Ideal Fluid Flows Using Complex Analysis
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Lecture 4
Elementary Flows: Uniform Flow, Source and Sink, Free Vortex

So, let us do a quick recap of what we have covered so far in the previous lectures. So, recall that in this course now that we are dealing with ideal fluid flows and ideal fluid flows are inherently irrotational. We have so far been working with the definition of fluid velocity in a certain way.

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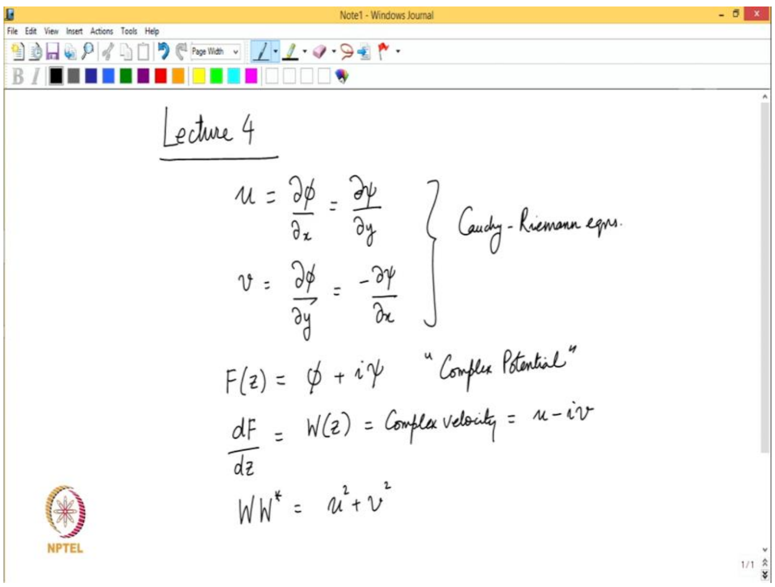

We have said so far that the x component of velocity which is u which should be d phi d x could also be written in terms of the stream function which would be d psi d y and the y component of velocity which is v which is d phi d y can also be written as minus d psi d x. So, in the last lecture we discussed that these are the Cauchy Riemann equations for a velocity potential which we could write as phi plus i psi. So, that was the velocity potential or the complex potential that we had defined.

Further, we said that if we take the derivative of this function with respect to z then what we recover is the complex velocity which could be written as u minus i v. More importantly, if you have to recover the magnitude of the velocity, we could just take W times its complex conjugate that would become u square plus v square. And clearly the square root of that number will give us the magnitude of the velocity at any given point on the X Y plane.

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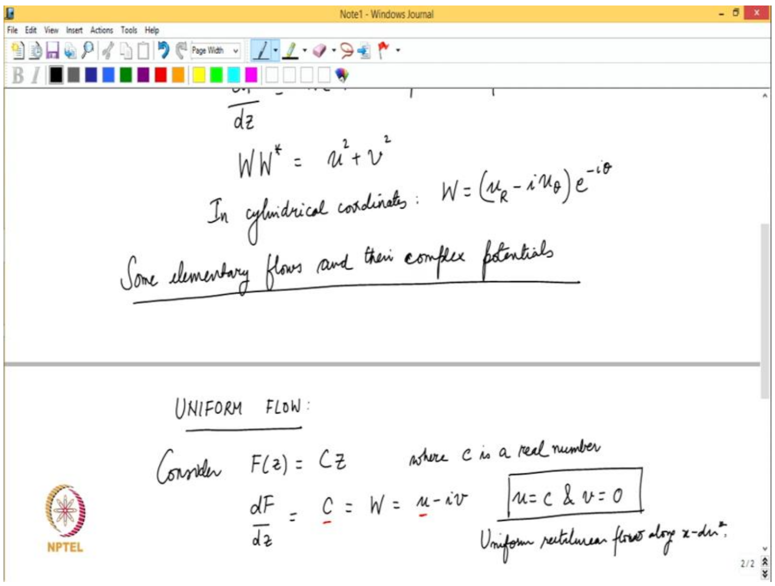

Finally, we said that in cylindrical coordinates, we could write the same velocity or the same complex velocity W in terms of the radial and tangential components. So, we could write this as u R minus i u theta e to the power minus i theta. So, we exploit the polar form of a complex number in a complex plane. So, this is something that we have already covered. So, in today's lecture we will talk about some elementary flows and also discuss what are their complex potentials. So, that would be the agenda for this class.

Now, the simplest elementary flow that I want to begin with is what is called as a uniform flow. And the way I would explain these potentials is that I would first propose a particular form of complex potential and then show you how it transforms to the flow pattern that we expect or that we are trying to come up with. So, let us consider a function F of z to be C times z where C is a real number.

For now, I am not going to put any restriction on whether C is positive or negative. We will see that the sign of C will eventually dictate the direction of the flow but it is important to note that C is real. So, what is d F d z. Well that is easily seen that is just C which I am going to write as the complex velocity W which could be written as u minus i v.

Now, if I know that C is real and that the complex number W is u minus i v, I could compare the real parts and the imaginary parts of the two sides and what we would get is u is C and v is 0. Now, what does this mean? This is characteristic of a flow which has a constant X component of velocity and the Y component of velocity is 0. So, this is a uniform rectilinear flow along the X direction.

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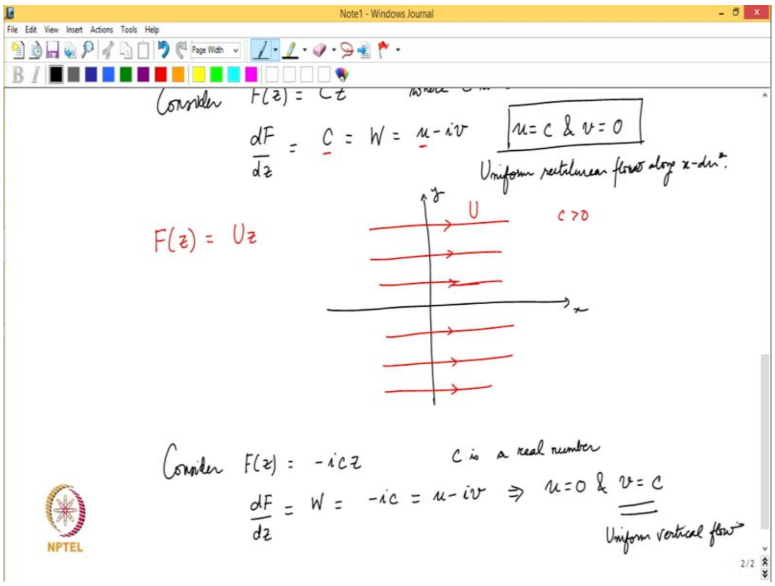

In fact, if we want to draw this velocity field, let us show on the complex plane, what we would have is a velocity field which should be pointing in one or going along one direction and if C is say positive then the flow would be from left to right. So, that is a uniform flow and the complex potential for it would be C times z.

To be more precise because typically we specify velocity in terms of the variable u. If I want a uniform velocity field with velocity capital U, then I would say the complex potential for this elementary flow has to be U times z and then we get flow field which has uniform X component of velocity and the Y component is 0. Now, we could also construct an elementary flow which has a Y component of velocity only.

So, consider F of z to be say minus i C z where again C is a real number, then we can again take the derivative of this potential with respect to z, write it as W and that is minus i times C which would be u minus i v and clearly this means u is 0 and v is C which is the same as a uniform vertical flow or flow in the vertical direction.

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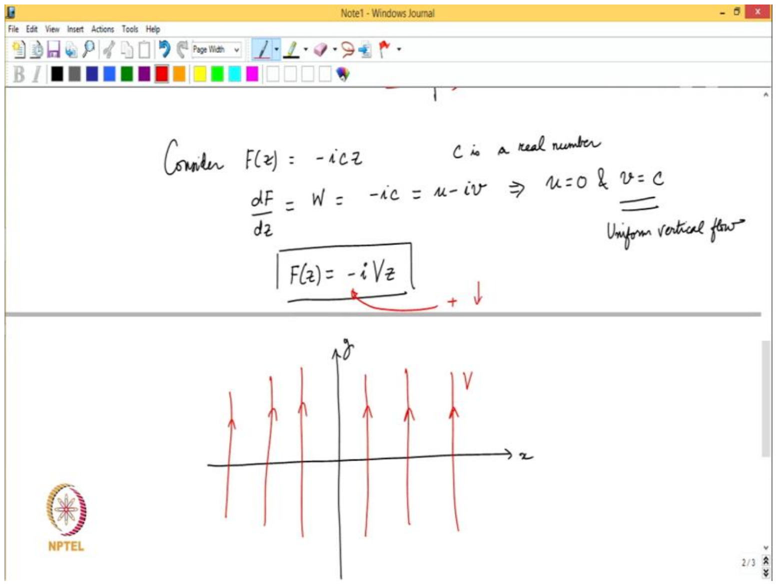

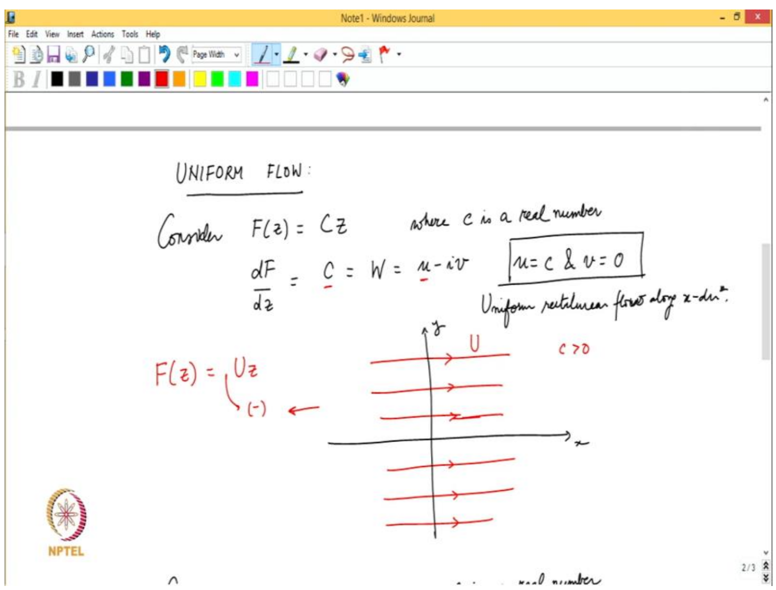

If I say that the velocity component along the vertical direction is say capital V then I can say that for obtaining a uniform flow along the vertical direction the complex potential must be minus iota V z. That complex potential will give us a velocity field of the following type. So, we will now have a field which is vertically oriented at all places with the magnitude of velocity V.

Now, it is clear that F C was negative, for instance, if I was to chain this and make this positive then the field would be downwards. And similarly, if I was to put a negative sign in front of here that would mean the flow is from right to left. So, those are very simple ways of managing these two elementary flows. There is a chaining the sign in front of the constant gives us the direction in which we want the flow to be going. Now, let us take a general case then. Now, we have looked at scenarios where the flow is either going from left to right or from bottom to up.

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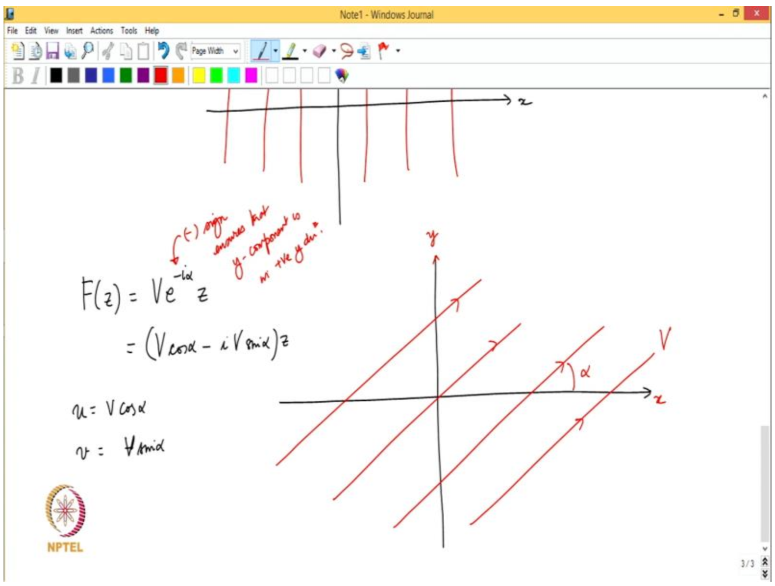

So, what if we want a general flow but uniform flow, may be pointing in some direction to the x-axis. So, let me propose a system or a situation where I want the flow to be going in this way but should be uniform. So, say I want the velocity to be V magnitude to be V and I know this flow makes an angle alpha with the x-axis.

So, you can think it over what would be a suitable complex potential for this. You can then verify that F of z here the more suitable way it will work out is that F of z should be V e to the power minus i alpha into z. That is a complex potential which will give you this flow pattern. Now, the reason being that we could write V e minus i alpha to be V cosine alpha minus i V sine alpha times z.

So, clearly if I take the X component of velocity u will be V cosine alpha and v will be V sine alpha and that is a significance of choosing a negative sign here. So, the negative sign ensures that y component is in positive y direction. So, that would become one of the a more general case of defining a uniform flow in a 2-D plane.

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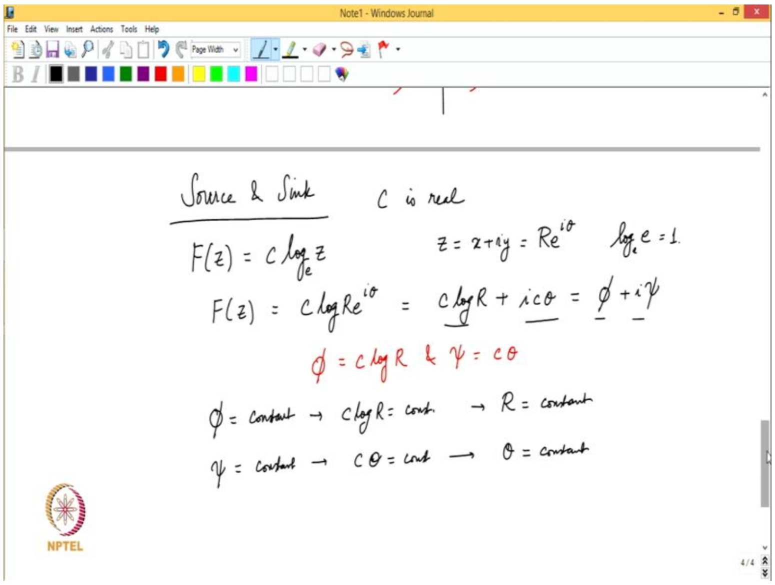

So, now, let us look at another elementary flow. So, these are source and sink. So, let us say we define a complex potential F of z as C log to the base e of z. Let us say that is my proposed complex potential. Now, because the dependence on z is logarithmic, it makes more sense here to define z not in terms of Cartesian coordinates but rather in terms of cylindrical coordinates.

Because what I want to exploit is that log of e is 1. I am going to exploit that property. So, let us do it this way. So, we can write F of z as C log of R e i theta which if we use the manner of taking logarithms, we can say this would be C log R plus C times i theta. Now, this is a powerful step here or this is one of the important steps that we have to be very careful about. I could write this as the sum of the velocity potential plus iota times the stream function. And I can compare both sides to say what represents what here.

Now, note that though I forgot to mention C is again a real number. So, if C is real, I can say that phi is C log R and psi is C theta. So, at this point, let us just spend a minute thinking about what we have got so far. We know with this choice of complex potential, phi is C log R, psi is C theta. So, what I am going to do is? I am going to now plot the flow field just by knowing these values without even having to calculate the velocities.

The way to do it is to identify what would be the locus of constant velocity potential and what would be the locus of constant stream function. Say, I want to look at conditions under which phi is constant. This means that C log R must be constant, C being anywhere constant, this translates to R being some constant value. Similarly, if I want to say psi is some other constant, we would say C theta to be some constant number which would again mean that theta is some constant value. Now, what do these representations mean on X Y plane?

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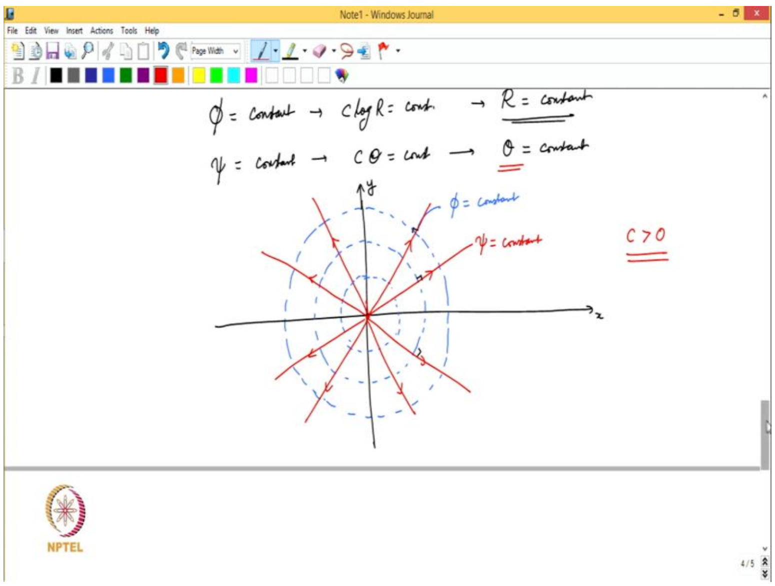

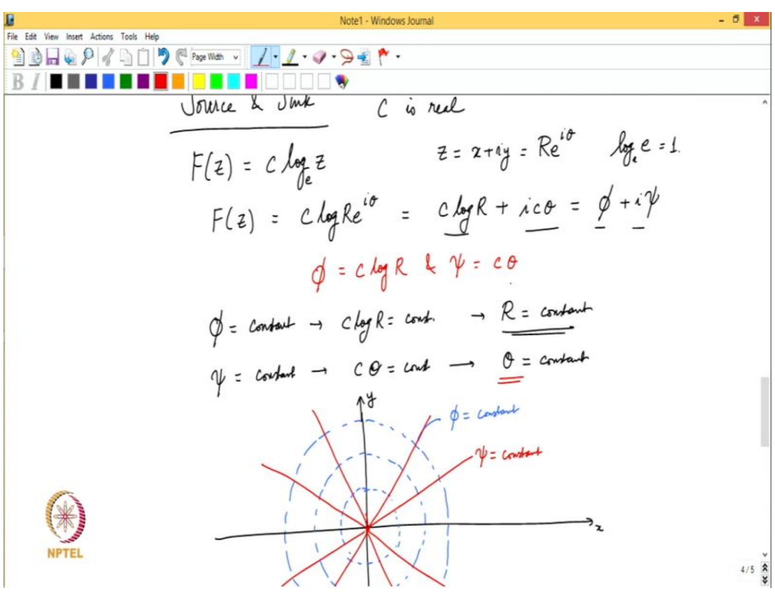

So, let us try and work this out. So, if we have the X Y plane here saying R constant, this means that the locus is going to be a circle where R basically means a distance of that point from the origin. So, on all points on that circle, the radius is the same because of which phi would be constant.

So, we can draw and let me use a different color for representing this, the lines corresponding to constant potential would be circles centered about the origin. So, these are lines of constant velocity potential. What about lines corresponding to constant stream function? Now, lines corresponding to constant stream function are theta constant. That means along a line which goes radially outwards from the origin, the theta remains constant.

So, along these lines now we could say that psi is constant. So, these are lines of constant psi. Remember, so, what do we know about stream function? We know that the derivative at any point in the streamline gives us the direction of the velocity field. Now, because the streamlines are actually straight lines, clearly, the flow must be along these streamlines.

So, the flow must be going along these streamlines either readily outward or readily inward depending on the value of the function C and whether it is positive or negative. So, let me say this now that if C is greater than 0 then the flow would actually be going radially outwards. So, we will have this scenario where the flow would be leaving the origin and it would be going only readily outwards, at any point where the stream lines and equi potential lines intersect. You would know that they are actually normal to each other. So, this is anywhere 90 degrees. Now, let me also tell you why when I say C is greater than 0, the flow is going readily outwards. We can prove it very easily which I am going to do now using the velocity definition.

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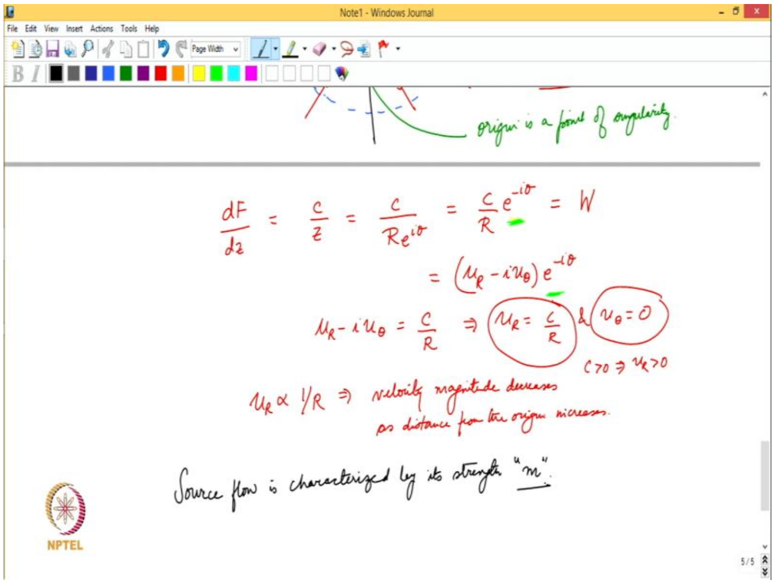

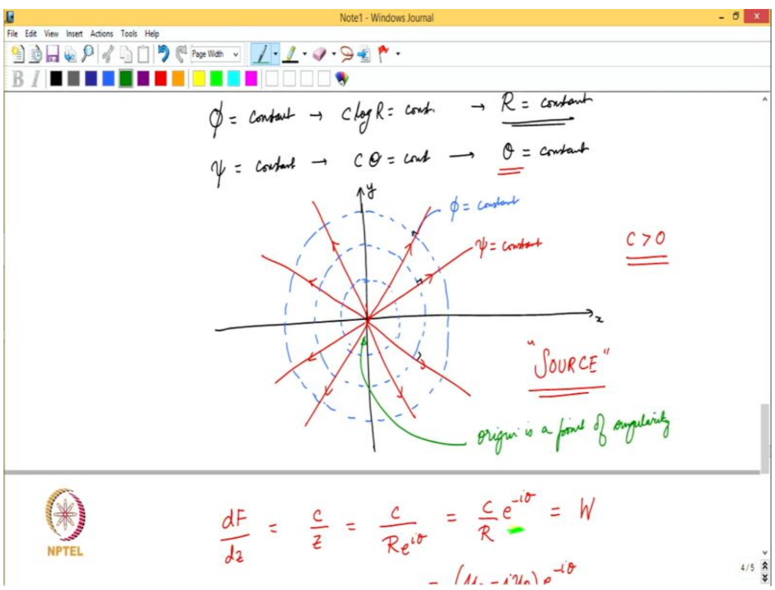

So, say we write now d F d z knowing that F is C log z. So, we can say d F d z would be C by z. And in the complex plane again z is R e to the power i theta. So, I can write this as C by R e to the power minus i theta. Now, it makes lot of sense here to actually write the complex velocity which is what this is W in terms of its representation in cylindrical coordinates.

What I mean by that is that it makes sense to write this as u R minus u i theta e to the power minus i theta because I see that there is an e to the power minus i theta here and I can straight away get rid of it by using this definition of the velocity, complex velocity. So, I can use this to write that u R minus i u theta is C by R. And for C being real this means u R is C by R and u theta is zero.

So, do you now see that what we have achieved from this is that we have a flow which has only a radial component of velocity and the tangential component is 0. Which means that the flow must be in the radial direction. And for C greater than zero u R becomes positive which means that the flow is going radially outwards which is what we had drawn here in this picture, where you can see now that for C greater than zero the flow is moving outwards.

More importantly, if you look at it functional dependence of u R which is 1 by R, this means that the fluid which is moving radially outwards, its velocity magnitude decreases with distance from the origin. So, the velocity magnitude decreases as distance from the origin increases. Now, since in this flow if C is positive and the flow is moving radially outwards, we call this as a source flow.

So, this type of flow is called as a source flow and it is also one of the more prominent elementary flows that you would have come across at some point. But the important point to take away from here is that its complex potential would be this function. And whether C is positive or negative, will dictate whether it is a source, or as I will show later or whether it is a sink. Now, note that origin is a point of singularity.

We should not lose sight of this fact that this origin is a point of singularity for this function. Then we will talk about these at some point later. Now, sources are typically characterized by volume of fluid that they can supply and so they are characterized by their strength. The sources or our source flow is characterized by its strength and the parameter chosen to define strength is m.

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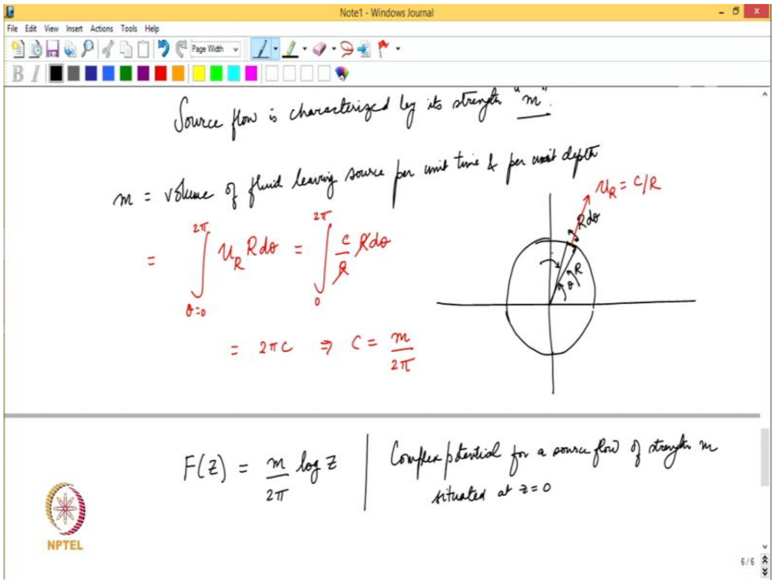

And the way we define the strength of the source is that this is the volume of fluid which is leaving the source per unit time and per unit depth. What I mean by that is, let us again go back to the schematic that we had. So, if we take at a distance R, if we take a isopotential surface. So, this is a distance R at some angle theta. What I want to calculate is how much flow leaves this surface per unit depth.

Remember, in the real system this would actually be a cylindrical surface, per unit depth means that we sort of take the depth to be let us say one unit. So, what we can do is we can take an infinitesimal element and for visual clarity I am going to draw this Infinity element slightly larger.

So, say the infinitesimal element has angle d theta because of which the arc becomes of length R d theta. Across this arc, the velocity is flowing radially outwards which is u R which is also C by R. So, I need to calculate m by saying this should be the velocity times the arc length or the length of the element which is R d theta over the entire circumference. So, I will go from theta 0 to 2 pi.

Now, note that this would be C by R into R d theta. So, R cancels off here and what we are left with this 2 pi C. Which means if I have a source of defined strength m, C will be m by 2 pi. So, using this definition now, we can write that the complex potential for a source of strength m is m by 2 pi log of z when this source is located at the origin. So, this is the complex potential for a source flow of strength m which is situated at the origin.

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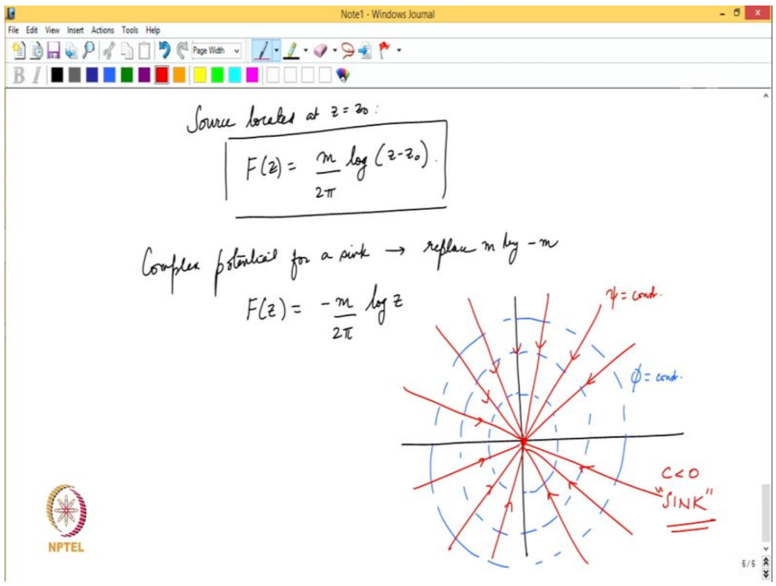

In a more general sense, if I say the source was located at some other location in the complex plane, maybe z naught we could write that the complex potential for a source located at z equal to z naught by just translating arc system. So, we could say this would be m by 2 pi log of z minus z naught. That would be the general way of writing a source flow when the source is not located at the origin but at some other point on the complex plane.

Now, what would be the complex potential for a sink? So, a source is something that provides flow, a sink will be one which sort of is consuming all the flow. Now, this is quite easy to see that you just replace m by minus m and you can recover the complex potential for a sink which is let us say situated at the origin. So, we will have the complex potential of a sink with strength m will be minus m by 2 Pi log of z.

And similar to what we did earlier, we will now have isopotential lines as circles. So, these are lines of constant velocity potentials, we will have stream lines which would be radially outwards. But this time these will be pointing towards the origin. So, this is with C less than 0 and this is what how we will describe a sink. So, this is how the picture would look like when we start dealing with sinks.

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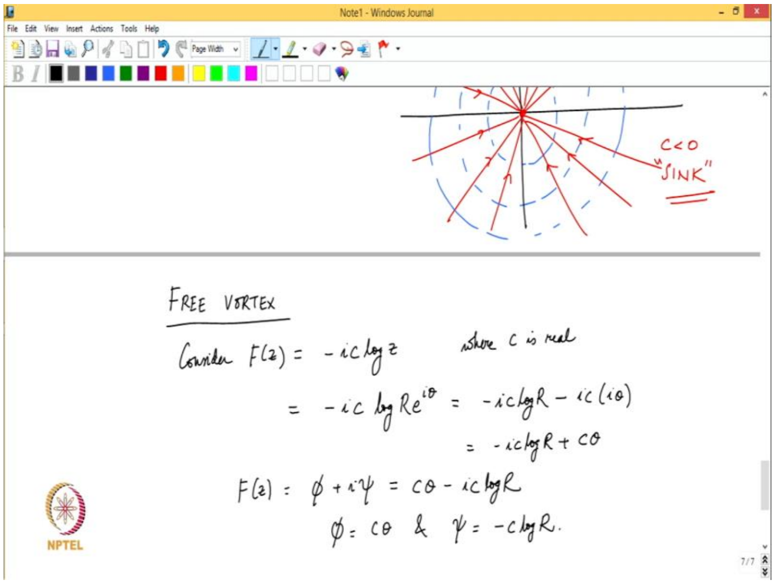

Now, let me now bring you to another very interesting elementary flow which is a free vortex. Now, let us consider in this case F of z to be say minus iota C log of z. Note that for source, we wrote the complex potential as C log of z, here we are writing it as minus iota C log of z. So, a very small difference can make a huge change in the kind of flow we can generate. As again, I would say C is real.

So, the way we worked out the previous case, we are again exploit the idea that we can write z in terms of polar coordinates. So, we can write F of z to be minus iota C log of R e i theta which we can simplify now to be minus i C log of R minus i C i theta which would be minus i C log of R plus C theta. Now, F of z is phi plus i psi which would be C theta minus i C log of R. Which clearly means that phi is C theta and psi is minus C log R. Now, let us do the same thing that we did earlier but with this flow.

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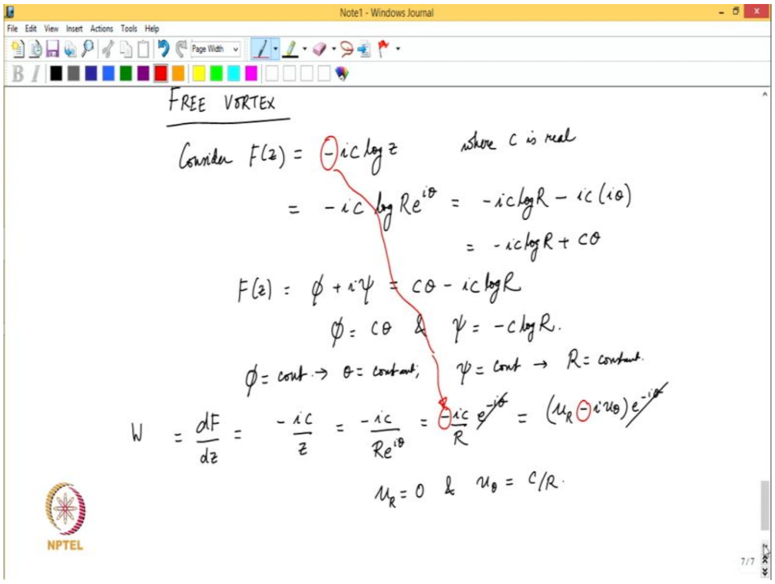

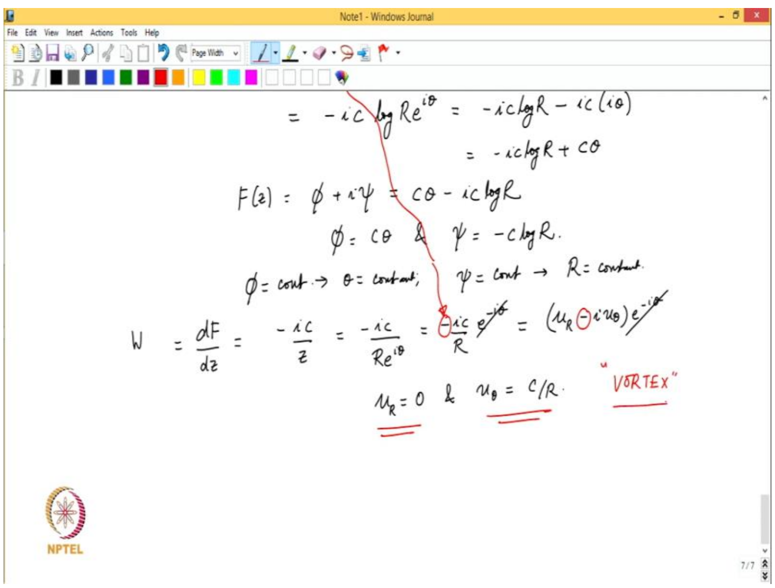

So, unlike this source, in this case, the situation is slightly turned. Now, we have constant for constant velocity potential, we require theta to be constant. And for constant stream function, we require R to be constant. More importantly, if you want to derive the velocity in the plane, in the 2D plane, using the complex velocity concept, we would say d F d z which is the complex velocity would be minus i C by z which we can write as minus i C by R e i theta which is minus i C by R e minus i theta.

Now, again I go back to the same principle of writing the velocity field in the cylindrical coordinates. So, we will say this is u R minus u i theta e to the power minus i theta. So, clearly u R is 0. So, if I cancel this, clearly, u R a 0 and u theta is C by R. So, now you can also see why I chose this as minus sign here. By choosing this minus I end up bringing this minus here.

And when I compare with this minus, these two will actually cancel. So, we get u theta as C by R. This is minor observation but an important and a powerful observation. It may seem very strange that we put up minus signs and plus sign at will, but that is not the case. We know what we are trying to derive. So, some changes have been made before and when choosing the complex potential.

So, now, u theta is C by R u R is 0. So, what type of a flow is this? Now, clearly radial component velocity is zero and there is only a tangential component of velocity. So, this flow what we will call as a vortex because this is a flow which is along the theta direction and when we want to draw it which we should now, we can do a similar thing here.

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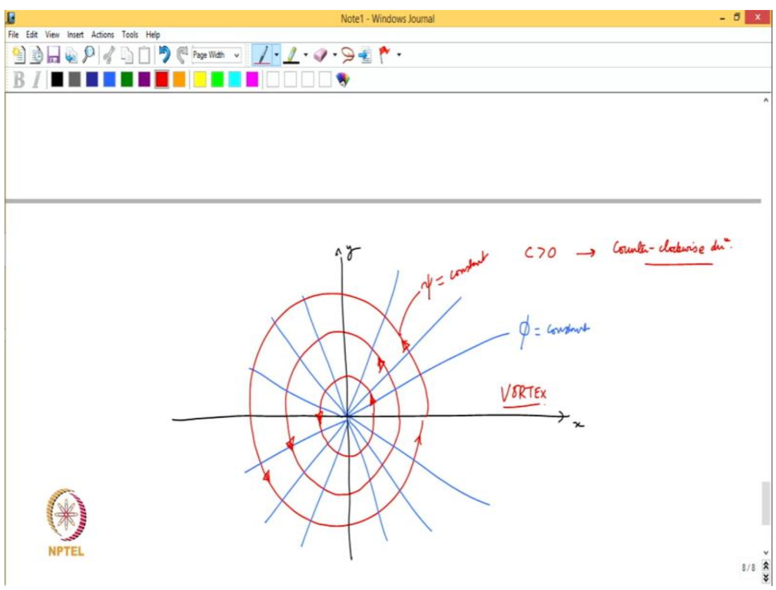

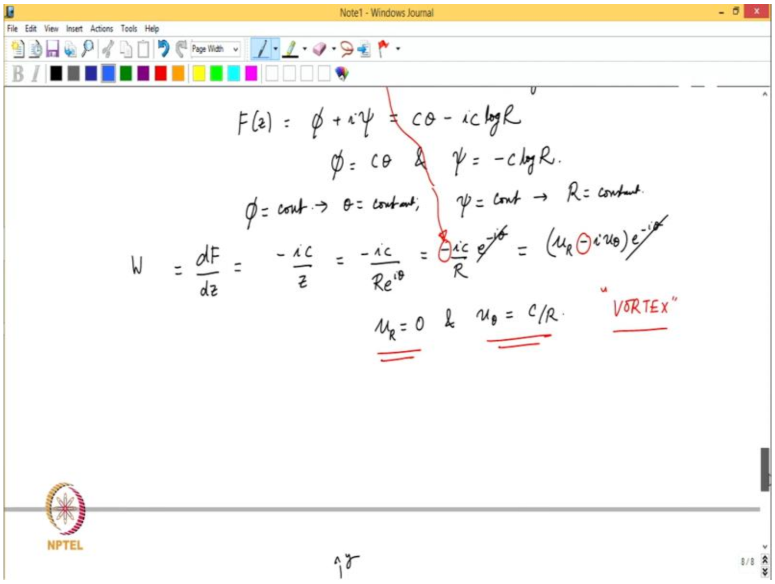

So, we say X and Y. So, what would be lines of constant potential? Note that constant potential would mean that theta is constant. So, we will have theta constant lines are these. So, these are lines of constant potential. So, along these lines, phi is constant and lines of constant value of the stream function would mean R is constant. So, now we will have surfaces which are circular which would denote the stream function. So, these are psi constant lines. And if C is positive, if C is positive, you can see that u theta is C by R. So, C is positive would mean that the flow would be in the counter clockwise direction which is a direction along which theta increases.

So, C positive would mean that if you use the right-hand screw rule, C positive means that we have to go around from theta 0 to theta 2 pi and so the flow must be along that direction. So, the arrows here show you the direction of the velocity. So, that is the elementary flow which is a vortex. Now, similar to the way we define the strength of a source or a sink, we also use the same concept here to define the strength of a vertex.

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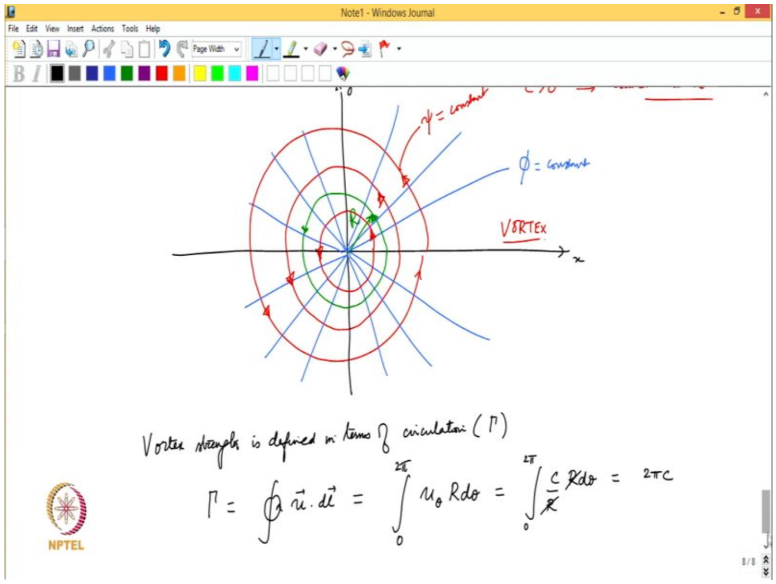

So, the vertex strength is defined in terms of a quantity called as circulation which will represent as gamma Greek capital gamma. So, gamma is defined as the line integral of the velocity along a closed path. So, in this case, we will calculate say for just to show you one case, let me say if I take one contour, circular say, I would go along this contour which is let us say at some distance R and calculate what would be u dot d l.

So, we can write this as. Now, note that along this path only one component of velocity exists which is u theta and that is so that is u theta. And if I take an infinitesimal element along this path, the element size will be R d theta which is along the vector u. So, u dot d l will be just u theta R d theta. And we will go from theta 0 to 2 pi along the entire curve. So, u theta was C by R and R d theta is anyway as it is. And so, we get R cancels here. So, we will just get 2 pi C and.

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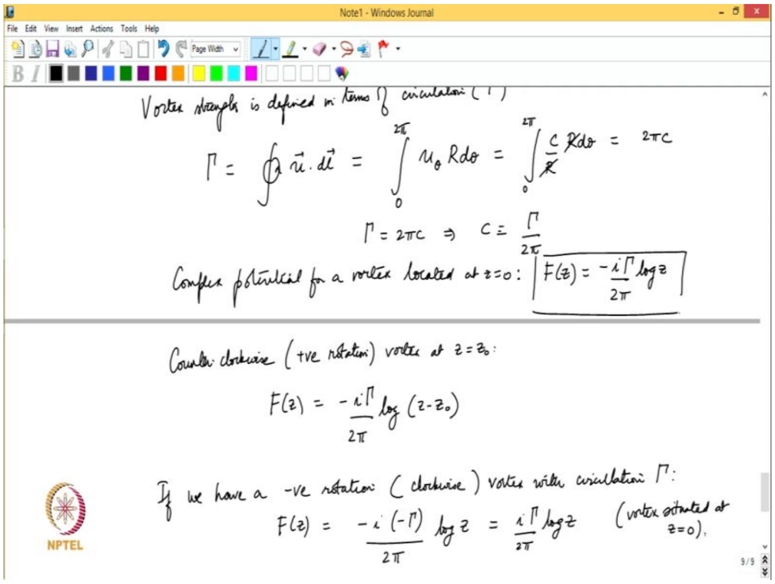

So, gamma is 2 pi C or we can say C is gamma by 2 pi. So, the complex potential for a free vertex which is centered at the origin would be, will now be F of z to be minus iota gamma by 2 pi log of z. If the same vertex was located at some other point, say z naught then you would displace the variable to that point z naught. So, we will say F of z there, will be just minus i gamma by 2 pi log of z minus z naught.

So, it is just a matter of translating our singularity or putting the function so that it becomes singular at point z naught. So, let us put that down. So, if you want to put the complex potential, let us say a counter clockwise which is what I am saying is a positive rotation vortex at z equal to z naught then we will say its complex potential will be minus i gamma by 2 pi log of z minus z naught. It is also now, I would say quite easy to see that the complex potential for a negative rotation vortex or a clockwise vortex, you can easily be obtained by replacing gamma by minus gamma.

So, if we have a negative rotation which I am saying is a clockwise oriented vortex with circulation say plus gamma, it is already has a negative rotation, then we will say F of z in that case would just be minus i minus gamma by 2 pi log of z which becomes i gamma by 2 pi log of z when the vortex is situated at the origin. If it situated at some other point, we again just translate that to that point. So, we say it would be just z minus z naught. Now, the singularity in this case, in the vortex is situated at the origin.

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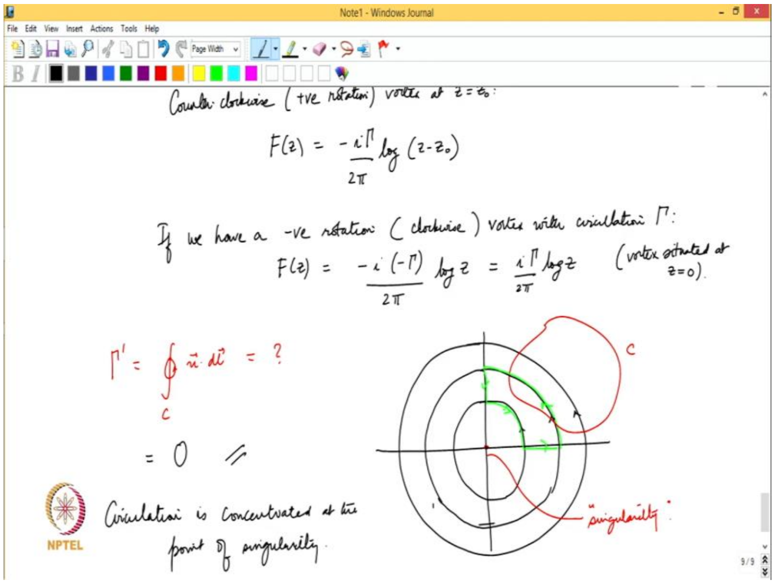

But I want to now ask you to think about this fact that say we had the streamlines given in this type, so, we had these stream lines in this flow, maybe with the positive vortex, if I was to take up contour in this flow which does not contain this singularity, which is the singularity is located here at the origin, so, if I was to take a contour C which does not contain the singularity and I want to calculate the circulation, let us say gamma prime over this contour C, what would be that value?

Now, C could be any arbitrary shaped contour but it does not contain the singularity and it is simply connected contour. Now, you can work this out maybe take a few special cases. I do not know. Maybe one case that I can propose for you to just work out is that if I take a contour let us say of this type, I go along let us say this path, come back down here, come down here, and then go by this path.

So, say this is the part that I choose, then you should be able to quite easily prove that along this path, this gamma prime comes out to be 0 because it does not contain the singularity. And consider this as an assignment for you to prove this to yourself. So, this happens for any contour C which does not contain the singularity and that happens primarily because the circulation of this vortex is concentrated at the singularity.

So, the circulation is concentrated at the point of singularity. And so around any closed path, which does not contain the singularity, we will still get the circulation to be 0. So, please, try this for yourself and I hope that you should be able to work out why this is the case. So, we will end this lecture here and see you in the next lecture. Thank you.