern and we would still get the same flow, that you are seeing on the screen. What have you achieved here? By doing this or by combining the doublet. So, by combining the doublet with uniform flow, we have achieved flow past a circular cylinder, let me say this for now, which I would elaborate a little more in the next lecture, this is without circulation.

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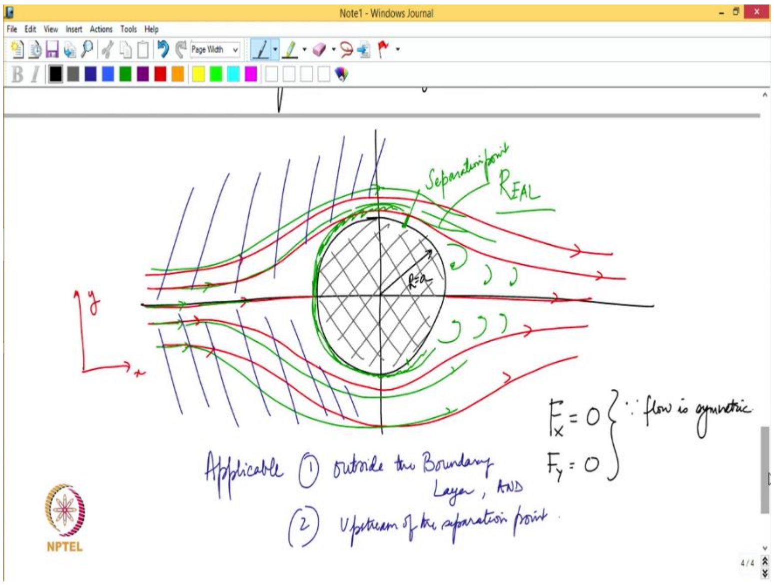

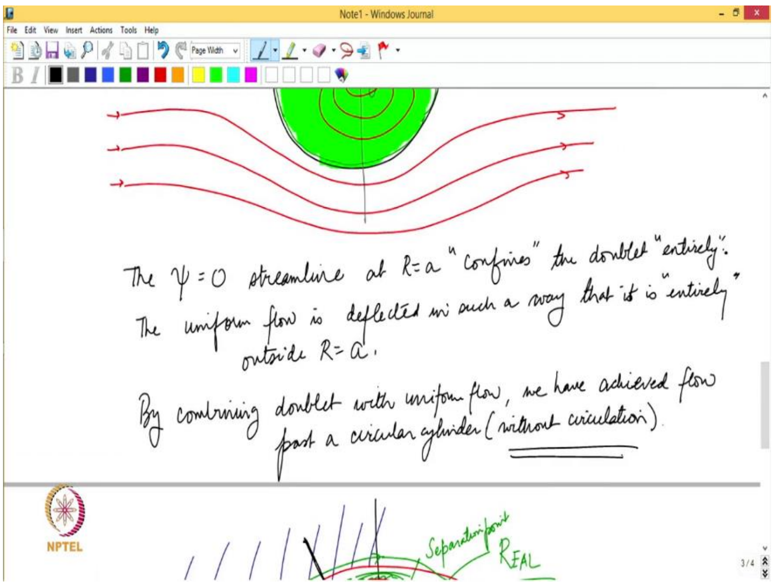

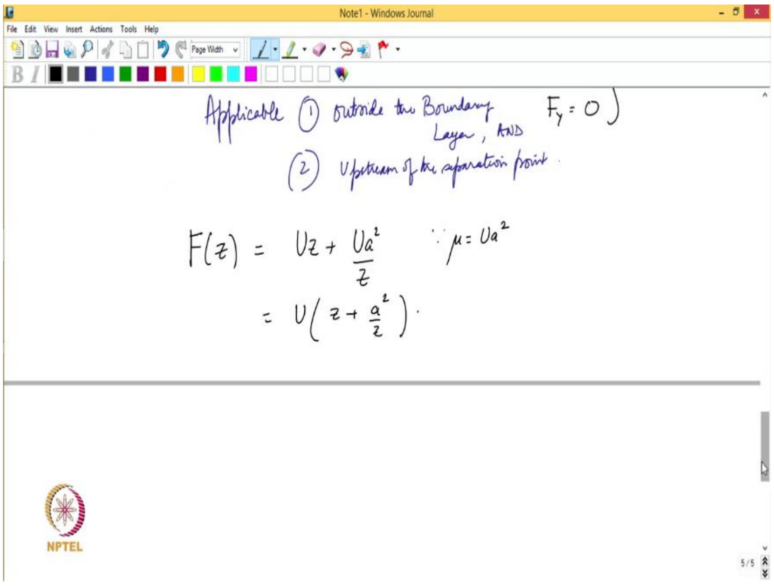

So, the flow pattern that we would have achieved now would be this, so we will have a cylinder sitting here at some distance R equal to a, and this is a solid cylinder. So, let me indicate it so, and you would have flow going around it so to say, let me also define my coordinate axes here, just to make another argument very soon.

Now, this is a very special solution because what it predicts is that on this cylinder since the flow goes over it, and we have two coordinate axes x and y, this solution predicts that force along x is 0, and force along y is also 0. And this happens, because the flow is symmetric about both x and y axis. So, it is the same flow. Like if you were looking from the x equal to 0 axis, which would be the vertical line if you see either side, you will see the same flow pattern. And if you see from y equal to 0 line, it would again be the same flow pattern, in the sense that it would be a mirror image.

So, because of the flow being symmetric, there is no force which is exerted on the cylinder. And more importantly it is also the case because viscosity has been neglected. We are not discussing any shear force in this case, but in reality, what you know is that a thin boundary layer develops. So, your thin boundary layer develops around the cylinder which also separates at some point, and we get a wake that is formed.

So, when the wake is formed, we get considerable drag force. So, this would be the real flow around the cylinder. So, when we have real fluid which was basically, which has not invested. So, you will get viscous drag and also pressure drag or form drag as it is called because, the flow separates.

So, there will be some point where the let us say, at this point the flow separates on either side. One on the upper half, the other 1 on the lower half maybe. So, there definitely effects in F y will not be 0. However, what we have derived even though it may seem that it does not match the real flow, what we have derived is actually a very unique case, because the solution is still valid in some regions of the real flow as I have shown you. So, even if the real flow is let us say, uniform flow coming over a circular cylinder, but there will be some differences, as I have shown you here, so there will be some differences, but even then, the flow that we have derived is still applicable outside the boundary layer.

So, it is applicable outside the boundary layer, and upstream of the separation point, that is very important. So, until the flow does not separate, our solution is still valid. So, what I mean by that is, our solution is still valid, so, to say in this region. So, here we can still apply this flow, but as soon as we get close to the boundary layer, or in the boundary layer, this solution will no longer be applicable.

So, conventionally, the way the earlier CFD solvers used to work, or they used to calculate flow outside the boundary layer using the same approach, that we would solve using a potential flow solution, which is what we are doing, we are solving for potential flows, a rotation flows.

So, they would solve outside the potential flow using this approach. And then they would sort of stitch the solution or marry it to the solution which is computed from within the boundary layer. So, and the pressure field basically used to come from outside So, the pressure comes from outside it is brought in at the edge of the boundary layer.

So, to say, so, you get pressure here in the normal direction, normal to the surface, let us say you get the pressure there, and then you bring it within the boundary layer, you impose it within the boundary layer. So, that is how the earlier CFD solvers used to work. Now, with fairly good advances in computational, a number of calculations we can do, and also because of better computers that we have, nowadays, we can do this, without even falling for the rotation flow outside, we can install the full scale Navier–Stokes outside the boundary layer.

But this is what is a key feature of this flow. I mentioned that this is without circulation, as we will see in the next lecture. If you have flow with circulation, then there can be a force that can be exerted on the cylinder and that would be specifically in the y direction, that we will call as a lift force, but that we will discuss in the next lecture.

So, just to summarise now, and to also give you some sense of how the, to complete the entire discussion on this problem, what we have come up with so far is a complex potential, which is given as U z plus U a square by z, recall that mu is U square. So, we can write this as U z plus a square by z, that is a complex potential for flow past a circular cylinder without circulation.

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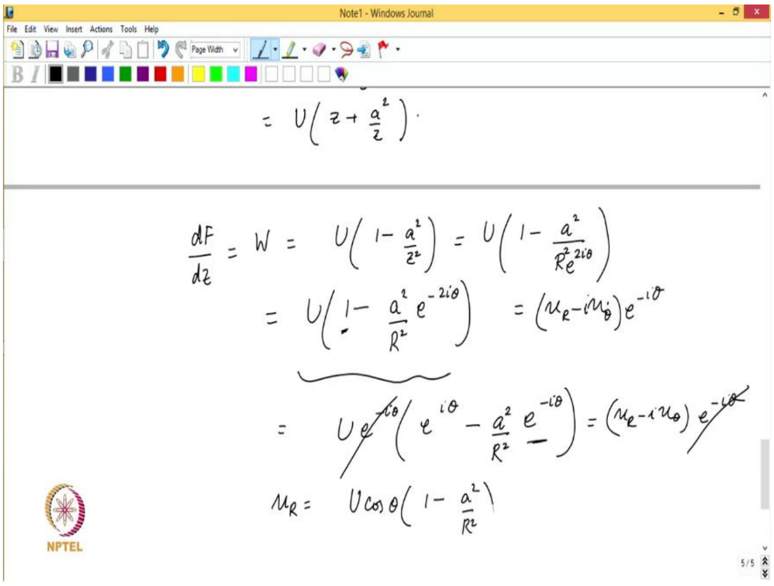

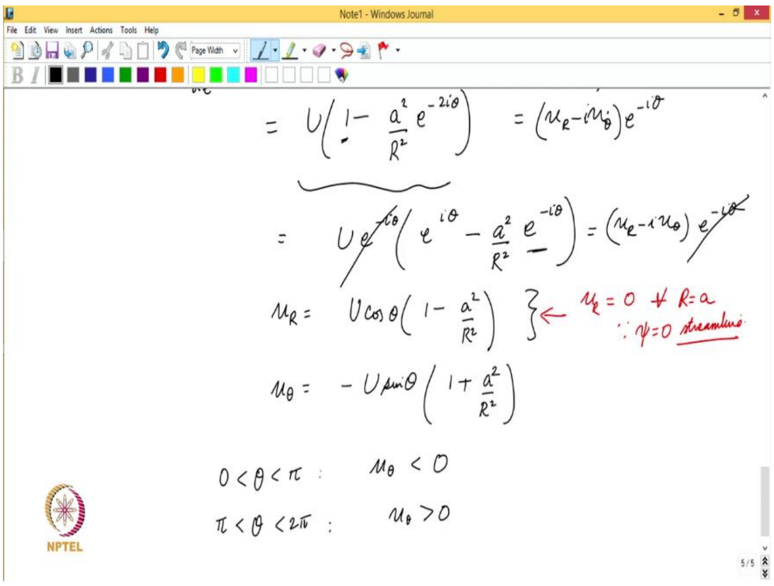

Let us try and calculate the velocity field for this flow now. Now, that I have shown you from the stream function approach, we can also calculate the velocity. So, we can take dF dz, which would be W. So that would be U 1 minus a square by z square. Now, using the fact that z is again in polar coordinates, given as R square e to the power 2i theta.

So, z would be R e to the power i theta. So, z would be R into e to the power i theta, z square becomes R square e to the power 2i theta. So, we can write this as U 1 minus a square by R square e to the power minus 2i theta. And because I want to compare this with remember u R minus i u theta into e to the power minus i theta. Well, I could sort of use this and write this as U e to the power minus i theta, times e to the power i theta minus a square by R square e to the power minus i theta.

Which is basically just pull out an e to the power minus i theta out, so you get e to the power i theta in place of unity, on place of 1. So now it is easy to compare, and separate out some terms, so these cancels, so clearly u R, would be, now remember, e i theta minus a square by R square e to the power minus i theta, will actually give me cosine theta plus iota sin theta, and this will give me cosine theta minus iota sin theta.

So, we will have to combine the real and imaginary part, we need to separate them, combine the real terms separate them out, combine the imaginary terms separate them out, and what we will get is u R to be U cosine theta 1 minus a square by R square, and u theta will be minus U sin theta 1 plus a square by R square.

So, though I have jumped a few steps here, I fully expect you to be able to work this out. So, if you look at this expression, note that u R is 0 for R equal to a. So, which is precisely the psi equal to 0 streamline. So, u R remain 0. So, there is no radial component of velocity across at streamline, which is good that we verified our earlier finding.

But let us also look at u theta. So, if I look at how u theta changes? So, let me say if theta is between 0 and pi, clearly u theta would be what do you think positive or negative? Note that 1 plus a square by R square is always a positive number, because R square would always be a positive number, a square is anyway positive. So clearly, the sin of u theta will depend on sin theta. So, when sin theta is positive, which is what it is between 0 and pi, u theta would be negative. And when theta is between pi and 2pi, where sin theta is negative, u theta would be positive.

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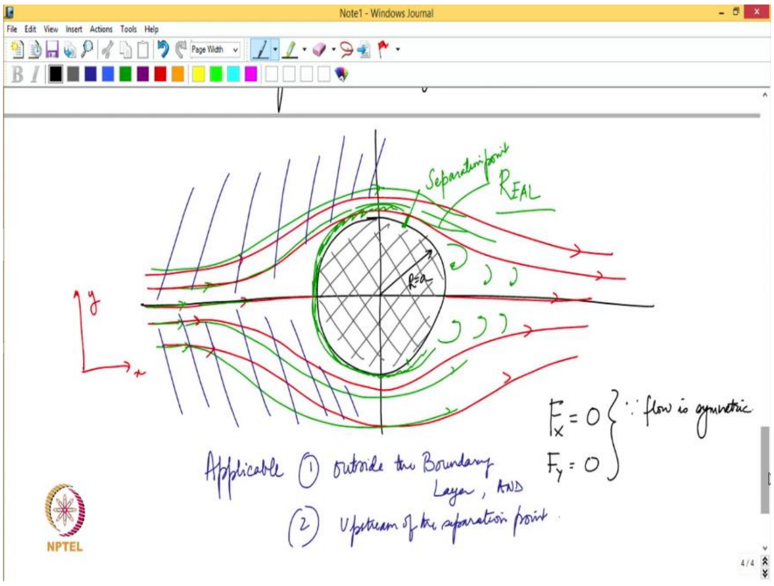

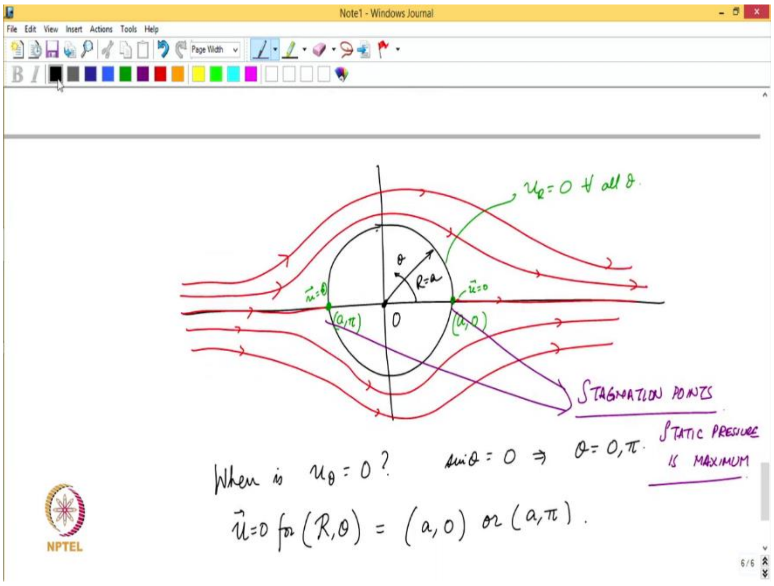

So, in this picture, that we have which we, maybe you can draw again. Now, that we have a lot of other things going on in that figure. So, what we are saying is from the point of view the origin, theta basically goes in this direction. So, between 0 and pi, u theta is negative which means that the flow must be in the clockwise direction.

So, that is what I have shown here, I am going to draw the velocity vectors going from left to right, they are actually in the clockwise direction. And when you go from pi to 2pi, so, pi to 2pi would basically mean you are in the lower half of the cylinder, then you get flow in the counter clockwise direction, with the positive sines of theta, and that is what we also see.

More importantly, on this surface R equal to a, as I said u R is 0 for all theta, but we also see that u theta can be 0 at two locations. So, when is u theta also equal to 0? This will be when sin theta is 0, which implies theta is either 0 or pi. When theta is 0 or pi, sin theta will also be 0, which means in terms of coordinates, say R comma theta, I can say the velocity U vector if you want to put it, this would be 0 for which values of R and theta. So, this would be 0 when R is a, theta is 0, or when R is a, and theta is pi. At these two locations, the velocity is 0, and where are those two points? So, those two points are, these two points, this point is a comma 0, because theta is 0 there, and this point is a comma pi.

So, these two points, U is 0, and these are called as stagnation points, these are stagnation points of this flow. So, stagnation points are those where the static pressure is maximum, in this case the static pressure would be the same at both of these two points, and that is the reason again why the cylinder does not feel any force along the horizontal direction. So, this is all about flow past a circular cylinder obtained by combination of a uniform flow with a doublet. And we saw in this case that the total force on the cylinder is 0.

So, there is no force along the horizontal or vertical direction. What we will do in the next lecture is to derive a solution in which there is some force over the cylinder, but that would only be because of pressure. As we are dealing with a rotation flows, there is no shear force on any solid object. So, there would be only form drag, or if you want to put it this way, and we will derive what the solution would be using the same complex potential method. So, I hope that this part is clear, and we will take it further in the next lecture. So, thank you.