Ideal Fluid Flows Using Complex Analysis
Professor Amit Gupta
Department of Mechanical Engineering
Indian Institute of Technology Delhi
Lecture 2
Stream Function, Orthogonality of streamlines and equipotential lines

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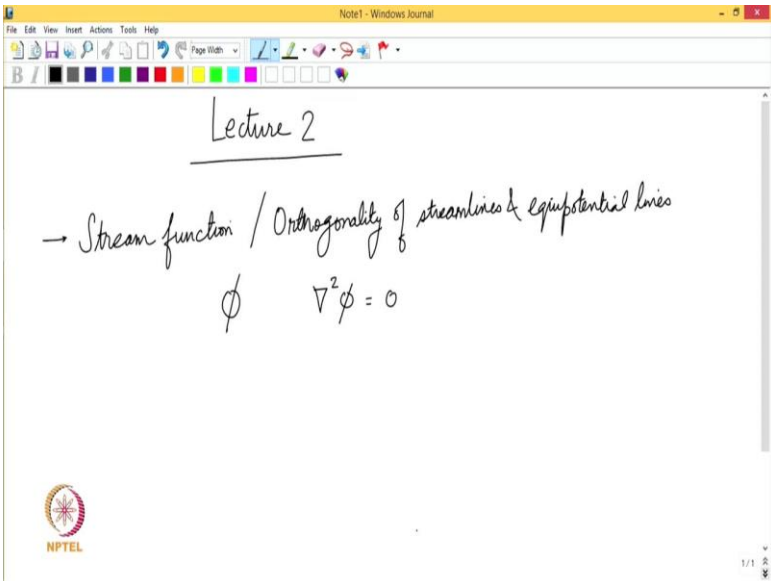

Welcome to this second lecture on ideal fluid flows using complex analysis. Now, recall that in the previous lecture, we looked at the concept of the velocity potential and how the velocity potential satisfies the Laplace equation. So, we have defined a variable or a function phi, which is governed by the equation del square phi is 0. So, in this lecture, I will talk about a second complimentary function which is stream function, and then I will discuss the concept of orthogonality of streamlines and equipotential lines.

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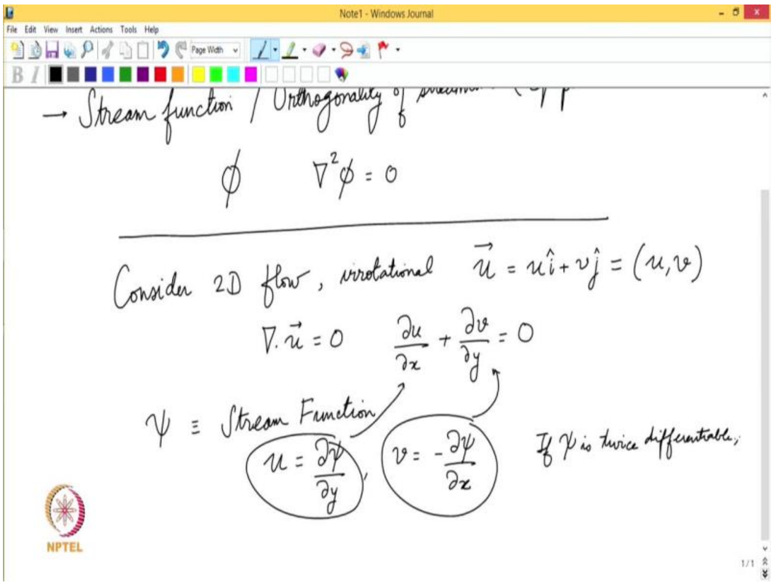

So, to begin, let us consider a 2D flow and let us say this flow is irrotational. Now, in this 2D flow, we can write the velocity vector u to be either say u i cap plus v j cap or I would use a shorthand notation where I would write this as u comma v. Now, for an ideal fluid, we know that this flow field must satisfy the continuity equation, which was del dot u is 0, which if you use in a vector sense will be d u dx plus dv dy is 0.

Now, what I will do is I will define a new function, which I am going to call as psi and I am going to call it the stream function so, we will define this new function psi which is a stream function, but it is given in a very specific form it is given in the following manner say I define u to be d psi dy and v to be minus d psi dx.

Now, if the function psi is twice differentiable, so, we can say if psi is twice differentiable which would mean we can take 2 derivatives especially the mixed type with this function then we can see that if I plug in these notations now, into the continuity equation.

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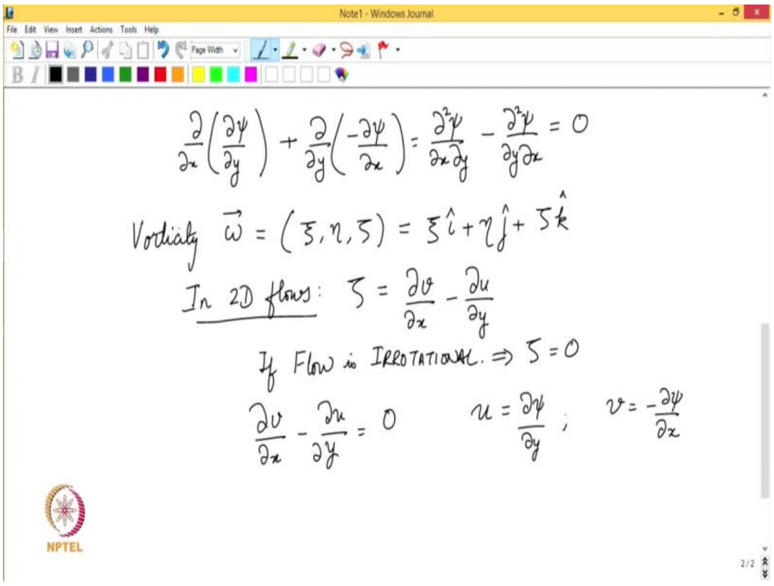

What we will get now will be dy by dx of d psi dy plus d by dy of minus d psi dx which would be d 2 psi dx dy minus d 2 psi dy dx which is 0. So, this function psi or this formulation of taking velocities components in terms of the function psi which is a stream function is such that in the 2D flow it automatically satisfies the continuity equation.

Now, though this function psi has nothing to do with flow rotationality or irrotationality so, this function psi would also be valid for a rotational flow in 2D flows. But for the purpose of this course, because we are dealing with ideal fluids, we will be stressing again and again on irrotation flows. And so, our focus will be on deriving properties of this function for a irrotation flows only. So, we will limit ourselves to the usage of this function psi for the case of irrotation flows.

Now, recall that vorticity which I denoted by the symbol omega, omega vector which would have three components. Let us say the 3 components are the following. We have xi, eta, and zeta as the 3 components of the vorticity vector, which I could write as for instance xi cap plus eta j cap plus zeta k cap.

Now, the only non-zero component of this vorticity function or this vorticity vector in 2D coordinates or in 2D flows it is easy for you to work it out and I am sure you would have also covered this as part of your undergraduate fluid mechanics that in 2D flows, the only non-zero component or the only the single component that would be possible to derive would be zeta, which would be given as dv dx minus du dy.

Further, if I was to say that the flow is irrotation, so, going beyond even the definition, if I say that the flow is irrotational what it means is that zeta should also be 0 that is the only way the flow can be irrotational what it means is dv dx minus du dy is 0.

Now, what I am going to do is, I am going to substitute the definition of the velocity components u and v in terms of the stream function. So, we will say u is d psi dy and v was minus d psi dx.

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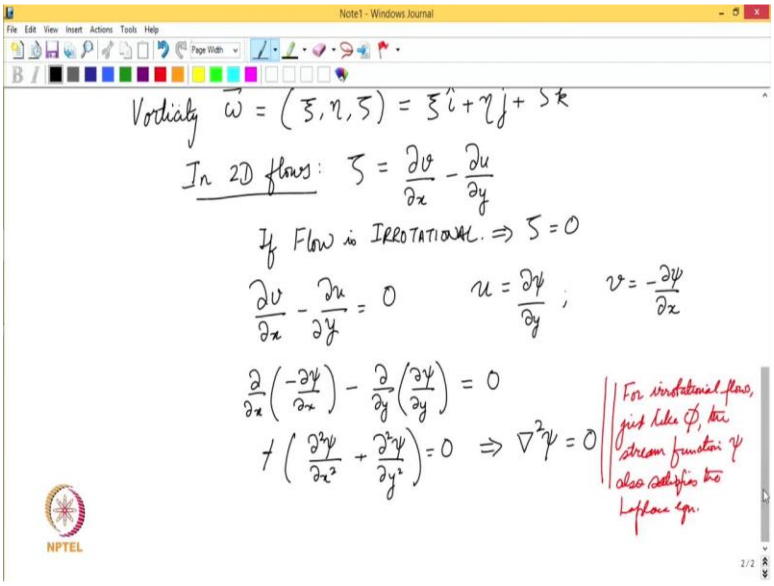

So, if you plug it in into the definition of zeta we get d by dx of minus d psi dx minus d by dy of d psi dy to be 0. And then if we take it one step further, we will get minus of d 2 psi dx squared plus d 2 psi dy square is 0 when the negative sign is immaterial, so, let me just cancel it off. So, what we get is in 2D flows del square psi is 0 if the flow is irrotational.

So, this very important result that we have derived now is that for irrrotational flows just like the velocity potential the stream function also satisfies the Laplace equation just like phi the stream function which is psi also satisfies the Laplace equation and this is going to be again a very important tool for us in the following lectures.

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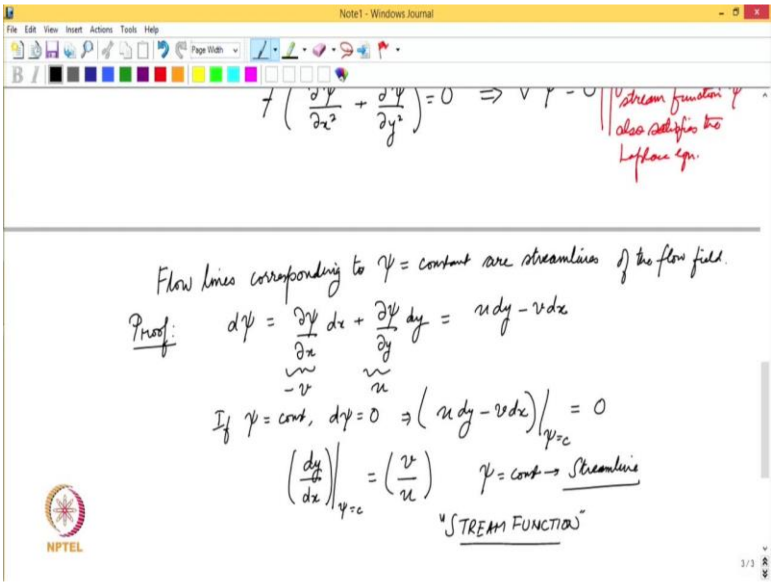

Now, let me state one key fact here and then we will try and prove it using the definitions that we have come up with so far. The theory that I am going to put up is or the phi that I am going to put up are or is that that flow lines corresponding to a fixed value of psi. So, for instance a psi is some constant. So, flow lines corresponding to a given value of psi are streamlines of the flow field.

So, if there is a line over which psi is constant, then that line is going to be a streamline of the flow field. And we can easily prove this by the definitions that we have come up with, we could say d psi, the total change in psi could be written as d psi dx dx plus d psi dy dy. Now, note that d psi dx is minus v and d psi d y is u. So, we can write this as u dy minus v dx.

Now, let us consider the case that the stream function is a constant. So, if psi was a constant, then d psi must be 0 or the change of psi on that line must be 0, which gives us u dy minus v dx over this line psi equal to constant must be 0, which if we do a little bit of rearrangement, now, you can see that dy by dx for a constant psi is given as the ratio of the two velocity components. So, v by u.

Now, this is precisely how the streamline is defined, if you recall we said the streamline is an imaginary curve, the slope of which or the tangent to which gives us a direction of the velocity vector. So, it is this which is the slope of the streamline. And so, we can say that psi being a constant corresponds to a streamline.

Furthermore, each value of that we assign to a stream line each unique value means or pertains to a new stream line. So, hence, we get the name stream function. That is the whole idea why this is called as a stream function that this this function can take a range of values. And for each value or each unique value, you get a unique streamline.

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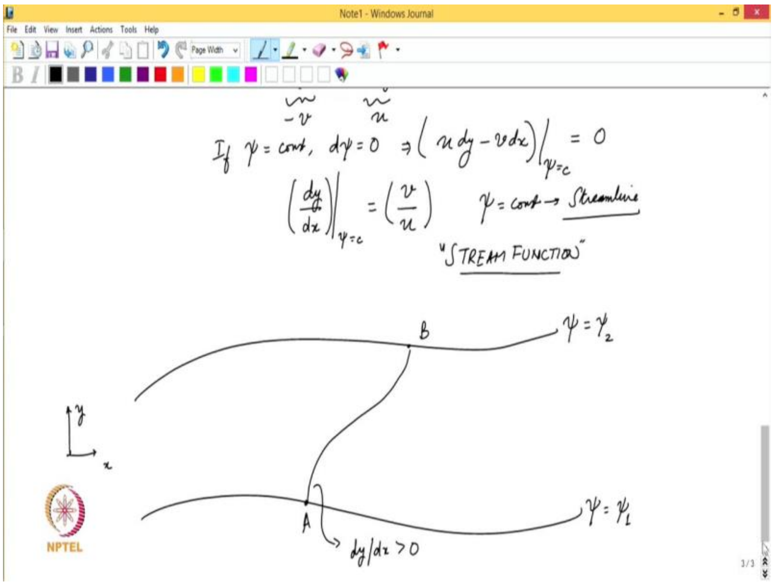

So, now, let us consider a scenario where we have 2 streamlines in a flow field, say psi equal to psi 1 and psi equal to psi 2. And let us define a coordinate system as well. So, let us say we have x here y here. So this is a 2D flow field. Now, the question that I am going to pose is this is the following. I want to know the flow rate of the fluid flowing between these 2 streamlines.

Note that we had said earlier in the previous lecture, we said the dot product of the velocity field with the normal vector on a streamline is 0. So that means that there must be no mass flow rate across a streamline. So between these 2 streamlines, there must be some fluid that is flowing. And you want to quantify how much is that flow rate.

So what we will do is, say we take any 2 points A and B, A on streamline psi 1, B on streamline psi 2. And let me draw a convex surface connecting A and B. So say we take a surface which has the following shape. And the shape is chosen specifically so that the slope on this surface is always positive. That is just for convenience, we could have chosen some other surface as well. But just to make a point just to prove it quickly. Let me say this is what I am going to choose.

So it is a curve, this line AB is of arbitrary shape, but it is such that it has a positive slope. For all we care, it could even be a line connecting in a straight line. But for now, let us just take a curvilinear section.

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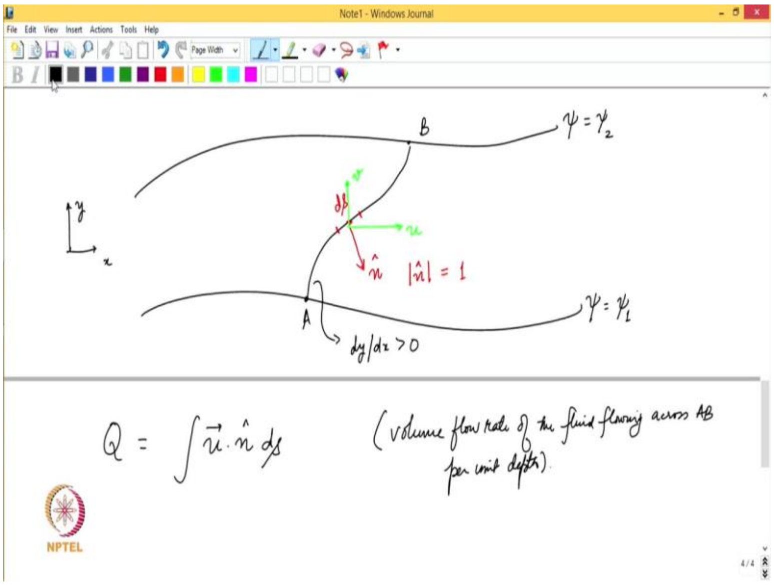

What do you want to determine? We want to determine the following quantity we want to know what is the flow rate across this line AB and that would be given by this function Q and let me now show you what Q is. So basically, if we take a small element, and which I am going to represent by a large element here for only for the purpose of for visual clarity, if I take an element which has a length of ds along this curve, AB, say we are going from A to B, so anywhere the ds will be pointing from A to B, the normal vector to this point would be in this direction. And that is a unit normal, which means modulus of unit normal is 1.

If I can decompose the flow at this location, I could say, this is, for instance, the x component of velocity, this is the y component of velocity, u and v. I am interested in knowing what is the flow rate across this element, small element infinitesimal element ds and that too per unit depth. So, this is the volume flow rate of the fluid flowing across AB per unit depth and that is what we are trying to calculate.

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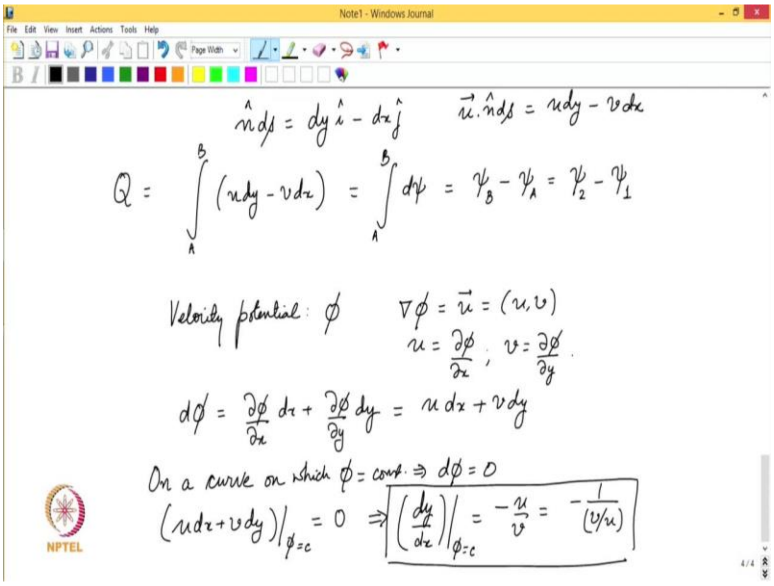

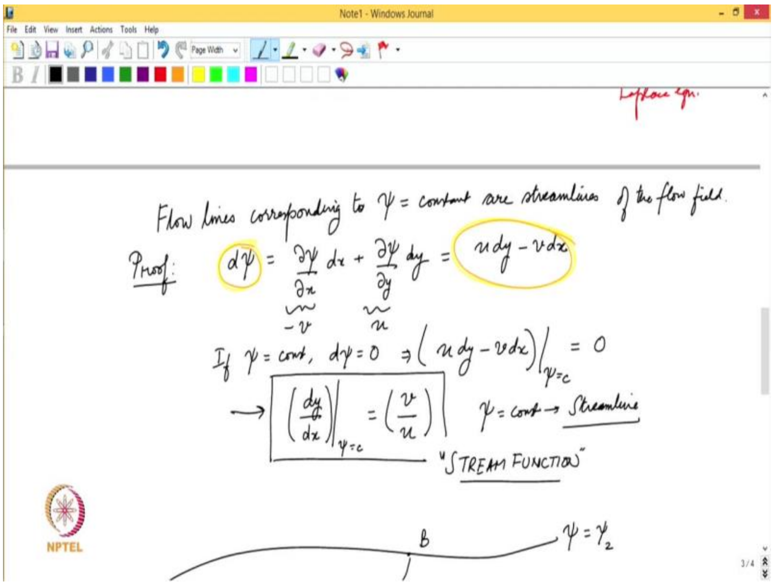

Now, it can be easily proved and which I am going to not do here, but you can easily work it out. That the normal vector are in fact, the normal vector times ds can be easily written as dy i cap minus dx, j cap where dx and dy are the length when you project ds along x and y direction.

So, knowing this, let us calculate Q. So Q would be, now remember, it is u dot n ds. So clearly u dotted with n ds would now be u dy minus v dx. So, u dot n ds will be u dy minus v dx. And now it can integrate over the path A to B so, we will get u dy minus v dx as the integral from A to B and now I note that u dy minus v dx is the total change in the stream function.

This comes from the definition that we had written right here. So, if you see this, this is what we are using now, with using this definition of d psi. So, we can plug this in and then we know that it is a total change in psi. So, that must be psi at B minus psi at A which would be psi at B psi 2 minus psi at A psi 1. And so, the total flow rate per unit depth between any 2 stream lines is the difference in the stream function values and that is the significance of the stream function. So, we can calculate the volume flowing between any 2 stream lines by the difference between the stream values or the stream function values.

Now, let us consider the, we bring we go back to the velocity potential and let us look at a specific relationship between velocity potential and stream function. Recall it velocity potential was phi and gradient of phi was u which is basically u comma v. So, clearly u must be d phi dx and v must be d phi dy that is just from the definition.

So, let us consider the total change in the function phi. So, d phi would be d phi dx dx plus d phi dy dy which we can now write as d phi dx being u I can write this as u dx d phi dy v, we can write this as v dy.

And we can now say that if I am on a curve on which phi is a constant on a curve on which phi is a constant would mean d phi on that curve would be 0 what it means is? That u dx plus v dy on this specific curve where phi is a constant is 0 if there is a curve in space of that type, which means, if you rearrange again we can prove the dy dx for this curve where phi is a constant is minus u by v, which I could also write as minus 1 by v by u.

So, I want you to put this specific result in perspective from what we have done just a little while ago, which is to talk about the stream function. So, together with what we have dones here, which is to prove that the slope of the stream function or slope of the stream line is given as a ratio of the y component of velocity to the x component of velocity, we now see that for a stream or for a potential function or for a potential flow problem or curve on which phi is constant, the slope is given by the negative of the x component of velocity versus the y component of velocity.

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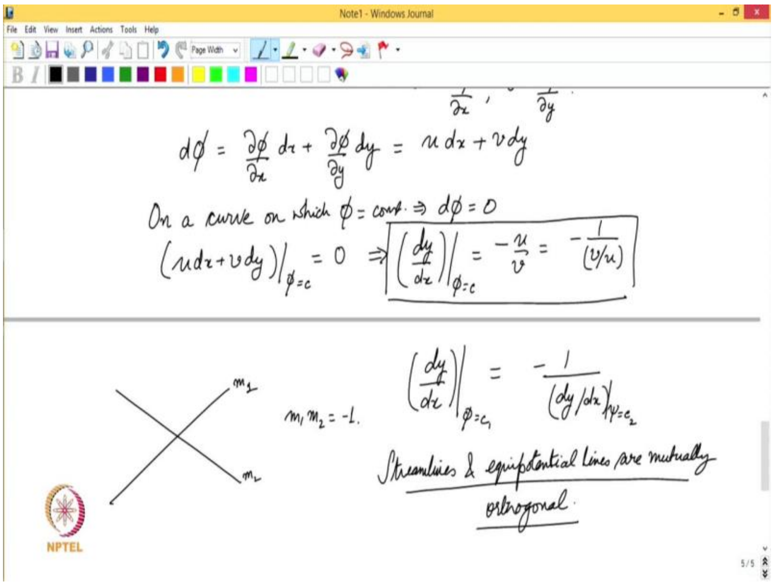

If you recall from coordinate geometry, if there are 2 lines across which let us say the 2 lines have slopes m 1 and m 2 the condition that these 2 lines are orthogonal to each other would be that m 1 m 2 is minus 1. The same thing has happened here that if you look at the slope of the line at this specific location on a given value of velocity potential then that is negative 1 by negative inverse of the slope of the line which is given by the constant stream function.

What it means is that streamlines and equipotential lines by equipotential we mean lines where the potential is a constant are mutually orthogonal or are orthogonal to each other. So, this will also be very handy when we start drawing flow diagrams in the next few lectures. So, I will finish this lecture here. And I hope to see you in the next lecture. Thank you.