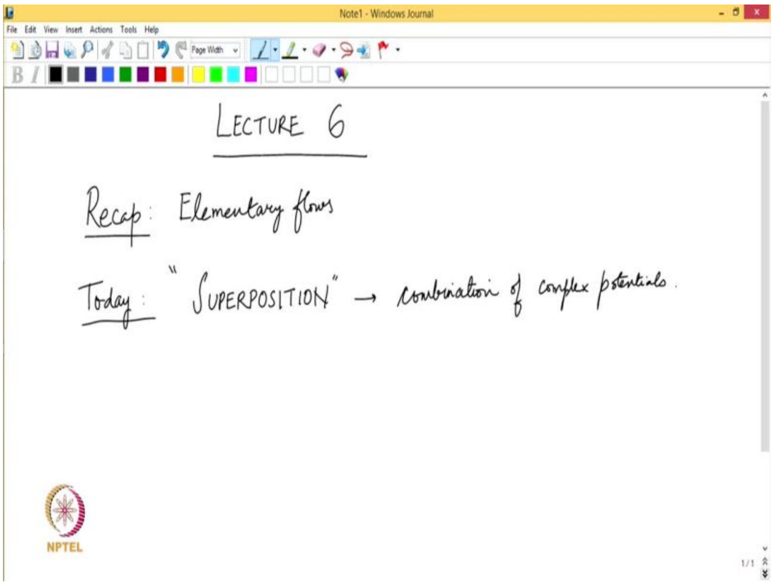
Ideal Fluids Flows Using Complex Analysis

Professor Amit Gupta
Department of Mechanical Engineering
Indian Institute of Technology, Delhi

Lecture 6

Superposition of Source and Sink: Double Flow

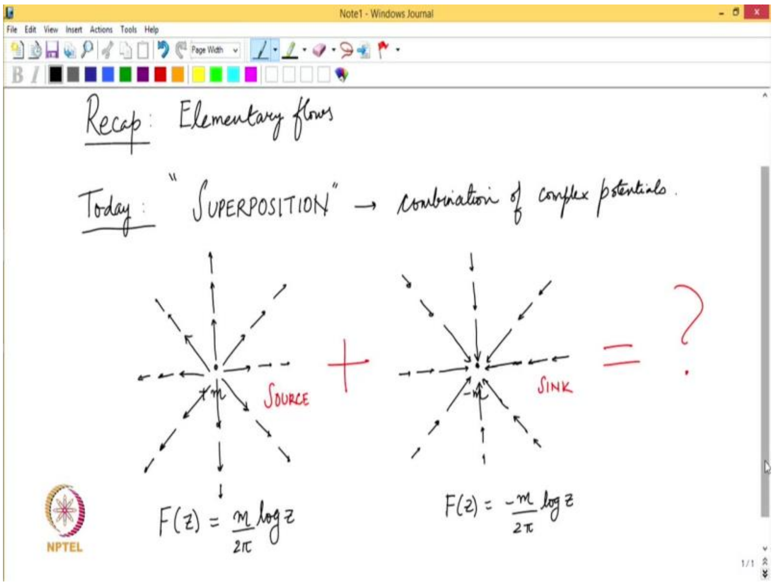
(Refer Slide Time: 00:17)



So, in the previous lecture, we looked at 2 more examples of elementary flows. Namely, we looked at flow near bend. And then we also looked at flow over a sharp edge. So, these are all examples where we had one single complex potential, and we could derive the flow field around that system or in that system.

So, today I will take the first example of superposition. And in the case of superposition, I would basically talk about what happens if you start combining complex potentials. So, the idea of superposition that we will cover today is looking at a combination of complex potentials when I mean by combination is linear combination.

(Refer Slide Time: 01:21)



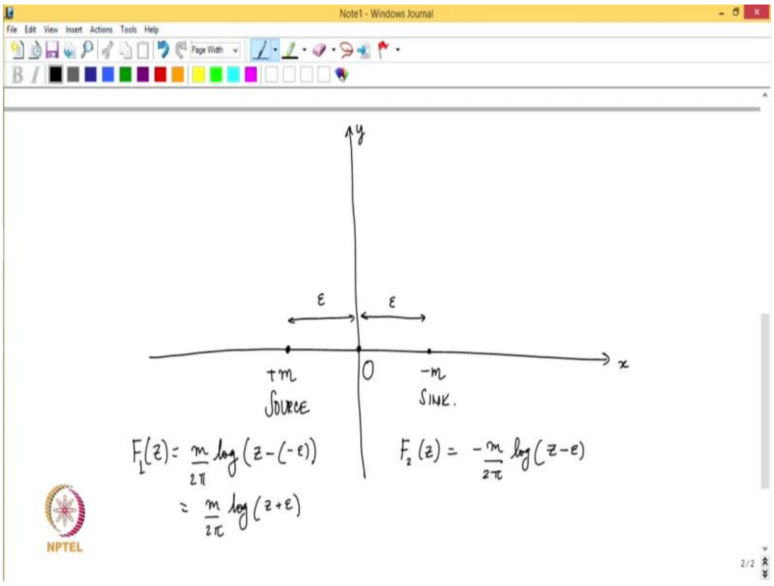
So, I will take a very classical case, which you would have seen in your undergraduate curriculum, but I would actually try and approach it from a slightly different perspective and also try and give you some insight into how do we develop this specific flow problem. So, let us say if you remember what a source is? A source is basically, let us say a source of strength plus m is something which gives out flow radially.

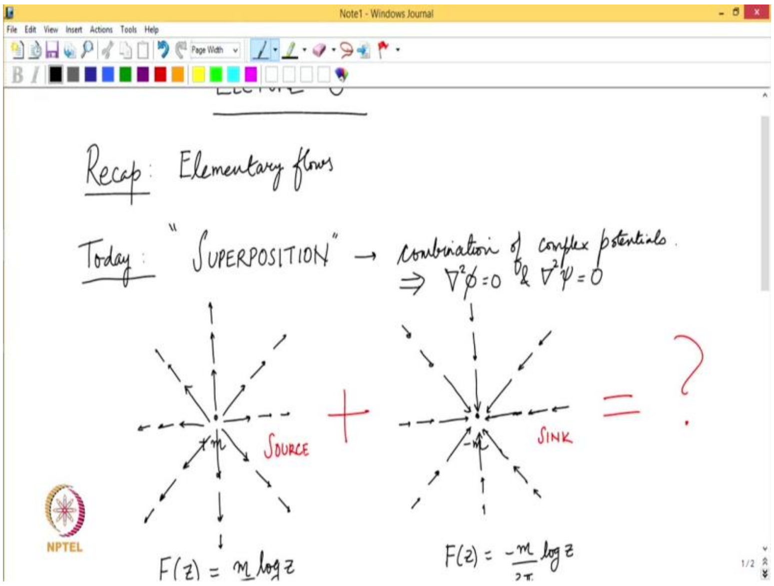
So, if I was to plot for instance, the velocity vector of flow going out of a source, you would see that the velocities are in a certain way of course radially and decreasing as we go further away from the source. So, something of this type maybe something like that, and what if I have a sink as well.

So, say I have a sink, maybe your strength minus m, then a sink is one which would attract flow. So, we will have velocities in this fashion. So, remember that a source will be a source for the flow to leave and the sink will be where you have all the flow coming in. Now, in either of these cases, the velocity is actually inversely proportional to the distance from the origin. Alright, and so, we knew that the complex potential for a source of strength m and located at say the origin is m by 2 pi log of z.

And similarly, the complex potential for a sink located at the origin would be minus m by 2 pi log of z. Now, this is something that we have already covered. Now, the first example of superposition that I want to pick up is to say that, what if I have a source and sink in the vicinity of each other? If they are very close to each other, what kind of flow patterns would you see? Alright, so in some way, what I am going to do today is to say what if I add a source and a sink? So, what would be the resulting flow pattern? So, I hope this is clear so far what I am going to do and that is the main agenda for this class.

(Refer Slide Time: 04:49)





Alright, so, let us do the problem in this way. Say, we have the x y axis and I am going to put the source and sink on either side of the origin. So, what I mean by that is that I put the source of strength plus m at a distance epsilon from the origin, where epsilon is a small number, and let me say I would put a sink of equal strength, denoted by minus m, at again the same distance from the origin. This is the problem that we want to work out.

Now, remember that, by itself, the source is a will radiate out flow, the flow will leave out the source along the radial direction, and the velocity will decrease as you go away from the source. And the same will happen with the sink that as you get closer to the sink, the velocity will be high. But as you go further away, the velocities will be smaller.

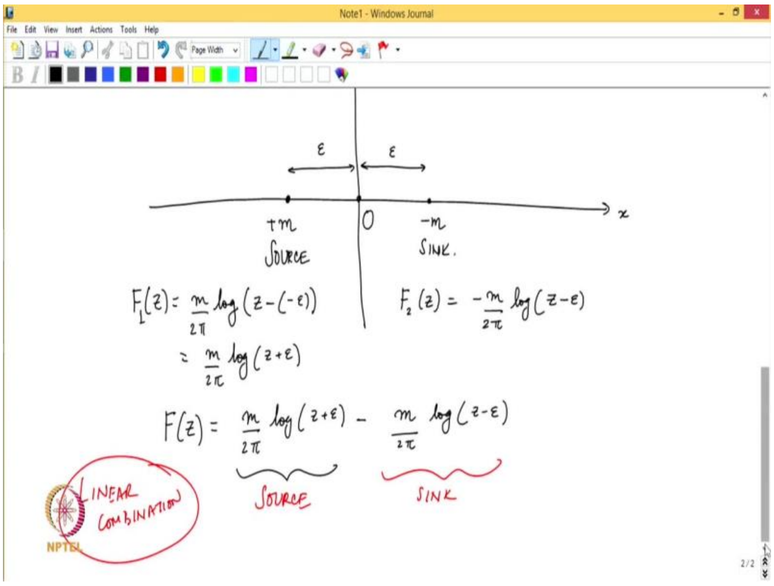
Now, there is a very classical analogy of this, what we are going to do, which is, if you look at, you know, in electrostatics, if you have 2 charges of equal magnitude, but opposite signs, if you put them close to each other, what kind of fields do you create? Something similar is going to happen here. Alright, something similar, but we will see the form of the complex potential is slightly different. So, the solutions will be slightly, at least physically, the problem will have some resemblance. That is what you want to bring about, or you want to illustrate it.

So, now the idea about superposition, just to remind you, again, when I am saying superposition, and I am saying we will combine complex potentials. This idea comes from the fact that you remember both our functions phi and psi satisfy Laplace equation. So, if I have 2 complex potentials, I can linearly combine them and still be able to satisfy these 2 equations for each of those complex potentials.

So, that is the whole idea of using linearity in combining complex potentials, so now for this problem that we are working on, what do we get? So, let us say we have as I said, we have a source here and a sink at this location individually, let me first write down what are their complex potentials? So, what about the source? What is the complex potential, say F 1 of the source that would be m by 2 pi log of z, it is not this m by 2 pi log of z. I hope that you understand this is not right.

Because the source is not located at the origin. It is rather located at some distance away from the origin. So, the distance is epsilon, but it is actually towards the negative x axis. So, we will have z minus minus epsilon as the complex potential corresponding to the source so we can write this as m by 2 pi log of z plus epsilon. What about the sink? Say F 2 z? In this case, this would be minus m by 2 pi log of z minus epsilon. That would be the complex potential corresponding to the sink.

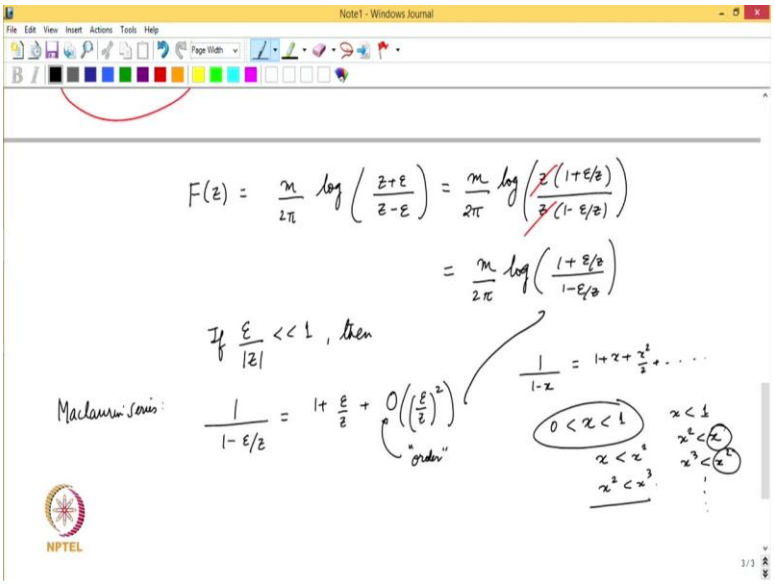
(Refer Slide Time: 08:29)

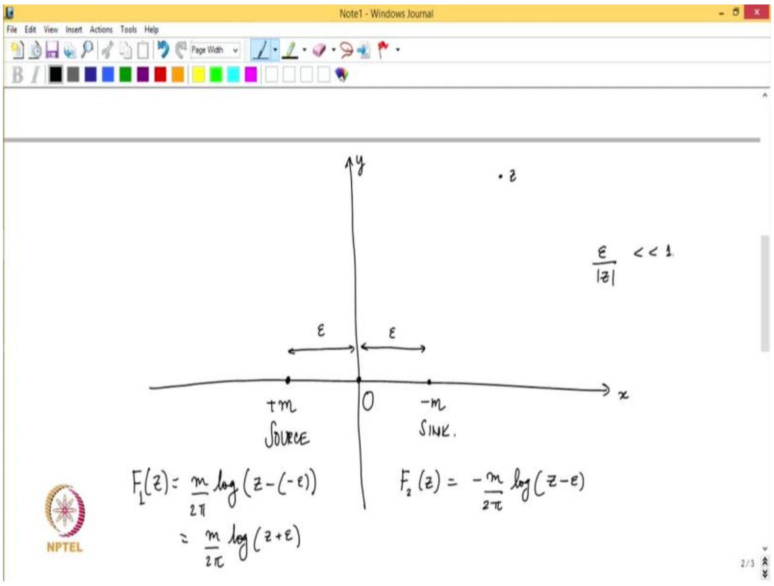


Now, what I am proposing is to combine these 2 potentials, linearly. So, let us say F of z, which is a complex potential for this problem that we are working with which is source and sink in the near vicinity of each other. This would be m by 2 pi log of z plus epsilon plus minus m by 2 pi log of z minus epsilon. I have taken great care in writing a plus sign here.

So, that you know that I am combining them linearly instead of you know, just it may appear that if I was to just put minus m by 2 pi log of z minus epsilon, it may appear to some people that I have sort of put 2 sources on maybe I am subtracting 1 from the other that is not the case. So, I can I can very well get rid of this here and write this as minus m by 2 pi, that becomes the complex potential coming from the source and the sink, so this is a linear combination of a source and a sink.

(Refer Slide Time: 10:10)





Now let us take it further. So, we will get F of z to be m by 2 by log of z plus epsilon by z minus epsilon, I can divide the insides of the logarithm by z, so I can write this as log of, for instance, I can say this is z times 1 plus epsilon by z divided by z times 1 minus epsilon by z by doing so, I get rid of these factors z. So, we have m by 2 pi log of 1 plus epsilon by z by one minus epsilon by z.

Now we get to an important consideration. Now, I say that what if I am trying to observe this combination of source and sink from some point z in the complex plane, which is maybe so far away from the origin? That the ratio epsilon by modulus of z, is much, much less than 1? So, I am saying that we are very far away from the origin. So, that as an observer to us from by the naked eye, this it appears that the source and the sink are almost at the origin, almost. Not exactly it, but almost.

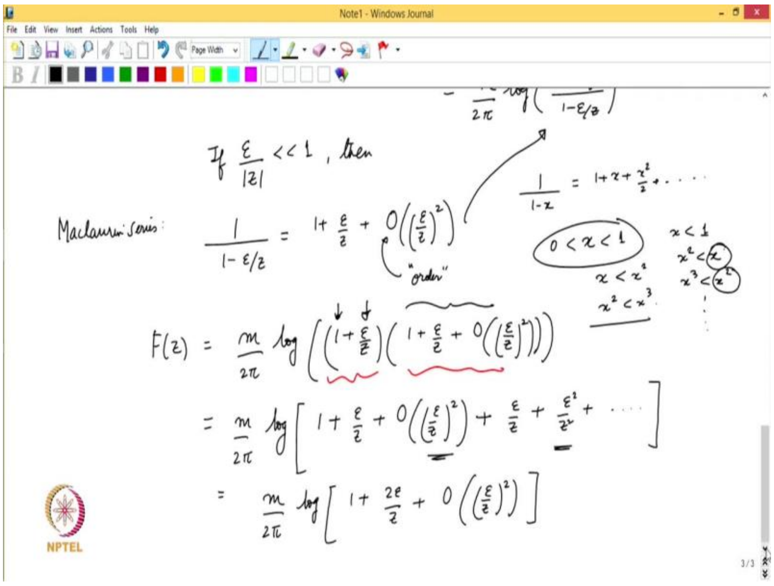
What I mean by that is that if we say, if epsilon by modulus of z could be considered to be much, much less than 1, then I could write 1 by 1 minus epsilon by z in terms of the well-known Maclaurin series. The Maclaurin series basically says that 1 by 1 minus x where x is a very small number, but greater than 0 could be written as 1 plus x plus x square by 2 so and so forth.

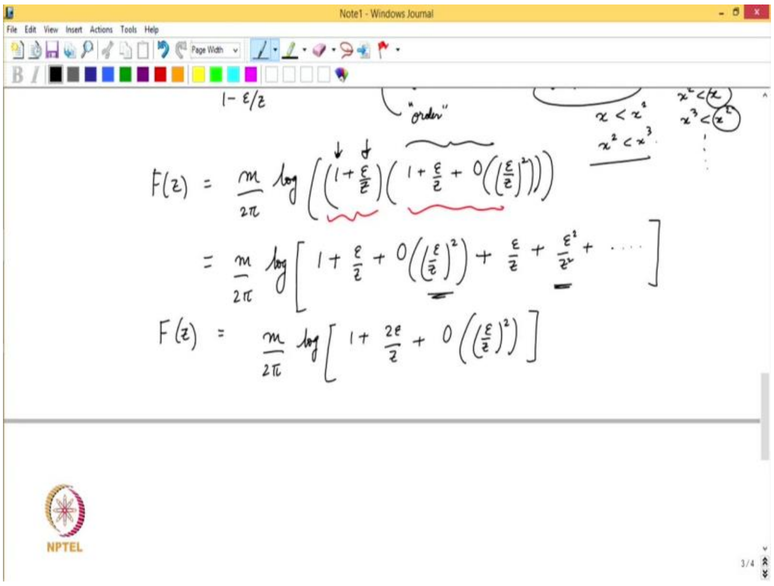
So, we can write 1 by 1 minus epsilon of z as 1 plus epsilon by z plus, you know, terms, which would be epsilon by z square, then there will be epsilon, by z Q, and so on, and so forth. So, what I am going to do is I am just going to write them as all terms, which are of the order of epsilon by z square and higher. By O, this means order that these terms are of this degree, it would be an epsilon by z square.

The next would be upside epsilon by z cube, and so and so forth. So, the, for a small number epsilon by z, the largest number will be epsilon by z square. In other words, if you say x is a small number, say x is less than 1 greater than 0, you should know that x would be less than x square, x square would be less than x cube and so and so forth.

So, that can easily prove by this only since x is less than 1, I multiply both sides by x, we will get x square as an x, if you multiply again by x, we will get x cube is less than x square, and so forth. So, the dominant term is basically in this case, clearly, if I truncate the series to linear part, and use higher order terms as the truncation error, then this is the highest order term. In this case, this is the highest order term. So, I am using the x squared term as a highest order term.

(Refer Slide Time: 14:31)



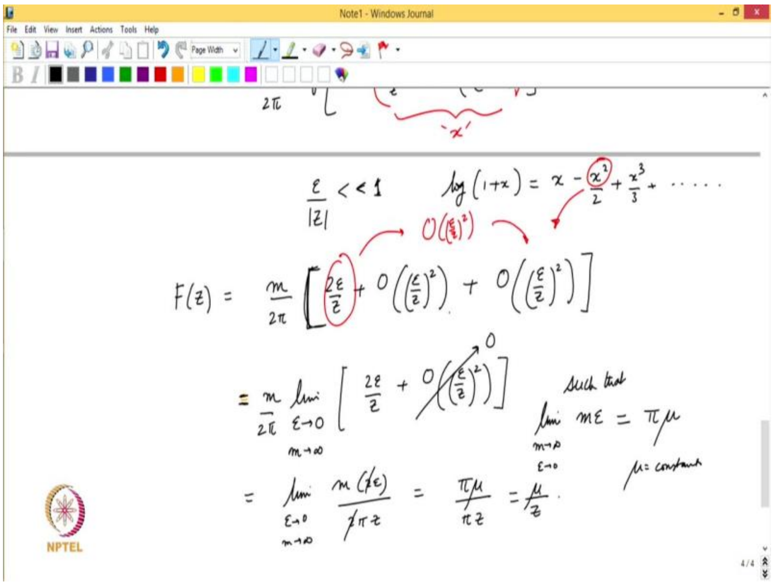


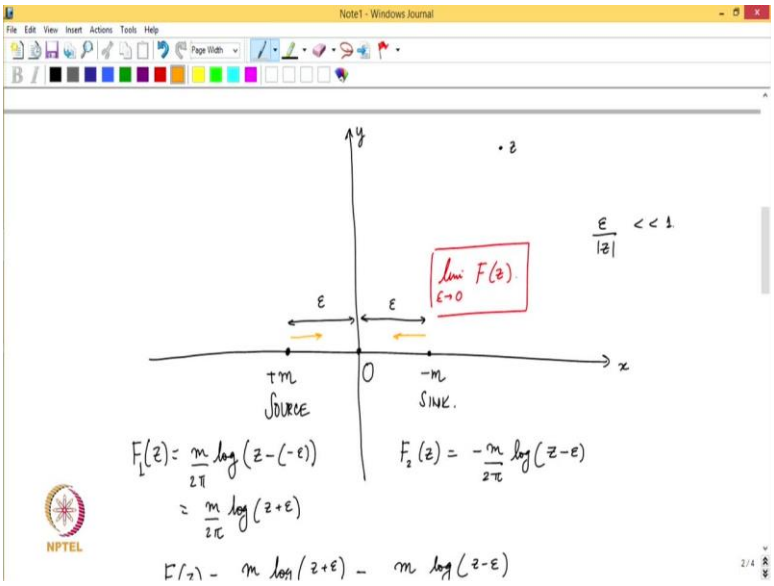
The whole idea of doing this is if I can plug this in here, then we can write that F of z is m by 2 pi log of we will have a 1 plus epsilon by z multiplied by now 1 plus epsilon by z plus order epsilon by z squared term. So, what do we get if we multiply this quantity with whatever it is here, what we will get is, so we start doing the cross multiplication, so we will have 1 plus epsilon by z plus order of epsilon by z squared terms.

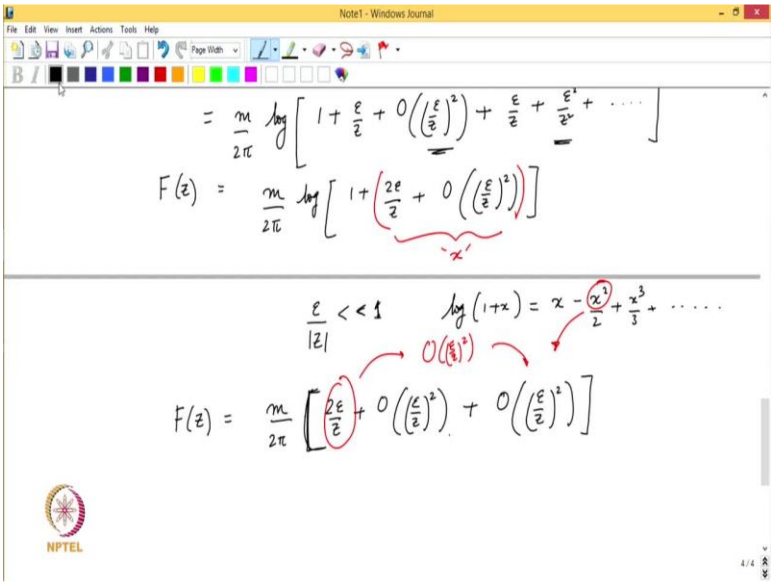
This is when I multiply 1 here, with everything that is appearing here. Then the second time, I will multiply this number with whatever appears inside the other bracket, so we will get epsilon by z plus epsilon square by z square, plus higher order terms, which I am not very concerned about.

But what I am now more interested in is the fact that this term and this are of the same order, so they must be they must add. So, what I can do is I can write this as 1 plus 2 epsilon by z plus again in order epsilon by z square term. So, I hope that so far, you have been able to get to this point in this algebra. Alright, so this is F of z so far.

(Refer Slide Time: 16:23)







Now, we apply one more aspect. Note that we said if epsilon by modulus of z is a very small number, it is a very, very small number. So, if I use this idea, that log of 1 plus x has an expansion as a Taylor series expansion, which is given us x minus x square by 2 plus x cube by 3, so and so forth. Right for small x, can we then write F of z using this expansion, so remembering that, or noting that this becomes a small number, so log of 1 plus 2, this sort of becomes our x, so to say.

So, can we write this as m by 2 pi. So, now we will have 2 epsilon by z plus order epsilon by z square. And let me put a square bracket here plus, the second term would be even of higher order. So, that would definitely have an order epsilon by z square term again. Well, why because when you square this to epsilon by z term, that would when you square it, it would again give you an epsilon z square, which is what you are seeing here, from this expansion, so this term corresponding to this term, we get another order epsilon by z square.

But we need not worry about it. That is the whole point that I am trying to explain here, that higher order terms will not have any impact on our result. And there is a reason why I am saying this. Now, let me say that, at this point, say I am so far away. That and I start to take this limit that the number epsilon is tending to 0.

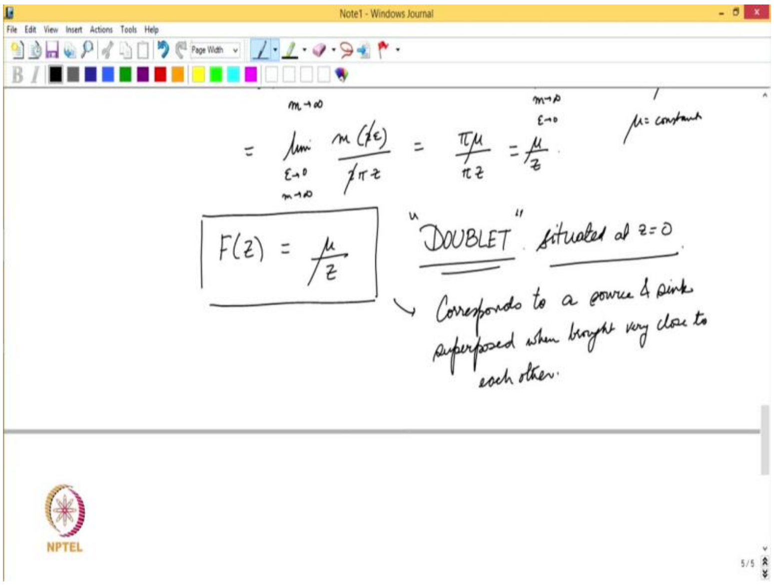
And what happens to our complex potential in that case? Which means that from either side, I am bringing the sink, ever closer to origin, and I am also bringing the source ever closer to the origin. So, they are both approaching the origin. So, that from a distant observer, it may start appearing that the these 2 the source and sink are almost one on top of the other. Not exactly, but still, you know, in the in the mathematical sense, there are one on top of the other.

In which case, I would now write this as m by 2 pi. Say limit epsilon goes to 0 of 2 epsilon by z plus order epsilon by z square. Now, in the limit that epsilon is tending to 0 of course, this will have no effect, this higher order term will be very small, so we can consider it to be negligible. But let me also say that as I am bringing them more and more closer to each other.

I also consider the scenario that the strength of these 2 elementary flows goes to infinity, which is that they become more and more I mean, it is it is, their strength is also very high that m is nearly infinity, but in such a way that this product m times epsilon in the limit that m goes to infinity and epsilon goes to 0 approaches a constant number.

So, let us say this is pi times some number mu, where nu is a constant. So, we bring them so close to each other, but their strengths are also going up, but the product of the strength to the distance remains a constant. In that case, I can write the complex potential as say limit epsilon going to 0 m going to infinity will have m times 2 epsilon by 2 pi z to be equal to say, 2 cancels here, m times epsilon because pi mu, so we will have pi mu by pi z which is nu by z.

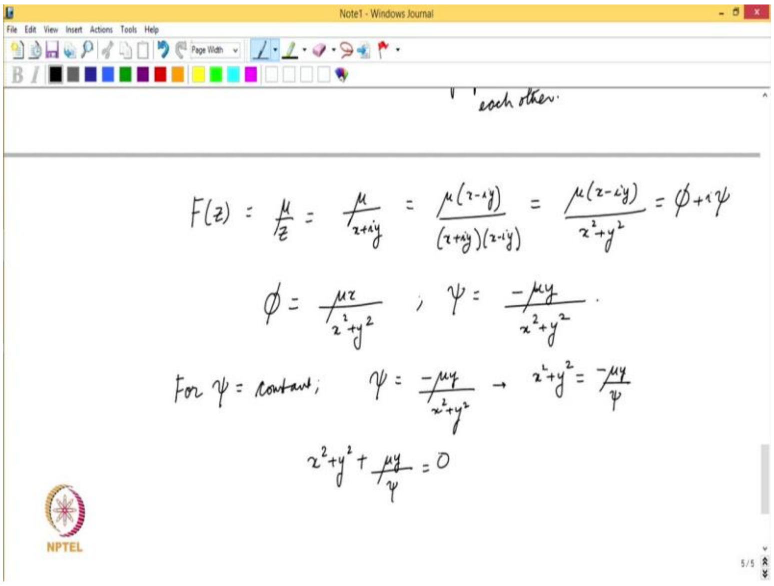
(Refer Slide Time: 21:15)

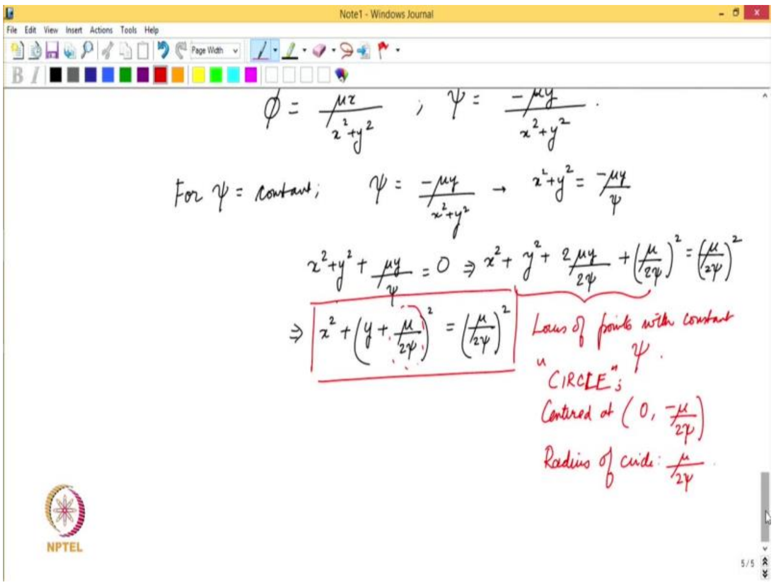


So, what we have reached is a complex potential F of z which is given by nu by z. This is a very important complex potential, which is what we call as a doublet. Doublet is made when you have a source and sink very near to each other. Now, this is also a doublet that is situated, I should say doublet situated at z equal to 0, what I mean by z equals 0 is the source and sink have been brought very close to each other, and that they are almost at the origin.

All right, this complex potential again, keep remembering that this is a complex potential which corresponds to a source and sink super post when brought very close to each other. Alright, so, what we will do is? We will look at the kind of flow pattern that you would see in this doublet, and also what kind of stream function and velocity potential that you get out of this doublet.

(Refer Slide Time: 23:02)





So, to derive the form the stream function, let us first write F of z, which is nu by z, but let us write it in Cartesian coordinates for a change. The answer can also be obtained what I am going to prove to you is also possible to derive using cylindrical coordinates. But for a change, let us do it using Cartesian coordinates.

So, we will write this as mu by x plus i y, which I can write as mu times x minus i y by x plus i y x minus i y. Basically, I want to bring the imaginary number on the numerator, I do not want it to be in the denominator, because eventually I am going to write this as phi plus i psi. So, we will write this as mu x minus i y x plus i y into x minus i y would just give me x square plus y square, the square of the modulus and which should be phi plus i psi.

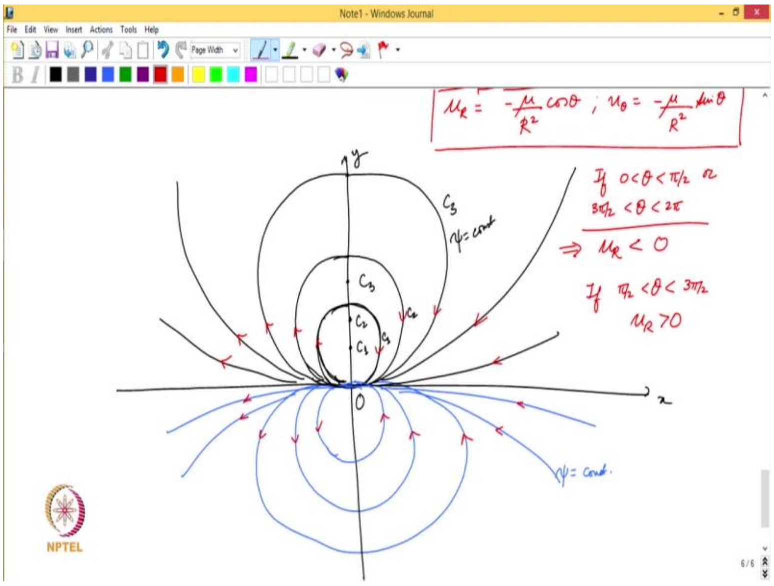
So, we can see that the velocity potential would be mu x by x square plus y square for this flow and the stream function will be minus mu y by x square plus y square. Now, as usual, let me define the psi 0 streamline. So, for psi to be say constant. Let me not do the 0 first. Let me just do the psi constant case first.

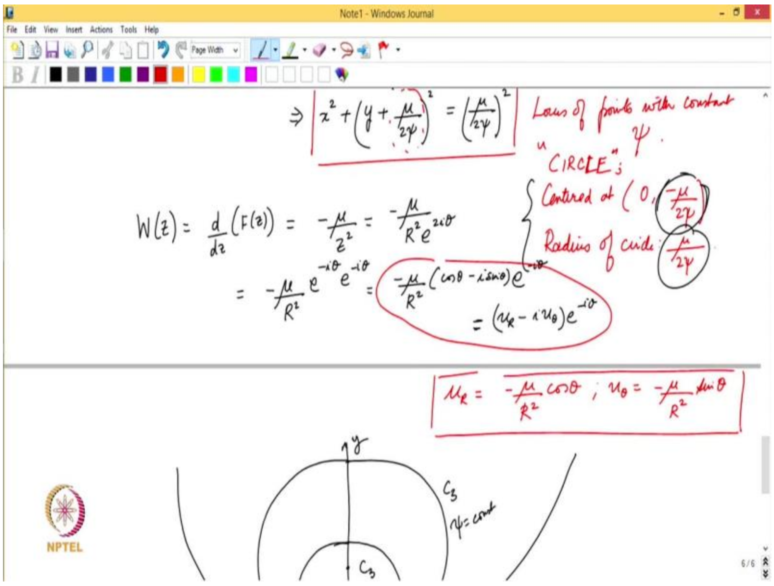
For psi to be constant, what do we get? If psi was constant, then since it is given by this number, or this dependence, we will have x square plus y square to be equal to minus mu y by psi where psi is a constant, I can write this as x square plus y square plus mu y by psi to be 0. Now, I notice that this is not going to be a linear function, but it has some non-linearity, but it has a very special form of non-linearity that it has a y squared here and then it has a mu y by psi.

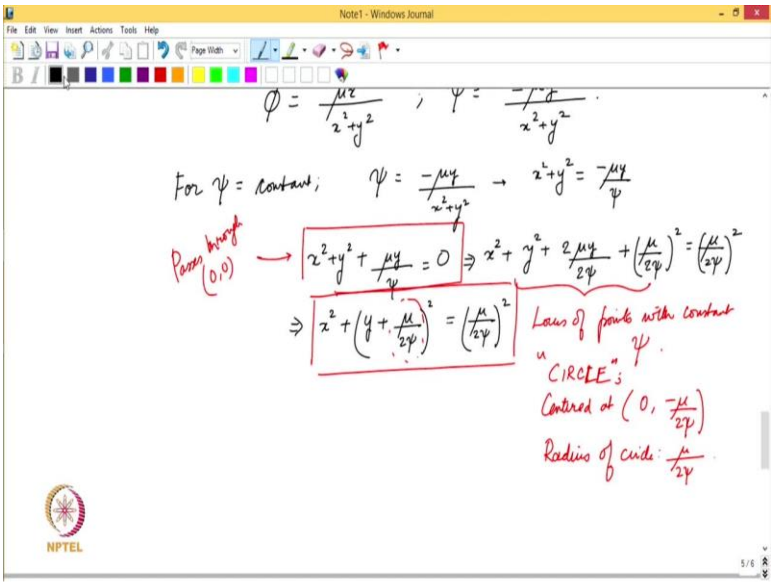
So, from my knowledge of coordinate geometry, I can see that probably this corresponds to the equation of a circle in which way let me show you this. So, this is let us say, if I write this as 2 mu y by 2 psi and I add for instance, mu by 2 psi square, and I add the same number on the right-hand side, I can write this as x squared plus now noting that this is just y plus mu by 2 sides square and the right side is mu by 2 psi square as well.

This then becomes a locus of points which have a constant value of the stream function with constant psi. And now I think that we should be able to make out this locus is a circle. And what is it where is the circle centered, the circle is centered at 0 comma minus mu by 2 psi, because I can see that this gives me the negative of that number gives me the location of the center, what is the radius of the circle? That is what appears on the right side of this equation. So, that was mu by 2 psi very interesting. So, we have a circle, which has a center on the y axis has a radius of mu by 2 psi.

(Refer Slide Time: 27:51)







And so, if we start now, putting this together or plotting this function, let us start putting this on the screen here. So, this is x and y. So, locus of constant psi are these circles, which are located at 0 comma minus mu by 2 psi. So, let us say if psi was negative, then centers would be for instance, let us say one sentence here, corresponding to it, the circle would be of this type, it would pass through the origin. Can you see that there is pass through the origin because if you look at the function definition itself, the functional form, it is clear that this equation here it passes through 0 comma 0.

The other way to look at it is if I look at this form the circle it is centered at minus mu by 2 psi and the radius is mu by 2 psi. So, if I add these two, if I add this in this it gives me a 0. So, clearly, the circle must pass through the origin. Let us say the center is C 1 corresponding to some value of psi. Then I can also have another circle C located at C 2 which would look like this one, so this is C 1, this is C 2. And then we can take maybe one more case say C 3, which would be somewhat like that. These are all lines corresponding to some constant value of psi, the same thing will also happen if psi is positive if psi is positive then the centers will be shifted down. So, we will have circles of this form.

We have this stream function on the on the bottom half as well, and in fact, the circles will grow in size. So, much so that probably the largest one which I can sort of show here would have a very large radius. So, it would appear like these two circles will have ever growing sizes on either side, so, this is again for psi constant.

Now, that looks to be the flow pattern that we will get out of this doublet, but what about velocity, we need to define the direction of velocity. So, we need to look at what is a velocity or the complex velocity in this case, which would be d by dz of F of z. So, this becomes minus mu by z square.

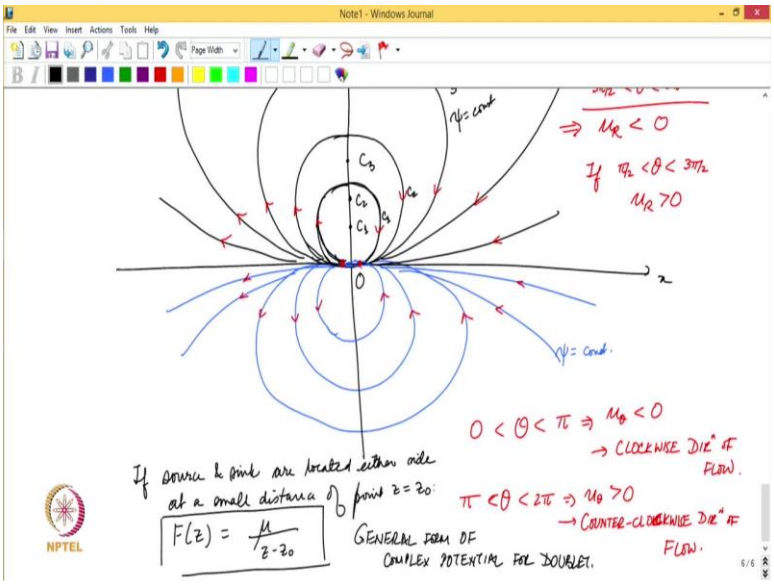
Now, here it makes sense to use a cylindrical coordinate to define the complex velocity. So, I will write this as minus mu by R square e to i theta and I would then write this as minus mu by R square e to the power minus i theta times e to the power minus i theta. So, this we can write as going further minus mu by R square of cosine theta minus Iota sine theta times e to the power minus i theta which should be equal to u R minus i U theta e to the power minus i theta.

So, now, if we compare the two sides of this equation, we will get u R to be minus mu by R square cosine theta. And we will get u theta to be minus mu by R square sin of theta. So, this is the velocity for this flow. Now, let us define the directions in which way is the flow in this in this problem, so, we look at say if theta is between 0 and pi by 2. So, we are in the first quadrant or if we are in the fourth quadrant. In either case, you can see that cosine theta would be positive. So, u R would actually be negative in this case.

Similarly, if theta is if theta takes up all other values, So, theta is between 3 pi by 2 and greater than pi by 2. So, basically which means second and third quadrant then u R will be positive because cosine theta will be negative. So, what does this tell us about this flow now, so, if we start labeling directions between the first and fourth quadrant u R is negative which means that the flow is inwards towards the origin.

So, it means the velocity vectors must be in this form or pointing in this direction in these two quadrants okay there is flow coming in. In the second and third quadrant, u R is positive which means the velocity must be pointing away from the origin. So, that must be give us the direction of flow in this fashion. So, we have you can see now we have if I can be a little more, you know, casual about it in the first two quadrants, we have clockwise flow. In the lower two quadrants, we have anti or counterclockwise flow which we can also verify using u theta.

(Refer Slide Time: 35:26)



So, for u theta. So, if I look at theta between 0 and pi, which is the first two quadrants, in this case u theta which is given us minus mu by R square sine theta right, sine theta is positive in the first two quadrants, so, u theta would be negative which is what is defined in terms of clockwise flow and if theta is between 2 pi and pi, u theta will be positive which implies the counter clockwise direction of flow. So, that is precisely what we have achieved.

Now, I hope this is so, far clear the key aspect that I want to now put up is that we had taken the scenario where the source and sink were very near the origin and we brought them very close to each other so, that they collapsed on the origin itself so, to say, but what if the source and sink are not at the origin but they are somewhere else. So, maybe they are at some other location they are at some location z naught or centered about z naught if source and sink are located either side at a small distance of point z equal to z naught and say they have equal strength of course, that is needless to say.

In that case the complex potential would be mu by z minus z naught. So, we need to remember this the complex potential in the general sense, for a doublet is mu by z minus z naught where z naught in our situation in the problem that we took up was actually 0. So, we just got F z to be mu by z, but when we go to some other location where this combination is located then this will be the general form of the complex potential for a doublet.

So, what we cover today is the first case of flow superposition and we took up the problem of linear combination of a source and sink. So, in the next lecture, we will look at a few more such scenarios where we can linearly combine especially the case where we can combine a doublet with one of the other elementary flows that we have taken and namely specially the source flow and we were try and derive what happens or how do you recover flow over a cylinder, a circular cylinder. So, that will follow in the next lecture. So, for now, I think we can stop it here. Thank you.