Ideal Fluid Flows Using Complex Analysis
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Lecture 3
Complex Variables, Analyticity, Cauchy – Riemann Equations, Complex Potential, Complex Velocity

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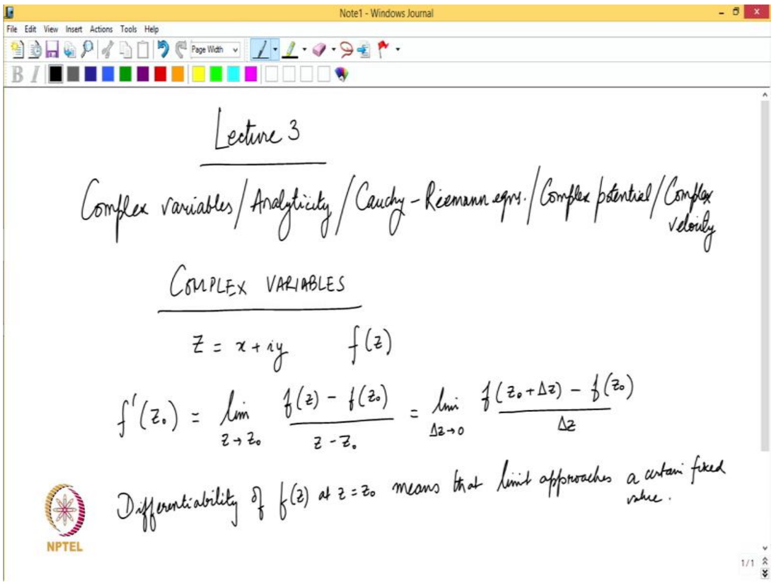

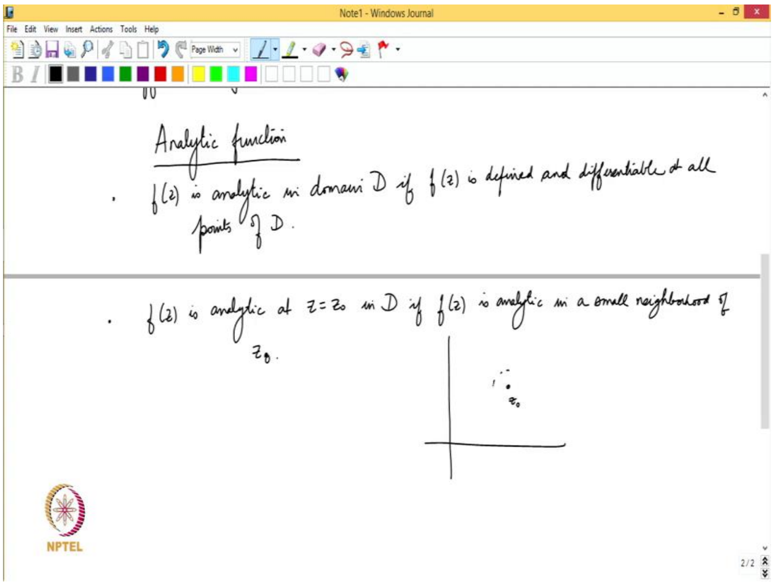

So, in the previous lectures, we have looked at the concept of the velocity potential, and the stream function. We also learned about the orthogonality of the iso-potential lines with the streamlines. So today we will go a step further and look at some preliminary concepts in complex analysis. So, we will start with the basic definition, here, and in this part, and then gradually move on to some more complicated concepts.

So, we will start with the analysis of complex variables. We will talk about what is an analytic, complex variable or complex function, we will introduce Cauchy Riemann equations. We will also talk about complex potential and a complex philosophy. So, let us begin this discussion today with complex variables. So, when we talk about complex variables, we inherently talk about a function of a complex variable. So, say a function variable z, which is x plus iy, y is a complex number. So, we will talk about functions f of z.

So, we define the derivative of a function f of z at some point say z naught. If you talk about derivatives, it is the same classical approach, which is the taking the limit from first principles. So, we can say that the derivative of a function f of z at the point z equals to z naught is given by the following its limit z approaching z naught f of z minus f of z naught by z minus z naught, which we can also write in terms of variable delta z to be f of z naught plus delta z minus f of z naught by delta z, in the limit that the variable delta z approaches 0.

Now, the point of defining the derivative of this function f of z is to talk about differentiability. So, we talk about differentiability of a function f of z at z equal to z naught and what we mean by differentiability or if a function of f of z is differentiable at z equal to z naught implies. So, we will say differentiability, or a function f of z is differentiable at z equals to z naught. If the value of f prime z naught approaches the constant value from whichever way, we approach z not starting in the neighbourhood of these points z naught.

So, we will say that the limit approaches a certain fixed value. If that is the case, then we say that the function is differentiable. The second definition that I want to introduce is about the function being analytic. So, we say the function f of z is analytic in a domain D, if f of z is defined and differentiable at all points of this domain. So, it should be defined and it should also be differentiable at all points in this domain for the function f of z to be called analytic in D.

Further, we can say that the function f of z is analytic at a point say analytic at some point maybe z equal to z naught in D, if f of z is analytic in a small neighbourhood of z naught, which basically means that, if we have a number z or z naught in the complex plane, then in a small neighbourhood around that point, the function should be analytic and then only we say that the function is analytic at that point z naught.

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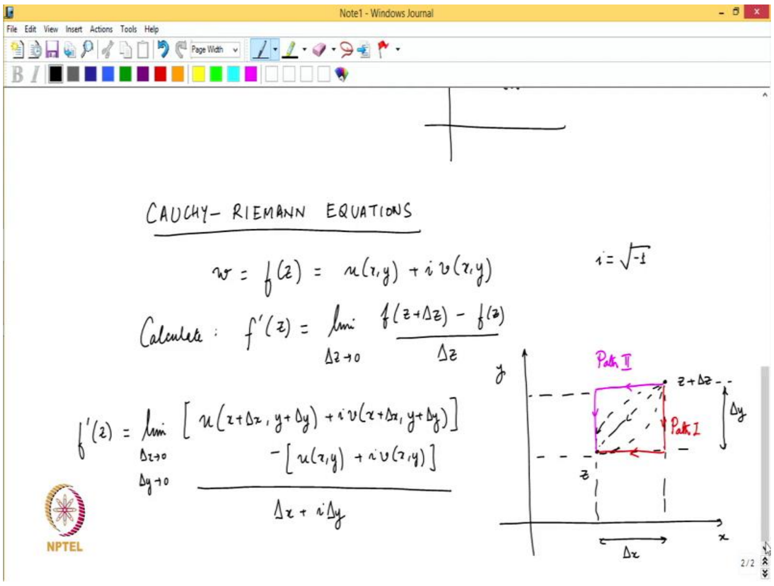

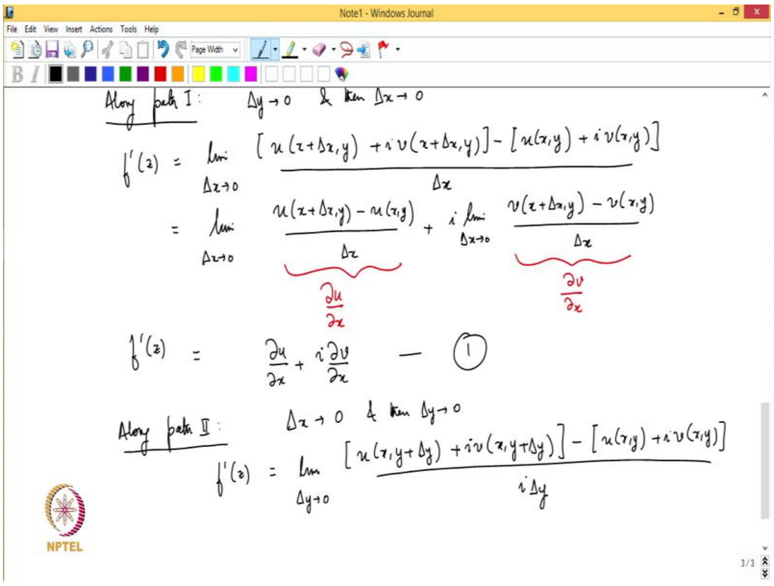

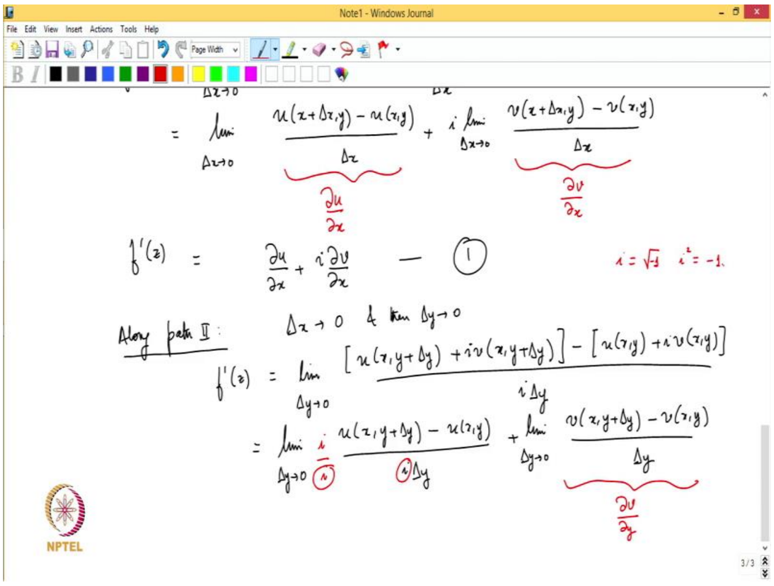

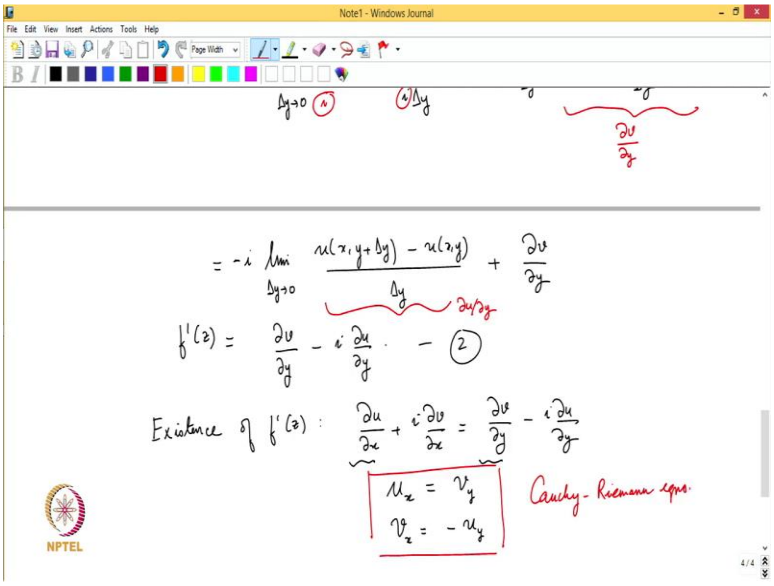

Now, knowing these definitions, let me now introduce you to the very powerful Cauchy Riemann equations which will form a strong basis for our analysis in this course. So, these equations provide a test for the analyticity of a function, or of a complex function f of z. And to show you what these tests are, let us begin with some variable w, which is a function of complex numbers z, which we could say right as u of x comma y plus Iota v of x comma y, where Iota is square root of minus 1.

So, u and v are real valued functions, but they are functions of the space coordinates x and y. Now, we will try and calculate f prime z the derivative of z, or the derivative of the function f with respect to this complex numbers z, which we can write as, again limit delta zeta approaching 0, say f of z plus delta z minus f of z by delta z. And let me graphically illustrate what we are going to do here. So, say we have the complex plane, we have the x axis here y axis on the side and say this point z is somewhere here. This is point z, and maybe we have point z plus delta z in its neighbourhood.

Now, if the function is analytic, then we can approach the point z from point z plus delta z in any direction. So, which means, we could take any arbitrary path to get to z from delta z. And in all those cases, the limit should lead this lead to the same unique value.

So, what we will do is, we will take a simplified case here, noting that the distance here is delta x and the distance here is delta y, we will try and approach the value along two paths. And these are paths that we are choosing because they are the simplest possible directions. So, we will choose let us say path one in this fashion and we will choose another path, say path 2 in this fashion.

So, we will try and take the derivative along these two paths. The special thing about these two paths is that if you look at path one, we first would reduce delta y to 0. Then later on, we will reduce that x to 0. Along path two, we will first reduce delta x to 0 and then we will reduce delta y to 0.

And let us calculate those limits. So let us calculate f prime z which in a more complete sense, would be limit of, say delta x approaching 0, delta y approaching 0 of u x plus delta x, y plus delta y, plus iv x plus delta x, y plus delta y, minus ux comma y, plus Iota v x comma y, and the whole thing divided by delta x plus i delta y.

So, let us evaluate this limit along these two paths, one and two. So, let us do this along path one. Let us say we evaluate this limit first. So, along path one will first send delta y 0, and then we will shrink delta x to 0. So, what do we get, we will get f prime of z to be limit, you can see that when I shrink delta y to be 0, we will get u of x plus delta x comma y plus Iota v of x plus delta x comma y minus u x comma y plus Iota v of x comma y in the denominator, we will just have delta x. And we can split this into now the real and imaginary parts.

So, we can say this is limit delta x approaching 0, we will have a u x plus delta x, y minus u x y by delta x plus Iota times limit delta x approaching 0 of v x plus delta x, y minus v x comma y by delta x. Now, individually, these two limits are quite easy to see, you see that the first limit here will give us du dx. In the limit, delta x goes to 0, the second one will give us dv dx. So let us write that down. So, we should get f prime z along path one to be du dx plus Iota dv dx. And let us say this is our first equation that we will use at a later stage.

Now let us do the same limit along path two. And along this part, going back to the picture here, along part two, we will first shrink delta x, and then we will bring down delta y. So what we can do is, we can write this as f prime z to be limit delta y going to 0, we will have u x comma y plus delta y plus Iota v x comma y plus delta y, minus u x comma y, plus Iota v x comma y and the denominator now, which originally was delta x plus delta y, which now only have an i delta y here, because you shrunk delta x to be 0.

So, we first took delta x going to 0, and then we will take delta y going to 0. Now again, we can split this into two parts, the real and imaginary part. So, we can write this as limit delta y going to 0. We can write this as u x comma y plus delta y minus u x comma y and denominator will be i delta y plus limit delta y goes to 0.

Now you can see that when I take the next term, I will have an Iota common in the v function. So, the Iota in the numerator and denominator will cancel. So we will just have v x y plus delta y minus v x comma y by delta y. Now it is easy to see that this function here will lead us to dv dy in the limit as delta y goes to 0.

But here on the other side, the first limit that we see, we just cannot make an assessment yet, what we can do is we can multiply the numerator and denominator by i. And then we can combine this i with this i to give us a minus 1. Remember, since i is square root of minus 1, i square would be minus 1. So, we can use that idea and write this as limit with an Iota outside delta by approaching 0 and there will be a negative sign that comes from the denominator, we will have u x comma y plus delta y minus u x y by delta y plus dv dy. And now, the second this number that we see here, looks like du dy.

So, r function f prime z along the second path will be dv dy minus Iota du dy. This is our second equation. Now, we can compare equation one and equation two. And if we say that the function is analytic and whichever path we take, should give us the same value of the derivative. Clearly, that means for the existence of f prime z we should have du dx plus Iota dv dx to be dv dy minus Iota du dy that is just comparing the two equations. And then we can also see that since u and v are real valued functions, we can say that clearly du dx must be the same as dv dy.

So, we can say u subscript x to denote du dx is the same as v subscript y which is dv dy and similarly, we can say v subscript x is minus u subscript y. So, these two equations now are essential for saying whether a function is analytic a complex valued function is analytic or not and these equations are called as the Cauchy Riemann equations.

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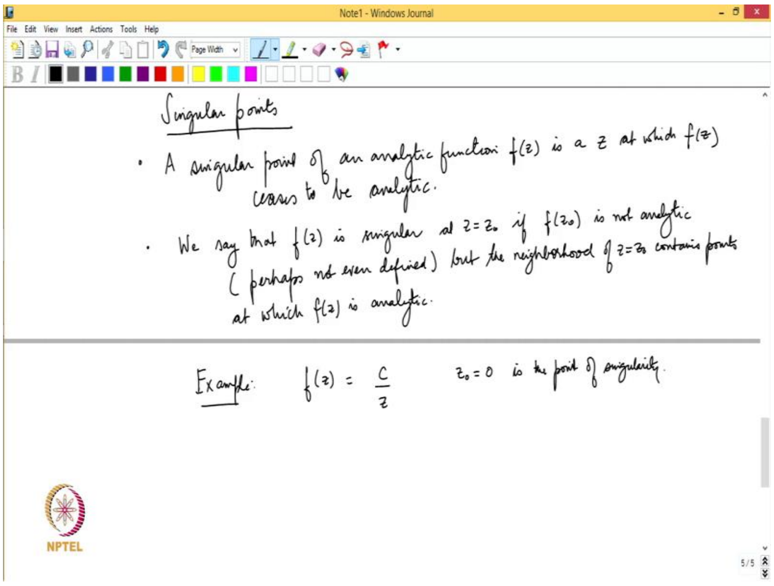

So, going further from here let us now also talk about another concept about singularities or singular points of a complex valued function. So, we define a singular point, or a singularity of an analytic function f of z is a specific value of this variable z at which f of z ceases to be analytic.

So, if there is a particular value of z, where the function is no longer analytic, we say that is a singular point further, we can say that the function f of z is singular at some value z naught, if f of z naught is not analytic or perhaps not even defined perhaps the value is not even defined at that point, but the neighbourhood of this point contains points which or at which f of z is analytic. So, even though the function may be singular at some point z naught in its neighbourhood you can have points where the function is still analytic.

In that case, we say that the function or the point z equal to z naught is the point of singularity for this function f of z and then it is not defined at this point. As an example say we have a function f of z to be c by z. So, it is easy to see that for this function c by z the singular point is the origin so the point origin is the point of singularity. Because at this point the function is not defined and nor are its derivatives.

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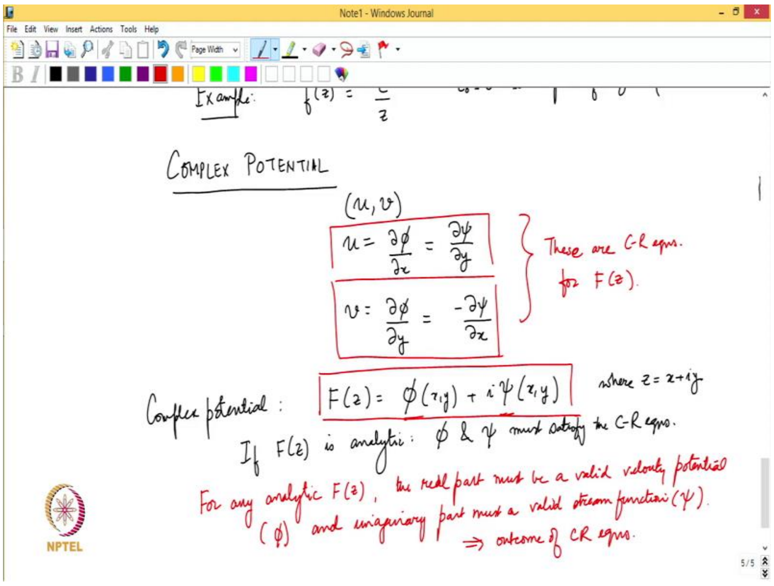

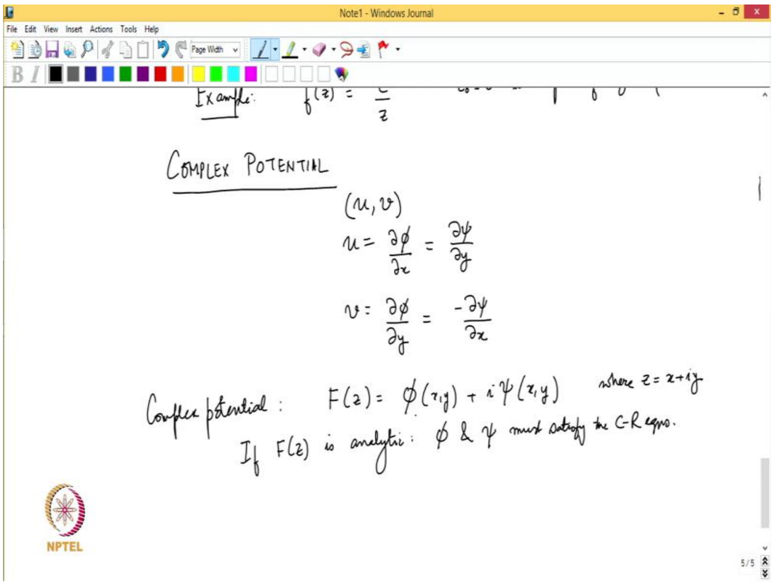

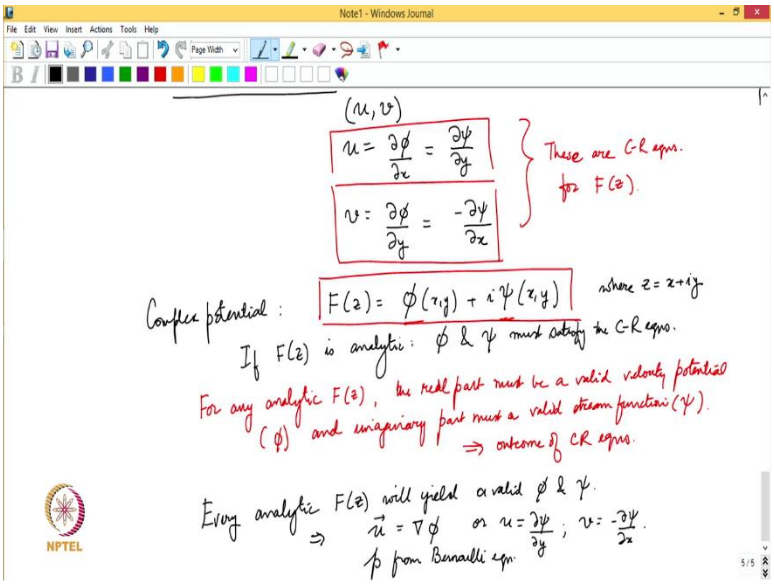

So, let us move on now, and let us define the most important item for this lecture, which is the complex potential, and which is why we have spent some time reviewing some definitions from complex analysis.

Now, notice that we were dealing with dealing with ideal flows and specifically 2d ideal flows, which are irrotational. And in 2d ideal flows, we have said that the two components of velocity which are u and v of the fluid flow could be written as u could be written as d phi dx. And v could be written as d phi d y in terms of the velocity potential, but it could these two can also be written in terms of the stream function, u could be written as d psi dy, v could be written as minus d psi dx.

So, keeping this in mind, let me define, and I will make it very clear why we are doing this way I am putting it in this fashion. Let me define a complex potential or a complex function of the following type. Say f of z is a complex potential, which I am saying is a complex valued function. But for now, let us say I put the name it as I name it as complex potential, and then it will become very apparent why I am using the word potentially here. Let us say f of z is phi of x comma y plus i psi of x comma y, where z is some point in the complex plane.

Now, if f of z was an analytic function it means then that phi and psi must satisfy the Cauchy Riemann equations. Now what were the Cauchy Riemann functions or the Cauchy Riemann equations that we just considered. So, we said if there is a function f of z, which is u plus iv, then the Cauchy Riemann equations mean that the derivative of the x derivative of the real part must be equal to the y derivative of the imaginary part and the x derivative of the imaginary part must be equal to the negative of the y derivative of the real part.

Now, if that was true, then we would say d phi dx must be equal to d psi dy, which is precisely what is given here. That is the first Cauchy Riemann equation. The second Cauchy Riemann equation will say that the y derivative of the real part must be equal to the negative of the derivative x derivative of the imaginary part which is also given here. So, these are the Cauchy Riemann equations for f of z the velocity potential.

So, now, it should become very clear why I am calling this function f as complex potential because it has this velocity potential and the stream function the derivatives are which basically give us the velocities. So, the potential term the name comes from the usage of velocity potential and the stream function in constructing this function.

So, since the Cauchy Riemann equations are automatically satisfied if the function f was analytic that is an inherent advantage that we get by choosing f of z of this form. Now, secondly, and the most important fact that I would like to put up here now is that is that if we have any analytic function f which has the real and imaginary part then clearly since the function is analytic, its real part must be valid velocity potential and its imaginary part must be a valid stream function.

So, for any analytic f of z the real part must be a valid velocity potential which is phi and its imaginary part must be a valid stream function which is psi, this is an important outcome of the Cauchy Riemann equations and this is I would say what are the most important properties that we would be using in this course.

The reason I say that is the following now, we need not solve the Laplace equation for the stream function or the velocity potential rather if you can construct any analytic function f of z then the real and imaginary part must be valid velocity potential and stream function for some flow or for some 2d potential flow.

So, every analytic f of z will yield a valid phi and psi and if that is the case, so, we need not solve the differential equation, but rather we can then just take up any of these two functions phi or psi calculate velocity using the gradient of phi for instance or using the stream function that gives us the flow field and then we can obtain pressure from the Bernoulli equation. So, we could derive the flow field in very simple terms without having to solve even the Laplace equation.

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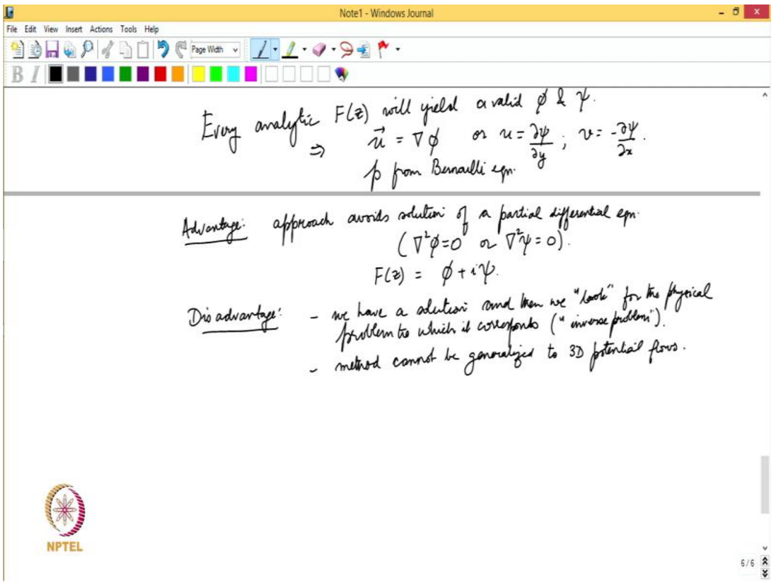

Now, there are advantages and disadvantages to this approach, let me first lay down the advantage. The advantage is that this approach that I have now explained, and which will take up as some examples very soon. This approach avoids the solution of a differential equation. So, we need not solve del square phi 0 or del square psi is 0 as long as we can define f of z which is analytic function in terms of phi and psi, we have a valid flow.

The disadvantage of this approach is, which I am sure some of you would have probably guessed by now is that first is that we now have a solution if we can build up an analytic function f of z we then have a solution and then we look for the problem where the solution can fit in or the problem that it corresponds to.

So, we have a solution and then we look for the physical problem to which it corresponds. But as I will show you, that this approach is not that bad, considering that once you have some experience constructing some of these functions, things will become much more easier to be able to define how a function of specific analytic function what kind of flow fields does it apply to.

So, we are working with some in some sense in inverse problem that we have the solution now, we are looking for the problem that it corresponds to the second disadvantage that we now have is that this method cannot be generalised to 3d potential flows but we do not lose much in fact, in the first set of design calculations, a 2d analysis is a good approximation to understand different aspects of a given flow field including for instance, if you have a bluff body in a flow, what kind of forces or what magnitude of forces would you expect to encounter on the bluff body.

So, those things can be easily done using analytic means and using this 2d assumption. So, this was the segment about Cauchy Riemann and how Cauchy Riemann helps us with coming up with new solutions.

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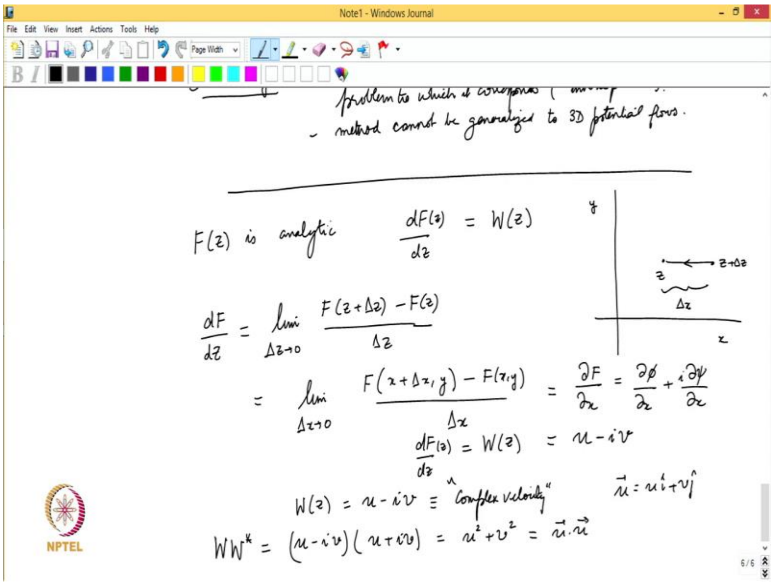

Now, let me take you through to the next most important concept, which is of the complex velocity. So now say that we are dealing with some function f of z which is analytic and same, we take the derivative d f of z versus dz. So, we take the derivative of this function with respect to z and I call this as a function w capital W.

Then, if the function f is analytic, then we can approach or we can take the derivative from any direction to this point z. So, we can evaluate dF dz. In a very specific direction, if I was to say from a limit point of view, this could be limit delta z approaching 0 f of z plus delta z and minus f of z by delta z.

So, say I take a path which is moving towards the point z. So, this is a point z on the complex plane. Say I take a path which is approaching z along the horizontal direction. So let us say this is delta x and we approach along this path for simplicity only, not for any specific reason. So, what I mean by that is that we could replace this function in the following way on this limit in the following way we can approach along the x axis and say what would be the limit calculated in the following way, which is f of x plus delta x comma y minus f of x comma y by delta x.

Now this limit would be df dx. Now recall that f is the complex potential which is comprised off the real part is the velocity potential, the imaginary part is a stream function. So, I can write this as d phi dx plus i d psi dx, which, if I use a definition of the velocity components d phi dx is u and d psi dx is minus v. So, we have this function W of z, which we wrote as dF z dz, giving us u minus iv.

Now, because this looks very similar to the way we would write the velocity field in vector notation, for instance in pictorially, we would say velocity vector is u i cap plus vj cap. Similarly, we are writing W z as u minus iv. So, W because of the correspondence with the way the velocity vector is written is called as the complex velocity of this flow field, we know that the real part and imaginary part in this case u and v can in no way can be added they are always two distinct parts of this complex number. So, u minus iv becomes a complex velocity.

In fact, if you look at this quantity, W times its complex conjugate W star is a complex conjugate of W this would be u minus iv times u plus iv, which would be used square plus v square, which could also be written in vector form as u dot u. So, this number W W star is the way we would calculate the magnitude of the velocity vector. So, the square root of W W star will give us the magnitude of this velocity.

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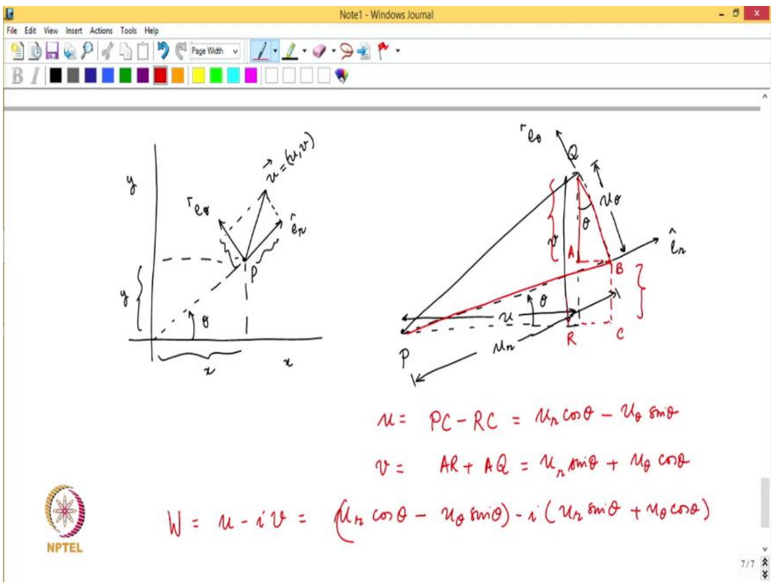

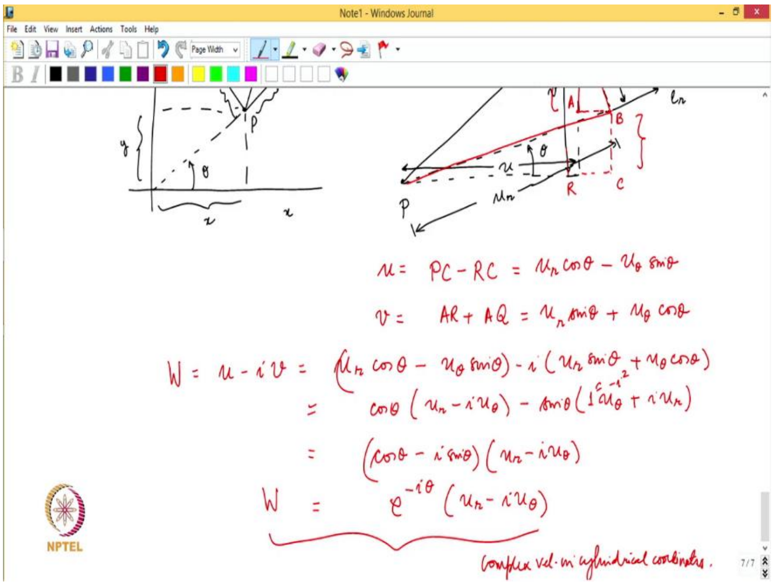

Now, frequently we will also work with cylindrical coordinates. So, not always would we be relying on Cartesian coordinates where x and y are axis, we may actually also have to work with cylindrical coordinates in two dimensions. So, in two dimensions the cylindrical coordinates are r and theta.

So, consider this case now, what we are now going to try and do is to say how the velocity vector will change in cylindrical coordinates, how the complex will actually will be defined in cylindrical coordinates. So, say at a point P in our complex plane, we have this point P where the coordinates abscess coordinate are defined as x and y. We know that the radial direction would actually be one pointing outwards from this place.

So, this would be the radial vector and entertain initial direction will be corresponding to angle theta measured from the real axis. So, that would be perpendicular to er, say at this point P, the velocity vector is given by this function which is on this direction, which is u comma v.

The objective for us is to define the component of velocity vector along er cap and e theta cap or along the radial direction and the tangential direction. So, how this the velocity complex velocity transform to cylindrical coordinates. So, say we want to or in this case, we are looking for the components along this direction and along this direction. So, we are looking for this component and this component.

So, what we will do is we can take this point P and let me just draw for everything to a in a bigger scale just for visual clarity. Say we take the point P here, and we will draw the velocity triangle on a larger scale for visual clarity.

So, say the velocity vector was pointing in a certain direction say from P to Q at this point. Now, we also know the directions of the x and y coordinates in this triangle. So, we have the x axis and the y is now labelled out at this point P, the radial direction is along this direction. And the digital direction would then be corresponding to that direction. So, this is the direction of the radial unit vector. And this is direction of the tension unit vector.

So, we want to calculate u theta here and ur. So, ur is this component and u theta is this component further, we know that this is u and this component is v and we know the angle theta is this angle which means, this angle should also be angle theta.

So, if I label some points on these triangles, just to make our life easier, say this is point B if I drop this point along the x axis, say this becomes point C we define this as point R, where the y component hits the x axis and say I drop a normal from point B along this line, which is say point A, then you can see that u which is the x component of velocity must be the component of ur along the x axis which is PC minus the distance RC in vectorial terms or in scalar terms which is PC becomes ur cosine theta and RC becomes u theta sin theta from this triangle because RC is the same as AB.

Similarly, we can say the y component of velocity will be say AR plus AQ which is the same as the distance AR will be ur sin theta which is this distance plus u theta cosine theta which is this distance to the complex velocity W which we wrote as u minus iv could be written as ur cosine theta minus u theta sin theta minus i times ur sin theta plus u theta cosine theta. If we rearrange these terms we can see, we will get cosine theta times ur minus iu theta minus sin theta times u theta plus i ur.

What I will do is there is a unity here as a coefficient of u theta, which I would replace by minus i square. And if I then take the common factors out, you can see this would be cosine theta minus i sin theta ur minus iu theta which becomes cosine theta minus i sin theta in the complex plane becomes e to the power minus i theta times ur minus iu theta. So, this important result gives us the complex velocity in cylindrical coordinates.

So, we will use these ideas now, in the next lecture to start constructing some elementary flows and starting with an analysis of ideal fluid flows using the using complex analysis. So, thank you and see you in the next lecture.