Ideal fluid flows using complex analysis
Professor. Amit Gupta
Department of Mechanical Engineering
Indian Institute of Technology, Delhi
Lecture No. 13
Singularities and their transformations

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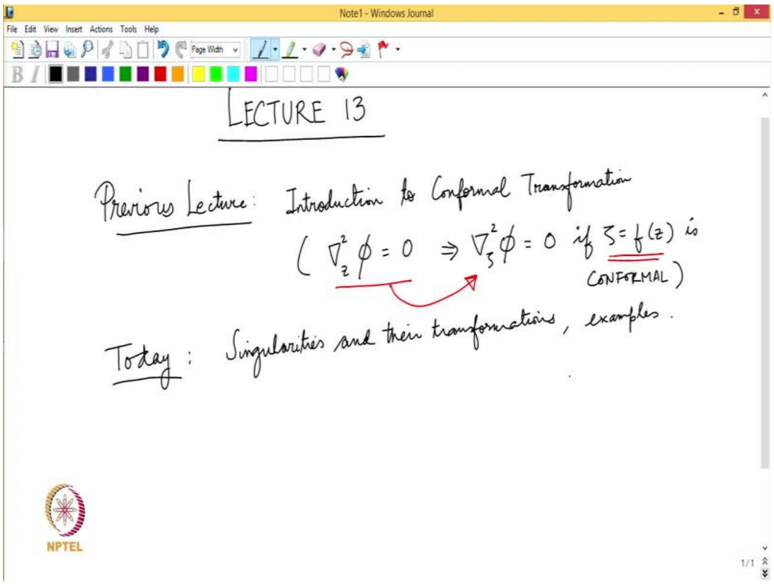

Welcome to this last lecture in this lecture series on ideal fluid flows using complex analysis. So, in the previous lecture, we briefly spoke about conformal transformations and the idea that we discussed was that in the Z-plane if we have a potential function phi or psi, which satisfies the Laplace equation, then it also satisfies the Laplace equation in a new plane zeta, where zeta is a transformed variable from z.

So, zeta is f of z which is the conformal transformation, which allows us to take from the first form to the second form, more importantly, this mapping is called conformal if the function f is analytic. So, we discussed these aspects in the last lecture. Now, in this lecture today I will talk about singularities and how do you singularities transform? And I will take a few examples a couple of examples to show you how conformal transformation actually is implemented.

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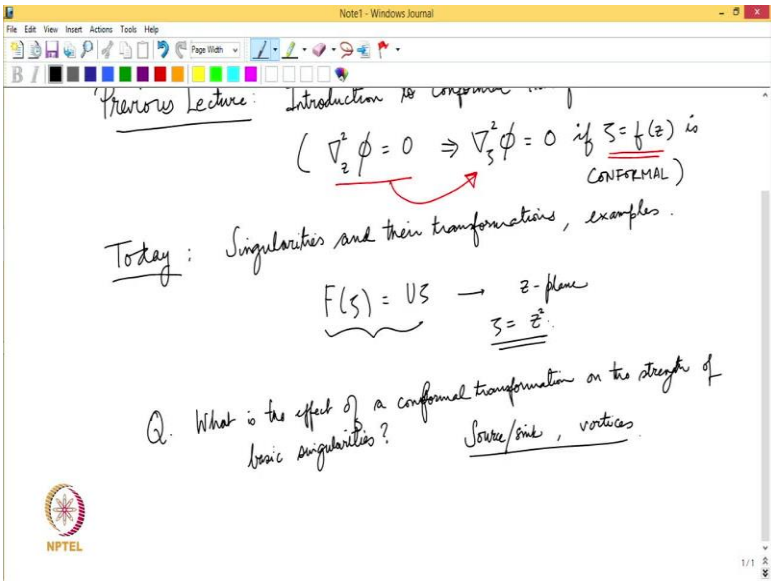

So, in the previous lecture, one thing that I did was to show how velocity potential which is say f of zeta which is U have zeta, how this transforms to the Z-plane and their conformal transformation, which I believe I wrote as zeta is z square, so we looked at this case, Now, note that this potential U of U times data is actually an analytic function and so the transformation is quite easy to accomplish, but, the things get a little more complicated if we are dealing with singularities in any of the planes.

So, for instance, let us look at this question or let us try to answer this question that what is the effect of say this a conformal transformation on the strength of basic singularities, some basic similarities that we have come across in our study of ideal fluid flows. To rephrase this question, what I am asking is how do these singularities say in the Z-plane transform to the zeta plane or vice versa? It could be either round.

So, the two singularities that we typically talk about in ideal fluid flows are due to for instance one is because of presence of source and sink and the other one is because of vortices because these two functions tend to be non-analytic when you go at the point of singularity.

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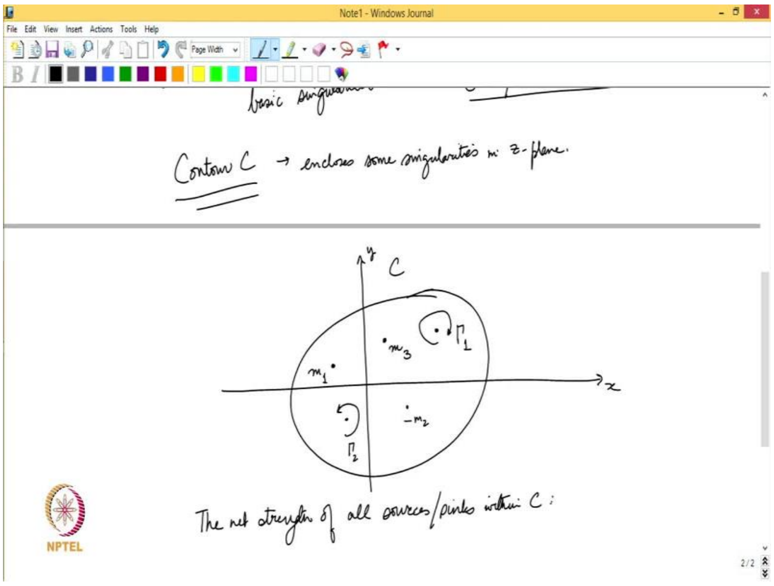

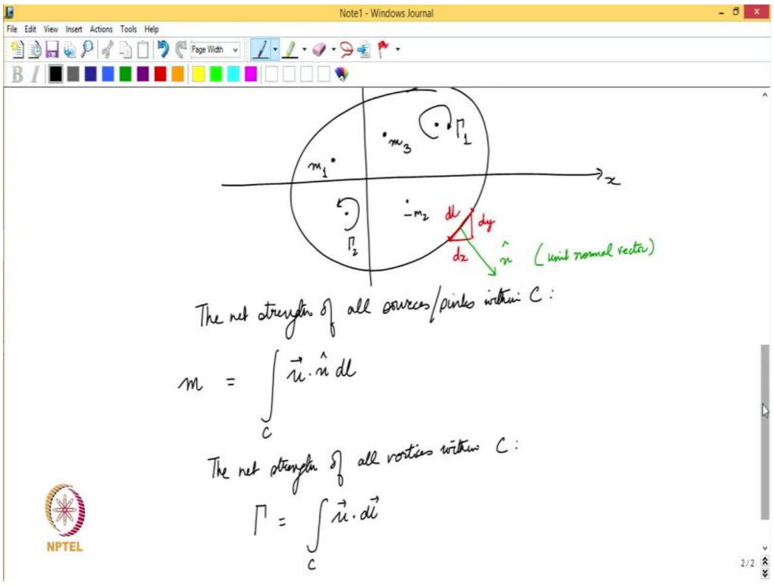

So, let us consider that we have for example in the z plane let us say we have a contour C, let me label a contour C in the z plane which maybe for instance something of this type, this is the contour close contour C, say it contains some singularities and the similarities that we are dealing with our sources and sinks are vortices.

So, I do not know so there is no specific manner in which I want to do this but let us say there is a singularity gamma 1 which is a counterclockwise or clockwise vortex, maybe there is another singularity of gamma 2 a counterclockwise vortex. Maybe there are some sources and sinks for instance, say this is a source m1, maybe there is a sink m2, there is a source m3 so and so forth. So, this contour C encloses these singularities C in the Z-plane.

Now, the question I am going to ask is, first of all in the Z-plane itself, can I calculate the net effect of these singularities? So, for instance, the net strength of all sources and sinks which are enclosed within the contour C we can determine this by calculating the number of sources or sinks the strength is defined in terms of the volume flow rate crossing the boundary of the contour.

So, we could write that the net strength for example m is given us integral u dot n dl when we would integrate this over the contour C, where I could say that the element dl is maybe an infinitesimal element shown rather big here for visual clarity. So, some element dl on this contour which has maybe components of dy and dx and the normal vector is defined as the unit normal vector, this is unit normal vector at this location is defined in a certain way.

So, the net strength of all sources and sinks would be given in terms of this integral which is u dot n dl. Similarly, the next strength of all vortices will be given in terms of the circulation which we define say in terms of a variable gamma that would be integral over the control C of u dot dl. So, say we want to first calculate these two numbers, in this contour C what is the net strength of vortices and what is the net strength of all sources? So, we know that gamma and m are given by these two integrals to show you how these are correlated to the complex velocity in the z plane.

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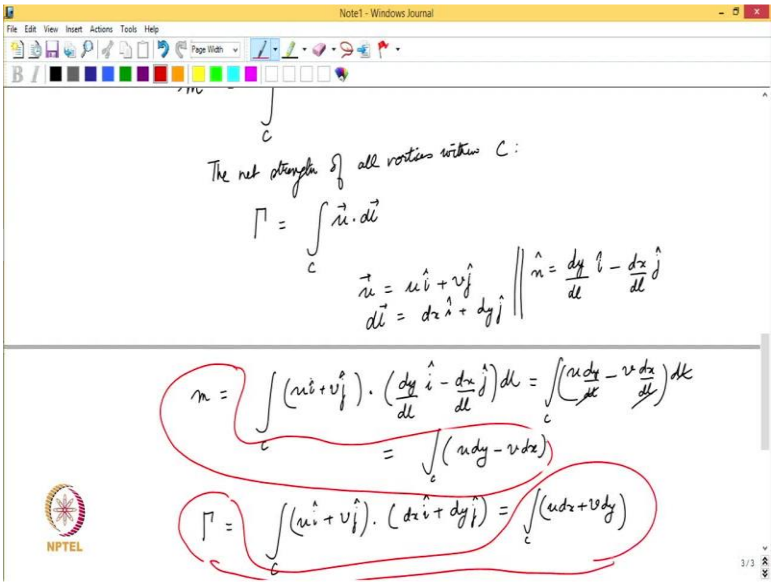

Let us look at first of all the definition of velocity, velocity vector u will be for instance u i cap plus v j cap in terms of complex in terms of vectorial notation. And I can write dl vector to be dx i cap plus dy j cap and more importantly in the normal vector n cap and this we I am borrowing from lecture 10, I think where I worked out the specific form of the normal vector, which you can derive actually, it is quite easy but just to do it in a quick manner of time. We could write end cap to be dy dl i cap minus dx dl j cap, where I basically consider the fact that n vector is perpendicular to dl. So, this is what the dl vector, so this is what we should get for the normal vector.

Now I can evaluate these two integrals, say let us look at m first. m would be u i cap plus v j cap dotted with dy dl i cap minus dx dl j cap times dl. Now, this is simply a dot product, so we will have u dy dl minus v dx dl, which I could write as well dl is a common factor in that sense, we could say this is just integral u dy minus v dx over the contour C in the z plane, which would give us a net strength of all sources within the contour C.

Similarly, we can write the net strength of all vortices to be u i cap, plus v j cap dotted with dl vector which would be just dx i cap plus dy j cap which would be just know u dx plus v dy. So, let us just keep this in mind that m is given by this integral and gamma is given by this integral.

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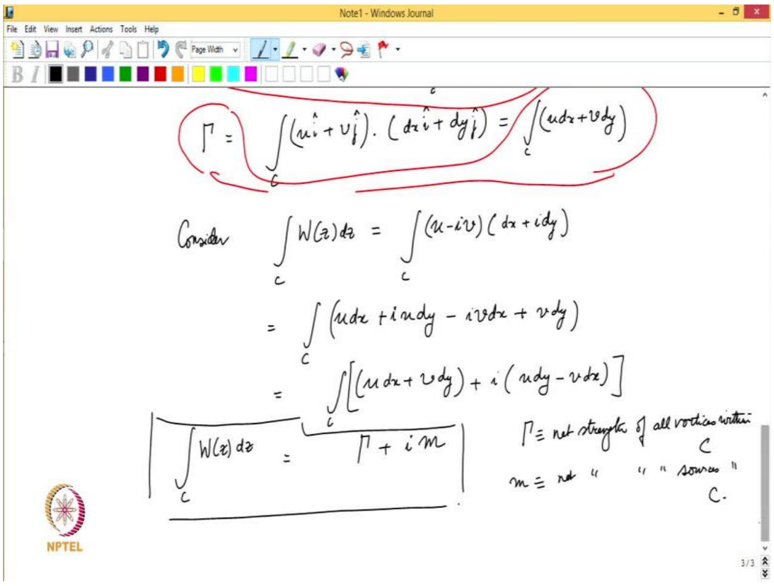

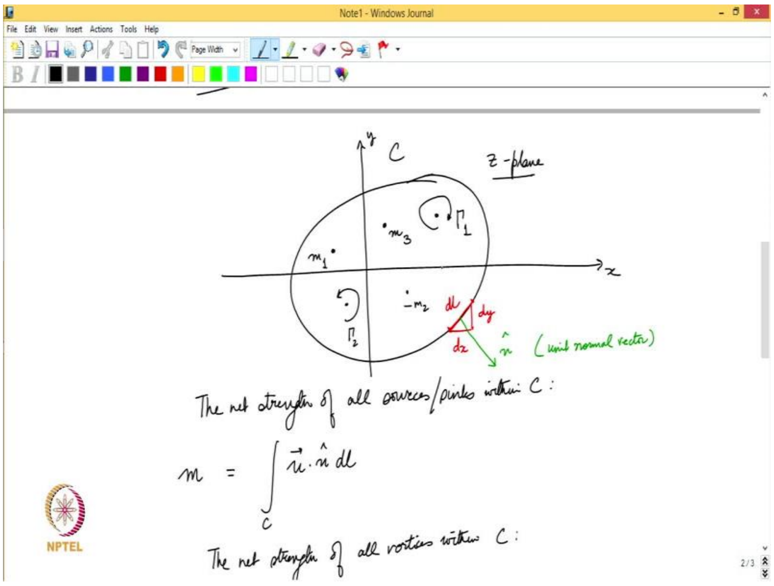

Now, let me show you how these integrals can be evaluated from complex velocity w. So, consider the following integral consider we will looking at W z dz over this contour C. This is given as remember W is u minus iv in the complex plane, dz would be just dx plus i dy. So, if we open up the brackets now, we will get u dx, let us say minus or other plus i u dy minus i v dx and then we will have plus v dy. So, this we can separate as a real and imaginary part.

So, we will get u dx plus v dy plus iota times u dy minus v dx which is integral u dx plus v dy over the contours C is given as gamma plus we will have iota integral u dy minus v dx will be m. So, this integral W z dz is gamma plus i m. So, gamma is a net strength of all vortices which are within C that is the most important part within C, and m is the net strength of all sources or sinks within C.

So, drawing a contour C around a bunch of similarities can give us the net strength of those singularities. Now, we will use this fact to our advantage, when we look at transformation of these singularities now which we have like the singularity that you see on the screen, you will look at how these singularities transform under a conformal transformation. So, when we go from a z plane to the zeta plane how these functions will now change.

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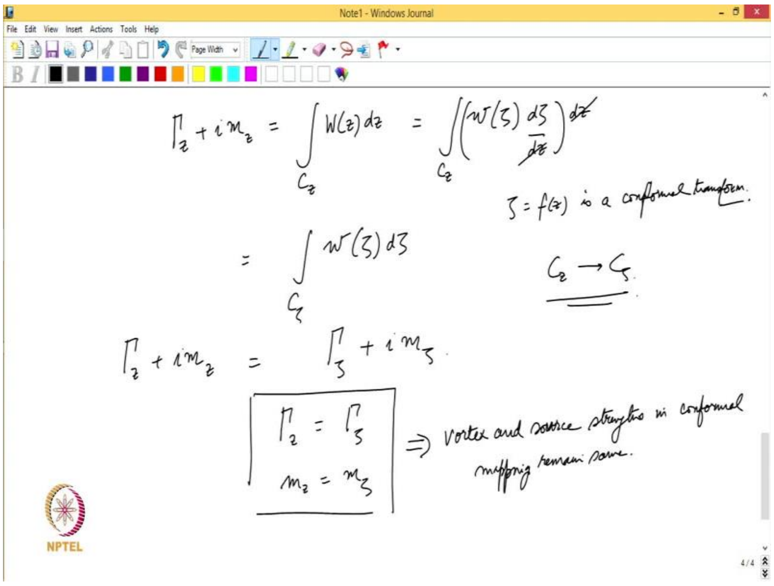

So, what we have come across now is that gamma let us say in the z plane plus Iota m in the z plane, subscript z denoting it is a z plane is integral W in the z plane dz integrated over a contour C in the z plane. I can write this integral by using the fact that W z which is the velocity in the z plane, it could be written in terms of velocity in the in the zeta plane. So, we could say this is for instance, w zeta times d zeta dz where zeta is f of z is a conformal transformation.

Now, this integral can now be written as w zeta d zeta, but this time the curve is no zeta or c zeta. So, we have basically transformed C z to C zeta in the conformal transformation. But I also note that once I write this as W zeta d zeta in the zeta plane, this should also be gamma in the zeta plus Iota m into zeta plane, that would be on that contour C in the zeta plane, the integral of the complex velocity will give me the real part as the sum of all ordinate vortex strength inside this contour and the imaginary part will give me the net strength of all sources within that contour.

So, what we have reached to is this point where we say gamma z plus i m z is gamma zeta plus i m zeta, which means gamma z is gamma zeta and m z is m zeta. And what it means is, it is a powerful result now that we see, it says that vortex and source strength in a conformal transformation or in conformal mapping remain the same or they are preserved, these strengths will now remain the same, the strength of the singularities.

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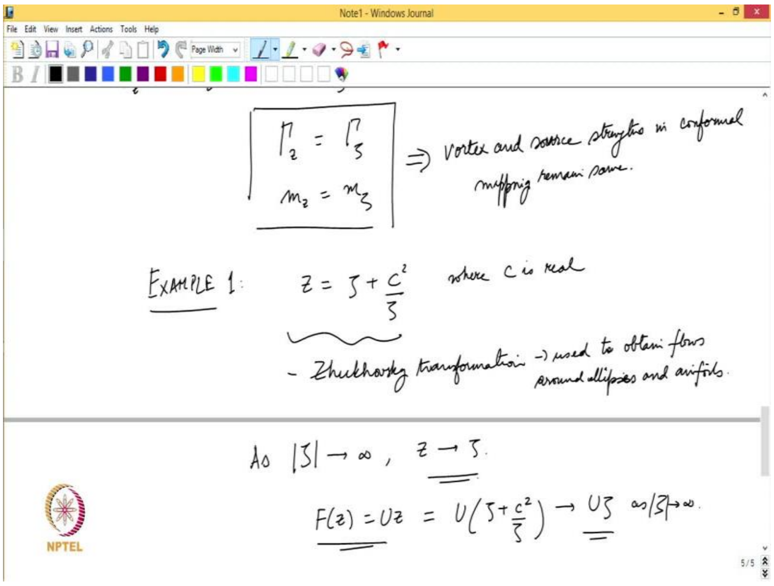

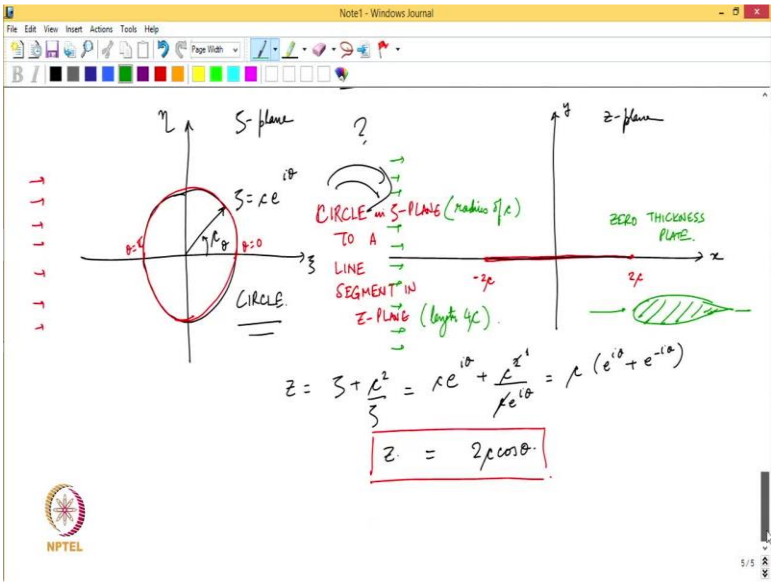

So, let us use an example to illustrate how this fact or does these ideas work out? So, let us consider an example the first example for today's lecture. Say we look at the following conformal transformation which is given as z or z is zeta plus c square by zeta, where c is a real valued number, this transformation is called as a Zhukovsky transformation and it is a very powerful transformation which is used to obtain flows around ellipses and airfoils.

A note that when zeta or mod zeta goes to infinity, which means that you are far far away from the origin, as mod zeta goes to infinity, z will actually go to zeta, when you go far away the function z starts going linearly with zeta. So, that becomes an important fact, considering that if you have a uniform flow, which is far away from the geometry that you are considering, then the uniform flow will transform exactly as uniform flow.

So, what I mean is that if you have for example, f of z to be u z, which is the complex potential for uniform flow far away from the origin, then we can say this will be u zeta plus c square by zeta and this will go as u zeta as zeta goes to infinity, mod zeta goes to infinity when you go far away from the origin. So, the velocity potential remains preserved in this specific transformation.

Now, near the origin of the zeta plane so, say we have a zeta plane which is something like this where the coordinates are z and eta. So, near the origin, say we take a circle of radius c, we are, say we traverse around a circle of radius small c, so that the functions zeta or the number any point on the circle happens to be c times e to the power Iota theta, where theta is this angular position along the circle in the zeta plane.

So, this is a circle in the zeta plane, and I want to now talk about how this transforms to the z plane. So, let us just draw this here, we are interested in knowing how this goes to the z plane. So, the question I am asking is, what does this shape turn out to be in the z plane. So, the transformation is fairly simple we know that z is zeta plus c square by zeta, so we can put zeta to be c e to the power i theta, what we will get is the following, we will get one c will cancel here.

So, we will get c times e to the power Iota theta plus e to the power minus Iota theta. We know that e to the power Iota theta is cosine theta plus Iota sine theta, e to the power minus Iota theta is cosine theta minus Iota sine theta. So, then imaginary terms cancel, what we get is 2c cosine theta. So, now it becomes very interesting we got in the z plane we actually have a real value representation of the complex number from the zeta plane.

So, let us look at what the shape resembles say we go from theta zero to theta pi in the zeta plane. When you go from theta zero theta pi, z will go from 2c to minus 2c. So, we have maybe 2c here and minus 2c some point here. So, when you go from theta zero to pi, you are essentially moving along this line you go from plus 2c to minus 2c and when you go from theta pi to 2 pi in the zeta plane then z will go from minus 2c to plus 2c, so you basically then go back to this point.

So, what have we done here using this Zhukovsky transformation, we have transformed the circle in the zeta plane to a line segment in the z plane. So, we have transformed a circle in the zeta plane to a line segment in the z plane. It is a line segment because it is a line which has zero thickness, which is zero thickness line or you could say it is a zero thickness plate.

Some mean being a limiting case of an airfoil which has zero thickness, so it is a flat plate with zero thickness.

And so, any flow over the circular cylinder has been transformed now to flow over a plate. So, just the only thing is that this is a circle of radius c. So, we have circle in zeta plane which is radius of c and we have a line segment in z plane which is length of 4 c. So, we have been able to transform this using this very basic transformation. So, that is the first way how in a flows can now be recreated.

So, we already have seen in this course how we can derive flow over a circular cylinder, now using that approach we can now derive flows over for instance in this particular case flow over a flat plate. There are some modifications which can be made to obtain even flows past an airfoil and it could be suitably modified to get flow paths and airfoil, I am just limiting myself not to go to this problem as of now, I think these are advanced topics which you can easily work out now once you know the basic foundation has been established.

Now, let us look at by the way just to complete this discussion, remember that we I said that in the data plane the flow way far away from the origin is uniform, so we had uniform flow. So, here as well, now in this case, we will have flow far away transforming to uniform flow. So, we have now the same approach or the same idea being applied. So, we have we knew the flow on the left side now we have derived the flow on the right side using this transformation.

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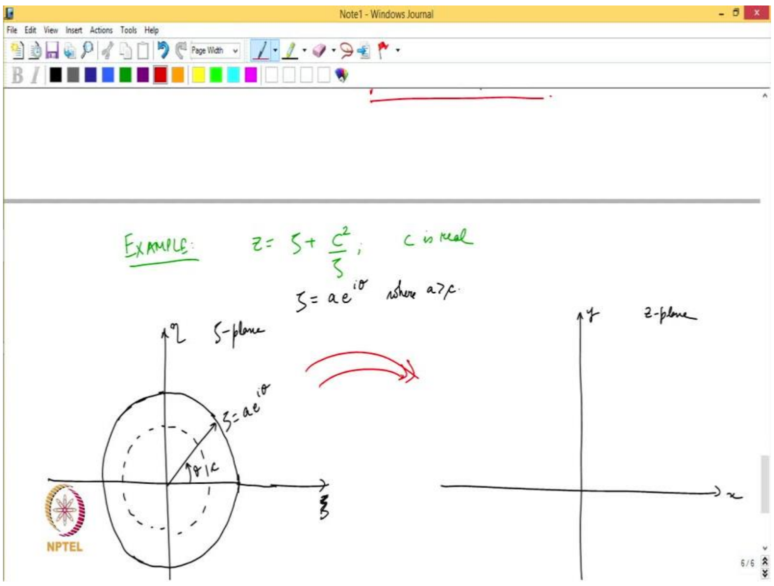

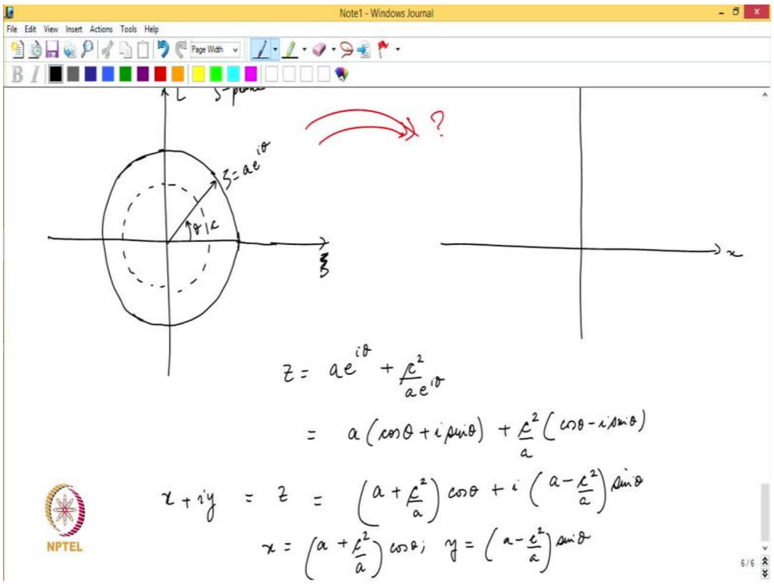

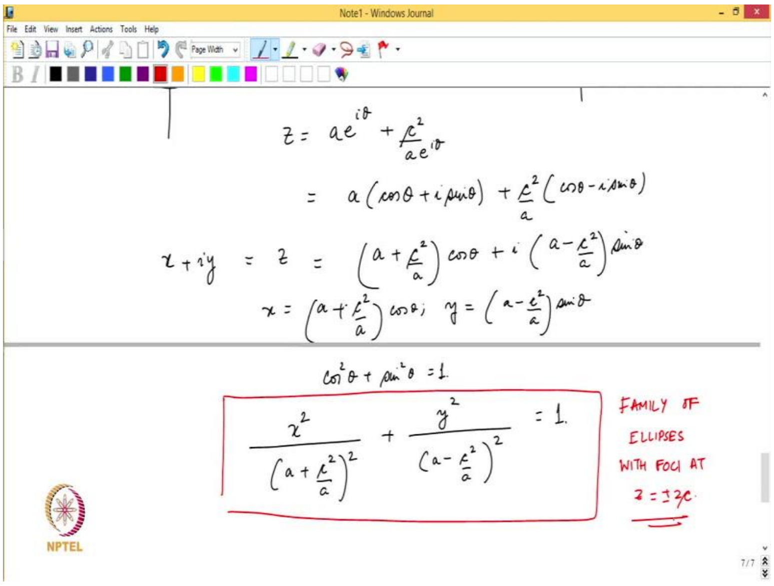

Now let us take another example with the same conformal mapping we still have z to be zeta plus c square by zeta, c is real. So, in the zeta plane we have zi in the eta coordinates. Now I want to look at how a new transformation works out in the jet plane which is given by the following. The transformation remains the same but except the representation I am going to use zeta is slightly different.

Let me say that zeta is a e to the power Iota theta. So, it is a point on a circle of radius a, where a is greater than c. So, the way to put it is that if this was c or the circle corresponding to radius of c, we now take a circle which is larger, which is something of this type. So, this is, at any given angle theta, we now have zeta to be a e to the power Iota theta, it is a circle, slightly higher radius. And I want to now find out what would be the transformation going to the Z-plane.

So, it is fairly simple, again we use the same approach or the same method, we say z is a e to the power Iota theta plus c square by a e to the power Iota theta, that would be the transformation. So, all points on this circle of radius a get transformed to some points on the z plane, so we can say e to the power Iota theta would be cosine theta plus Iota sine theta plus c square by a will have cosine theta minus Iota sine theta. Which we can now combine the real and imaginary part this would be a plus c square by a cosine theta plus Iota times a minus c square by a sine theta.

And this is equal to z which should be some point say x plus i y in the z plane, now come from the zeta plane to the xz plane. So, if I compare the real and imaginary parts, I can say x is a plus c square by a cosine theta and y would be a minus c square by a sine theta. So, far, I cannot say much about what this transformation looks like, but I can derive the locus of points that satisfied this transformation. And it is easy to see you would have come across this locus in coordinate geometry and I use the identity that cosine square theta plus sin square theta is 1, to derive the locus.

So, what I can say is x square by a plus c square by a whole square plus y square by a minus c square by a square is 1, that is a locus of points in the in the z plane. And what is this represent, this locus? This is a locus of a family of ellipses depend on the value of a and these are locus of family for ellipses which have foci at set equal to plus minus 2c, 2 foci are actually on the real axis.

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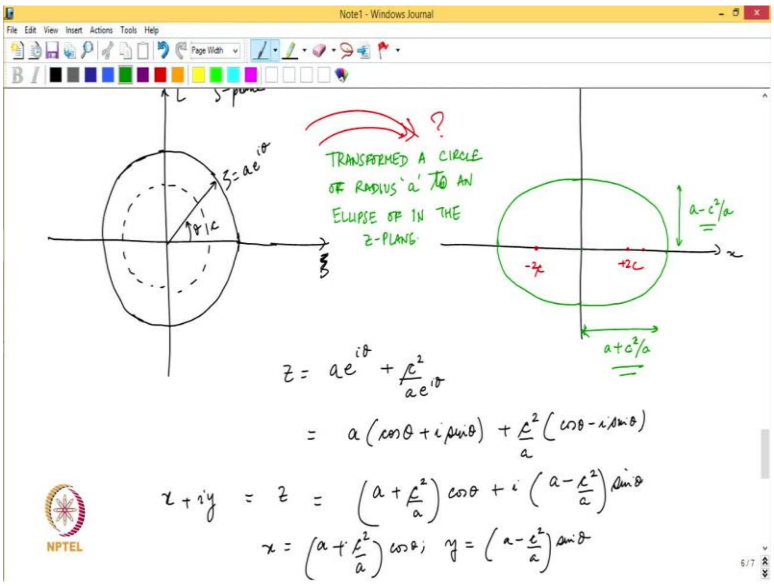

So, we will have for instance for a particular value of a, we will have maybe again I can draw it here to this say this is the focus plus 2c there is a focus minus 2c, what we have is a transformed shape which would look like an ellipse So, what we have done by this is transformed our circle of radius a to an ellipse of or an ellipse in the z plane and in for this ellipse the minor and major axis can be easily now written down. So, the major axis is a plus c square by a which appears below the denominator of x and then the minor axis is a minus c square by a.

So, now, flow past a circular cylinder of radius a has been transformed to flow past on ellipse with minor and major axis given off a minus c square by a and a plus c square by a as I have written here. So, this another example of where conformal transformation can be used to derive much more complicated flows that we otherwise could not have done just by using various potentials.

So, this is the last example that I wanted to take up off course there are wide variety of thinks that could be covered now, they could be you could look at how you could recreate or transform many of the other flow that we have covered in this part of this course but I believe that to get some basic understanding of ideal flows using complex methods this is a good representation for this short lecture series. So, I hope that you have enjoyed listening to these lectures and it is been a good learning experience for you as it is been for me as well. Thank you.