Ideal Fluids Flows Using Complex Analysis
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Lecture 5
Flow in a Bend, Flow Around a Sharp Edge

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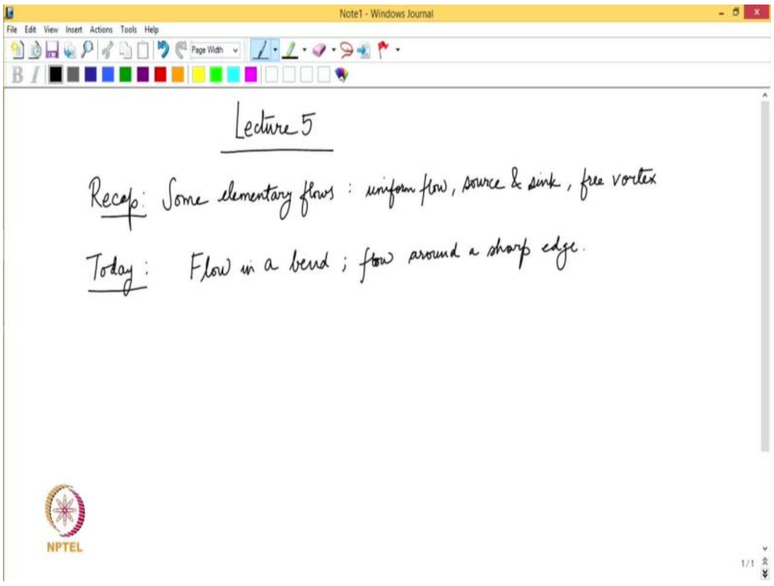

So, let us just do a quick recap of what we did in the previous lecture. So, as you would recall, we looked at some elementary flows, and namely, we did three flow potentials, uniform flow, source and sink and a free vortex. Now, in this lecture, I will talk about two more flow scenarios, one would be flow in a bend, other one is called a flow around a sharp edge. And these are namely flows, which you typically see around a solid boundary or a solid object. As compared to what we did in the previous lecture, which was away from boundary, so to say.

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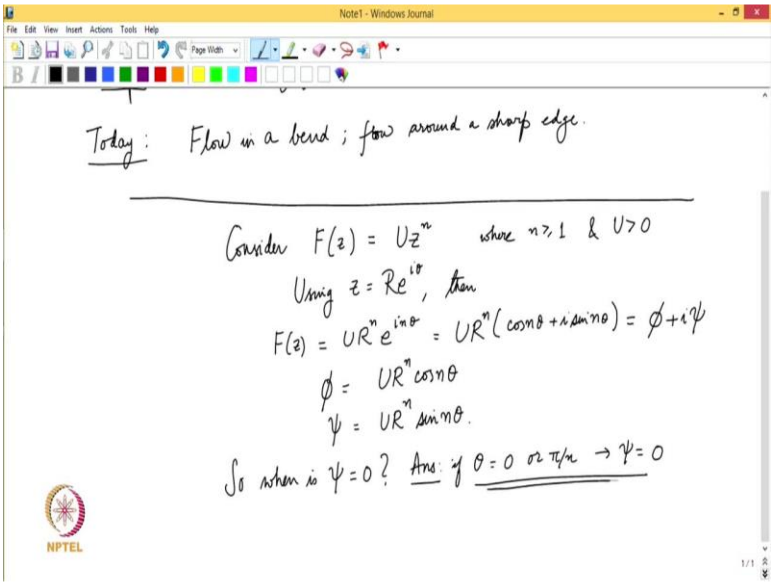

So, let us begin and let us start with a very simple, complex potential. So, let us consider complex potential F of z which I am going to define as U times z to the power n, where n is a number which is greater than equal to 1, it need not be an integer. And let us say U is greater than 0, u is real valued number, and it is greater than 0. So, let me now explain and I hope that the pattern is now developing on how we would tackle problems in this course, let us say we want to derive the particular scenario to which this complex potential is valid, and also the velocity field corresponding to this complex potential.

So, let us start and I will go to one of my favorite ways of representing a complex number which is using polar coordinates. So, we can say if I use z to be R e to the power of i theta then f of z would become UR n e to the power i n theta, which I can also write us UR n cosine of n theta plus Iota sine of n theta.

So, the first thing I want to do is to derive the form of the velocity potential and the stream function. So, I could write this f of z as phi plus i sin. And so, if we then compare the real and imaginary parts, phi becomes UR n cosine n theta, psi becomes UR n sin n theta. So, I hope this is so far, this is easy to do and I hope that this is clear as this until this point.

Now, let me take you to the next key step, which is to identify a specific dividing streamline. Now, there are various ways of calculating dividing streamline, I find the easiest one to be the case where psi, if I can begin with if I say psi is 0. So, I want to know, and it will become very clear why I am doing that. So, I am asking the question, when is psi equal to 0? Okay, that would be a particular streamline where psi 0. Clearly, the answer to this question is that psi will be 0. If, of course R is 0, that means you are right on the origin.

But more importantly, when n theta is a multiple of pi, including 0. So, I can say if theta is 0, or pi by n, of course, and 2 pi by n, but we need not go that far. If theta is 0 pi by n, these would be the first 2 roots of this equation. In this case, psi happens to be 0. So, what I mean by this important statement here is that if you look at the complex plane.

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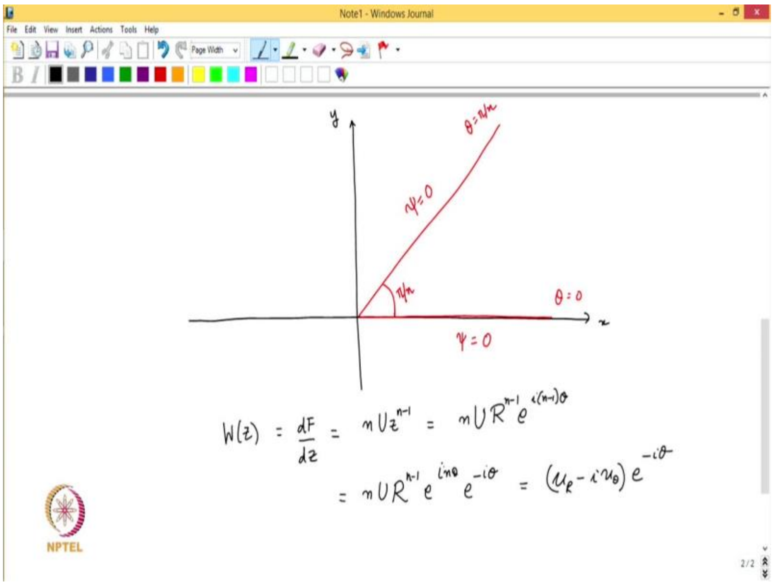

And let us start sketching out what we would get from this solution. If I look at this complex plane theta 0 is actually this line, this line corresponds to theta 0. And theta pi by n might correspond to maybe see this line. So, this is theta pi by n. So, this angle sector has an angle pi by n. So, these two lines correspond to the same value of psi. So, keep this in mind. And we will use this fact, very soon, we will we will come back to the sketch and complete our analysis in a little while.

Now, let us also calculate the complex velocity, which would be just dF dz. So, this comes out to be n times U z n minus 1 and again, if I use the definition of the complex number in polar coordinates, we will have this as R n minus 1 e i n minus 1 theta. And I can write this as n UR n minus 1, e to the power i n theta into e to the power minus i theta. That is what e to the power i n minus 1 theta would look like.

And I am doing this for the obvious reason that I want to calculate the radial and the angular velocity components from this equation. So, I need a minus e i theta, for obvious reasons. All right. So, now can we compare the two sides and derive the two velocity components? The answer is yes. We can do that right. So, let us do this.

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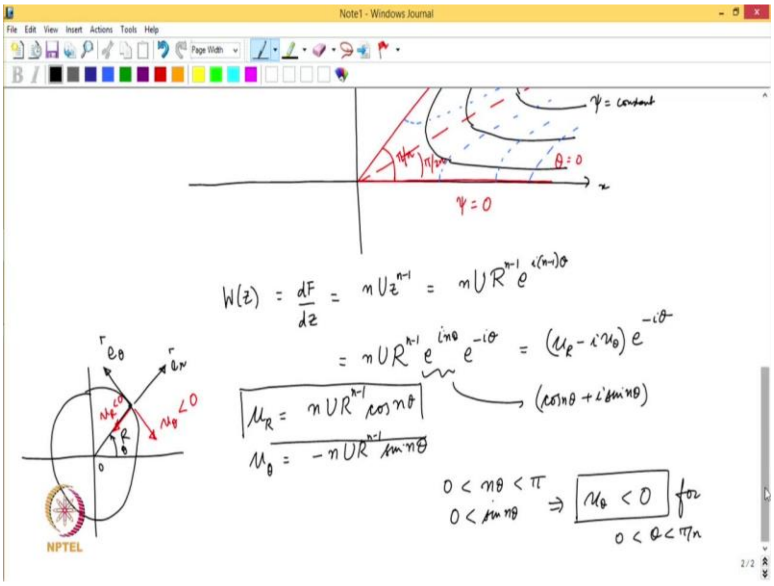

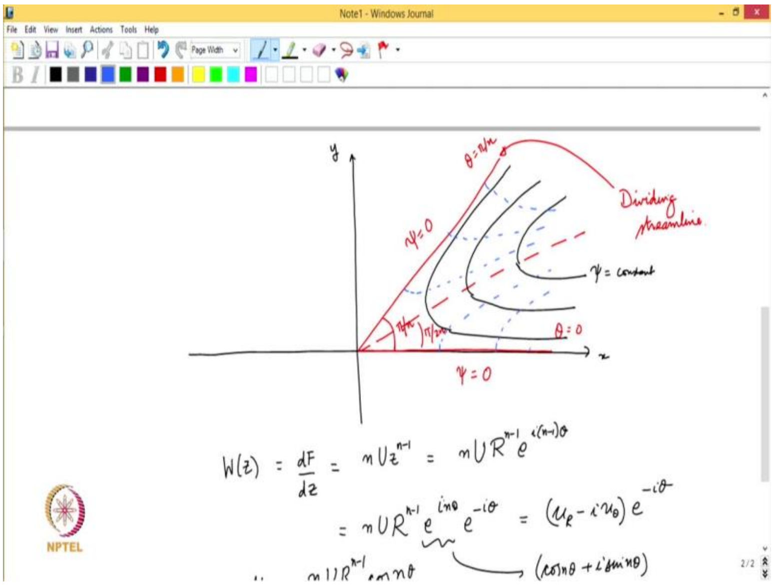

So, we will write u R is going to be n UR n minus 1 times cosine n theta. Right? because e to the power i n theta as I wrote earlier would be cosine n theta plus i sine n theta. And similarly, u theta would be minus n UR n minus one sine n theta. Now, note that I said theta is between 0 and pi by n, at least for now we have the dividing streamline. This, as I am saying is the dividing streamline, which is, in fact, the word dividing is, is a miss over here. But you know that if you go along this path, there cannot be any mass exchange, because streamline is 1 where the velocity vector is always tangent to it.

So, between this angle of theta and pi by n, you can notice n theta would be between 0 and pi. Which means sine n theta would actually be always greater than 0 and less than 1. Because in the quadrant, if you are taking the sin of the angle, n theta within theta lies between 0 and pi, the sine of that angle is always positive in the first two quadrants.

So, the important fact that I wanted to mention is that sine n theta is always greater than 0, which means that u theta would always be less than 0, between 0 and pi by n so keep this in mind. This is for theta between pi by n and 0. So, that will give us some idea of how the velocity field would look like in this plane.

More importantly, if I wanted to plot other psi values on the same plot, if I wanted to plot other psi values, and let me draw the bisector here of this angle, as we will see very soon, that there is symmetry about this bisector So, this angle is pi by 2 n then the other streamlines in the flow would look like in this following way. You will see the other streamlines of this type.

So, these will be lines of stream function having a constant value and the lines of constant potential would now be normal to these lines of constant stream function should we have something like even though my picture does not do justice to this orthogonality I hope you will get the point that these would be lines of constant velocity potential.

So, now how do we determine the direction of the flow? So, what I will do to explain this is to look at what is the sine of UR and the reason I want to do it is that if you look at the complex plane and maybe I draw a circle at some distance, so, you finally get the complex plane and draw a circle of some radius R then I know that the radial vector point is positive in this direction and the angular vector is pointing in this direction at a given angle theta.

So, the important thing to note here would be that e theta would be negative if it is pointing here if this any velocity component pointing is pointing in this direction u theta is in this direction right we could say this is less than 0 and if you have a component pointing inwards, then clearly u R must be less than 0.

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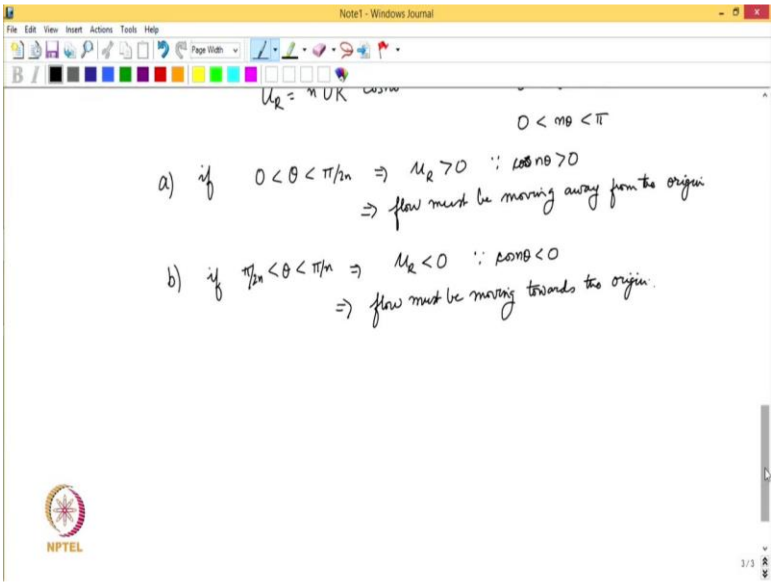

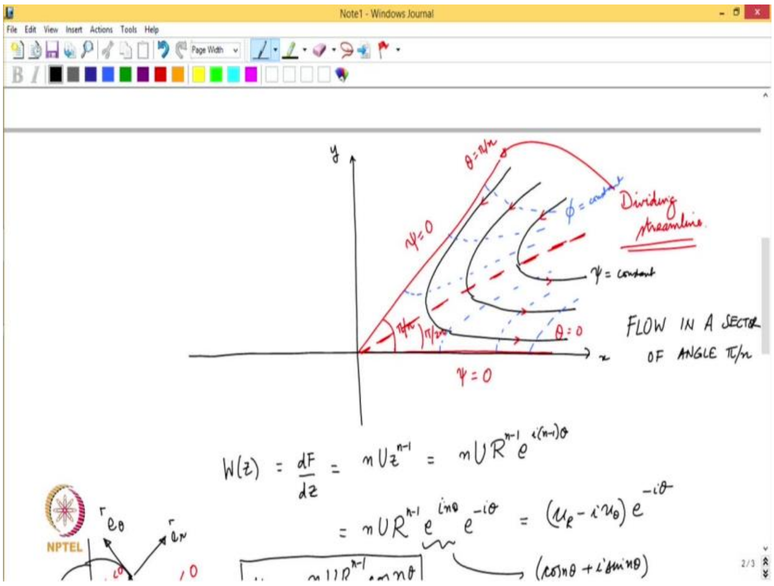

So, we use this fact now, so, if you look at U R let us put it here again. So, u R is n UR n minus one cosine m theta. So, between theta of 0 and pi by n or I could say n theta between 0 and pi the cosine function changes in sine. So, we can say if theta is between pi by 2 n and more than 0 then u R would be positive because cosine n theta would be positive because cosine of n theta is positive, which means, since u R as positive that means, that the flow must be moving away from the origin or should be pointing away from the origin and if theta is between pi by n less than pi by n but more than pi by 2n in that case u R would be less than 0 because cosine n theta would be negative which implies that the flow must be moving towards the origin.

Well I am being a little casual here when I am saying flow what I actually mean is a fluid that the fluid particles are actually moving in that direction. So, what does this mean now, and in fact, one more interesting fact that you should now observe is that u R a 0 when theta is pi by 2n so, clearly this line that I have drawn here the bisector is actually the line of like this is the line on which u R will be 0 but, u theta will be maximum on this line that you can prove to yourself now, which basically means that this should have some kind of a mirroring effect.

More importantly, because I said u R as positive when theta is between 0 and pi by 2n it means that the velocity vector in the lower half should be pointing to the right and on the upper half should be pointing inwards, so, then the flow is coming in towards the origin then it turns around at this angle of this angle pi by 2n and then it goes away, the fact that I have now used is that this psi 0 as I said is a dividing streamline.

So, importantly if I was to put two planes, if I was to bring two sheets of metal put them one at theta 0 the other one a theta pi by n, theoretically, it would not make any difference to the flow, because there have been no mass going across these plates, because of these two plates would actually become part of the streamline. So, the flow here remains confined within this angle of pi by n. And we call this as a flow which is in a sector of angle pi by n. So, this is sort of like a flow which is in a sharp corner.

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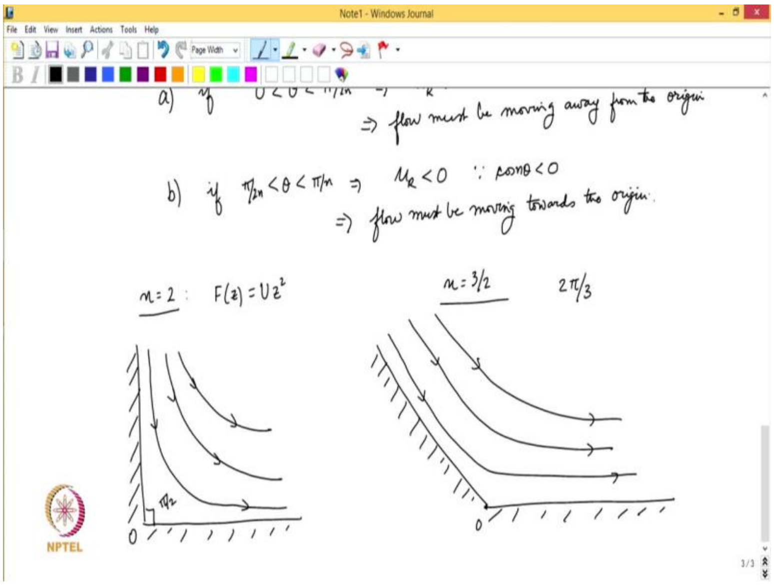

Now, the application of this, or if I can take two special cases of this scenario, let us do what happens with n is 2 n equals 2 would mean that we have a flow or we have a potential which is Uz square and the sector which can find the flow is of angle pi by 2. So, this would be a flow which would be confined within two walls, which are at 90 degrees to each other.

And then the flow comes towards one wall and then it turns around near the corner. So, that would be the nature of the flow. So, this is flow near a corner. Determined just from complex analysis, if you want to go one step further, we can even have as I said n need not be an integer, it could even be a non-integer, what if n is for instance, 3 by 2. So, that would mean that the sector angle would be now not pi by 2, but it would rather be pi by 3 by 2, so, that becomes 2 pi by 3.

So, that would be something like you can imagine how it would be flowing a sector of this type where it approaches the origin and then it turns around. So, this is flow down an inclined surface, and then it goes along the horizontal surface. So, what we have looked at are these special cases of flow near the intersection of two surfaces. So, that flow can be given by a potential of the type, Uz to the power n.

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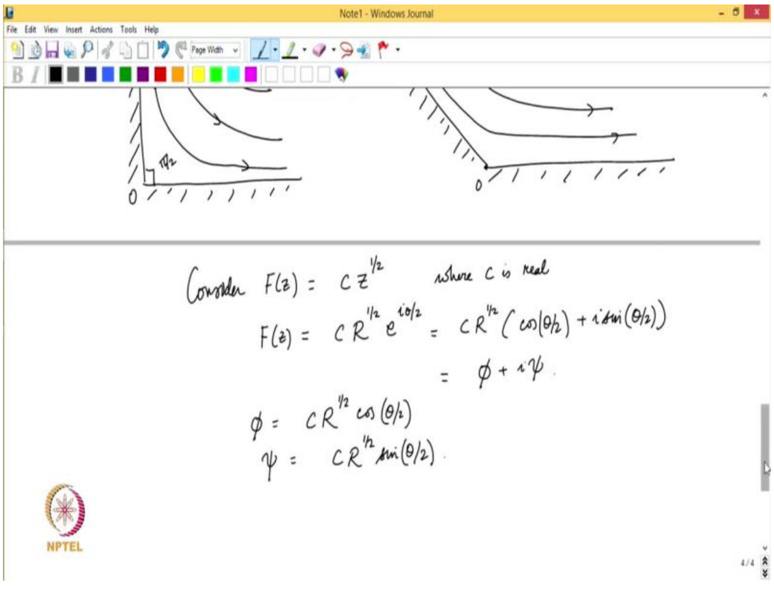

Alright, now, let me take another special potential, let us consider F of z to be some constant value C, into z to the power half or square root of z so, we will take here the condition that C is real. And you want to now determine which situation does this correspond to. So, again, we will start as I said, with my favorite tool, which is to use cylindrical coordinates. So, we will say this is C R square root into e, i theta by 2 which we could write as C into square root of R, I can see this would be cosine of theta by 2 plus Iota sine theta by 2. And that should be phi plus i psi.

So, clearly, phi is C R square root cosine of theta by 2 and psi would be C R to the power half sine theta by 2. So, now I will do the exact same thing that I did earlier, which is to calculate what would be a good dividing streamline and the easiest way to do that is to choose the particular value where psi is 0. So, when is the stream function equal to 0?

So, in this case, it would be when theta is either 0 or 2pi. Pi will not be a solution because that would become sine pi by 2 which is 1. So, that clearly cannot be the solution but we can have between 0 and 2pi where all the complex numbers or situated we will have only two solutions which correspond to theta 0 and theta 2pi. So, this corresponds to psi 0.

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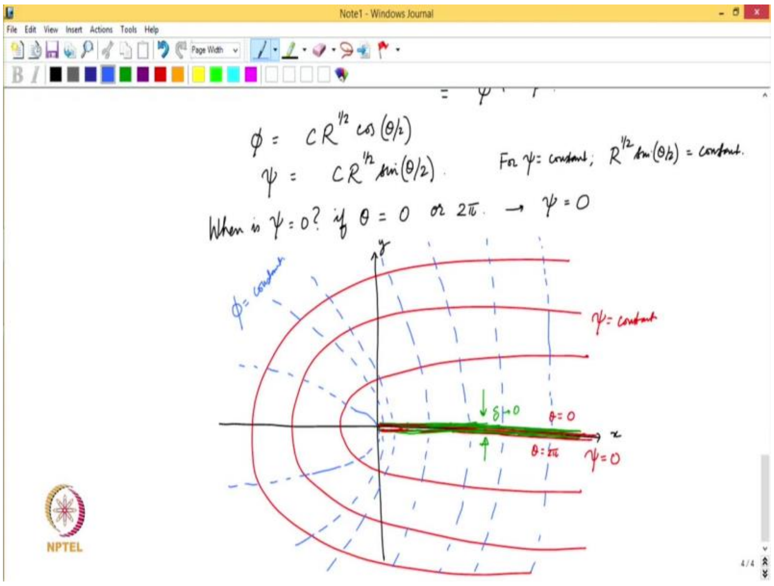

So, let us again start sketching it. So, this is the complex plane. So, what is theta 0 correspond to it corresponds to this line and when you go all the way around by 2 pi you reach the other side that corresponds to so, if I can say this is theta corresponding 2 pi, this is theta corresponding to 0 those are the two lines which correspond to psi 0, but now, what are we saying because, see these two lines are I even though I have exaggerated the gap between these two lines, these two lines are actually nearly on top of each other.

So, the point that I would make here is that I can hypothesize that there is a body this is solid object here on the x axis which has a very small thickness say maybe delta it has a very small thickness delta which approaches 0. So, it is like a very thin plate maybe a knife edge. And what we are looking for now, as I will show you is flow around this thin edge.

So, in the limit that delta goes to 0 we get the theta 0 and theta 2 pi are streamlines are basically the same streamline of this flow. Now, the idea is that for constant stream function clearly so, for psi to be constant R, to the power half into sine theta by 2 should be a constant although this locus is very difficult to come up with but that is what the definition of that locus of points should be needless to say but if you were to plot this these contours this is what you will actually get.

So, the psi constant lines would look like the following. So, they would resemble locus of this type so, these are lines of constant stream function including of course, the theta 0 and theta 2 pi line. If you want to plot now, the lines of constant velocity potential they would look like the following they would look like this we can just go a little further we can also take them on the other side keeping in mind that they have to be normal to the lines of constant stream function.

So, they would see something like this. So, these are lines of constant phi, but we have still not said anything about the direction of the flow. So, now we calculate the direction of the flow in this case. So, how do we determine that again we go back to the same method which is to calculate the velocity complex velocity.

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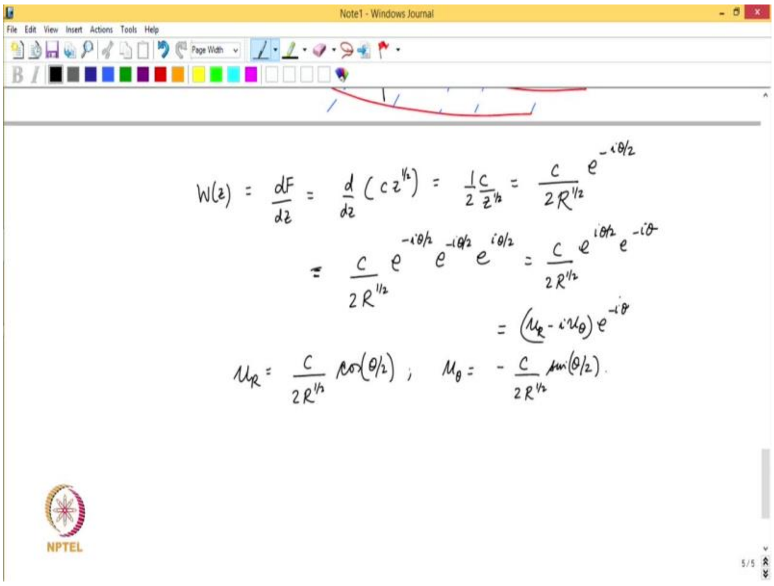

So, we say that W of z would be dF dz which would be d by dz of c square root of z which will be 1 by 2 C by z half. And that I could also write as C by 2 using z to be R e to the power theta I could write this as in this form. Now, I know that I want to equate this to the form of the complex velocity, which has e to the power minus i theta, but I do not see e to the power minus i theta I see e to the power minus i theta by 2.

Well, it does not bother me the least bit. I could write this is C square root of R, I could multiply by e minus i theta by 2. And also, i theta by 2, right, this makes it C by 2 square root into e to the power Iota theta by 2 into e to the power minus Iota theta. And this I should be able to write as uR minus i, u theta e to the power minus i theta. So, now, it is easy to see that uR should be C by 2 square root of R into cosine of theta by 2. And u theta should be minus C by 2 square root of R sine of theta by 2. So, these will be the velocity components in the complex plane. So, how do we determine the direction of flow?

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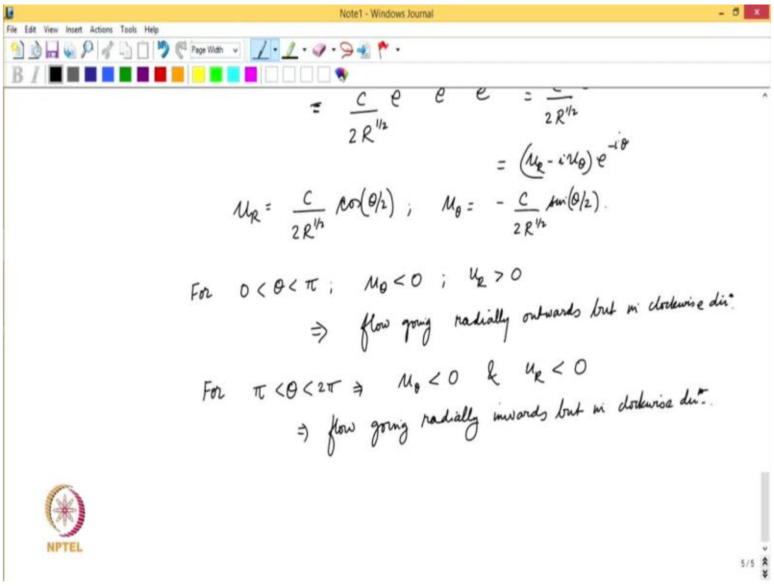

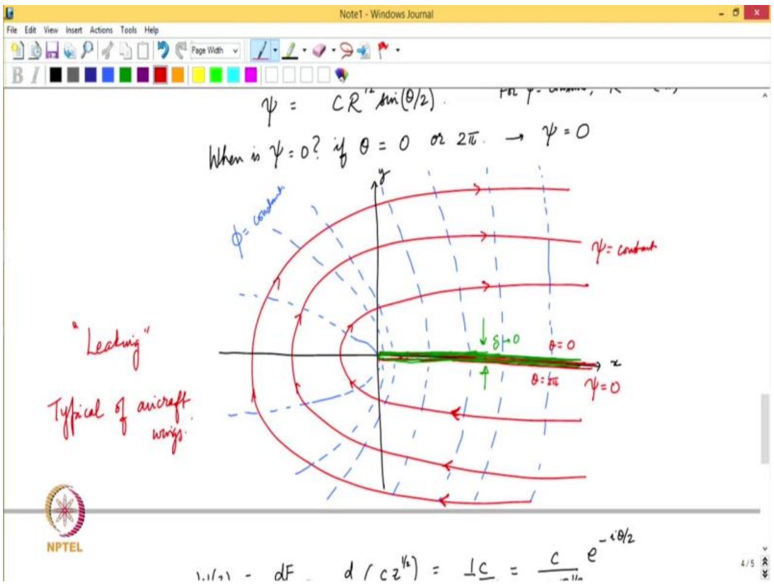

Well, again, we go back to the same way of looking at how these functions vary with angles. So, if I look at the range of theta, say going from 0 to pi, in this range, u theta would always be negative, because u theta has a minus sign theta by 2 and so, between 0 and pi sine theta by 2 or rather theta by 2 will go from 0 to pi by 2 and sine is always positive. So, u theta will always be negative.

Similarly, uR would also be positive because cosine of theta by 2 when theta by 2 goes from 0 to pi by 2 is always positive. So, in this zone between 0 and pi, the flow will be going radially outwards but, more importantly in the clockwise direction, why do I say clockwise? Because u theta is negative, which means that the flow must be you know, using the right hands screw rule, that u theta negative means that it is actually rotation in the clockwise direction.

Similarly, if I look at theta between pi and 2 pi, in this case, now u theta would still be negative because now, theta by 2 will go from pi by 2 to pi, that is the second quadrant where the sine function is always positive. And so, u theta will always be negative. But uR in the second quadrant, specially cosine of theta by 2 is actually negative, so, we will have uR is less than 0, which means that the flow will be going radially inwards when as mean inwards or outwards again saying with reference to the origin that it is either going towards the origin or away from the origin.

So, this is inwards, but in the clockwise direction. So, now can we draw the velocity vectors now that we know between 0 and pi, which is the first two quadrants or upper half the value of uR is positive. And in the lower half the value of uR is negative. So, I hope you could do this, but let us say for your convenience, let me let me draw this on the streamlines. So, we are saying on the top half the flow is going radially outwards.

So, it is going away from the origin. So, that is a direction that the flow is taking in the lower two quadrants the flow is coming radially inwards. So, it must be in this direction. So, that is the way this flow would appear for this complex potential. Now, what does this look like? To give you a physical interpretation? This is a very thin sheet. And the floor seems to be escaping from the lower half of the sheet to the upper half. This escaping right by escaping I mean it is leaking this, if I can be a little more, you know, non-technical, there is leaking of the flow.

So, flow basically is going around this edge, a very thin, very sharp edge. Now, where do we see this type of behavior? Now, this is typical of what you see in aircraft wings. That is the simplest example that I can give. And especially what I mean by wings is near the tips of these wings. So, the classical application where this kind of complex potential found its use was, if you look at how flow goes around the wings of a plane.

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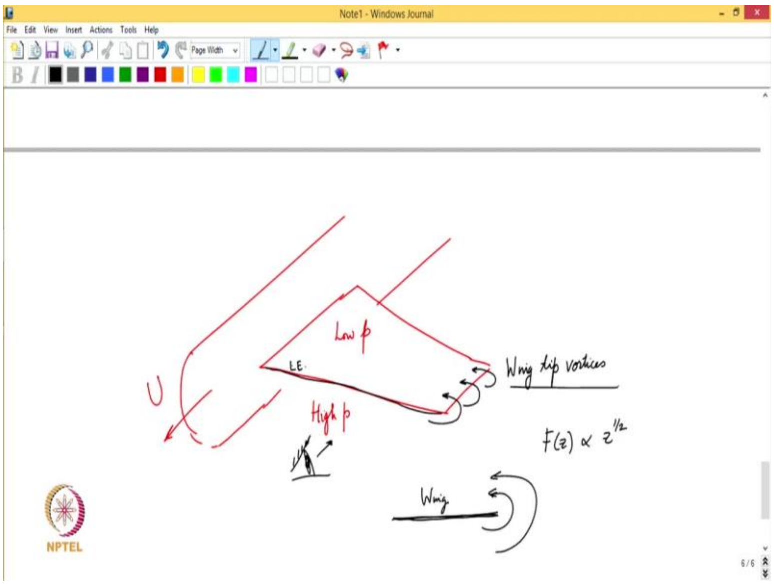

So, say we have a fuselage of the plane, right, so we have some fuselage and then there is a wing attached to the body of the plane. So, that is attached to the fuselage. So, that is it, this is the plane and this is moving in some direction, I do not know, maybe in this direction U with some velocity.

Now, what we typically have in a plane is that the reason you get lift is that you have a high-pressure region on the bottom, and you have a low-pressure region on the top of the wing. Alright, and say I am looking at this plane, or at this wing in this direction, say, I am looking as an observer. I am looking in this direction, which is right at the leading edge, this is the leading edge of this wing.

So, I am looking at this edge of the wing what I see is that the pressure under bottom is high and the pressure on the top is low. Due to which flow at the bottom will actually try and leak to the top surface. So, what we will get our wing tip vortices. So, to modeled this kind of behavior near a solid object in which case we are assuming that the solid is a thin structure we can use this idea that the complex potential to describe this type of a flow would be proportional to square root of z and that would give us the flow around this sharp edge.

So, when you see from this distance you will not see the depth but you would definitely see over the wing this kind of flow pattern which what we have. Now, wing able to modeled, so we stop this lecture here in the next lecture we will continue our discussion over few more elementary flows.

Specially we will look at now the idea of superposition, so far, we looked at complex potentials in a one at a time but now we will use the idea that we can add these complex potentials and obtain more and more difficult solutions which may not be so apparent from our regular analysis of fluid mechanics. So, thank you.