Ideal Fluid Flows Using Complex Analysis
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Lecture 11
Calculation of Forces Using Derived Flow Field

So, in the previous lecture we looked at the method of images as a way of analyzing bounded flows and we also looked at the Blasius theorem. So, today what I want to do is to use some of these concepts especially the Blasius theorem.

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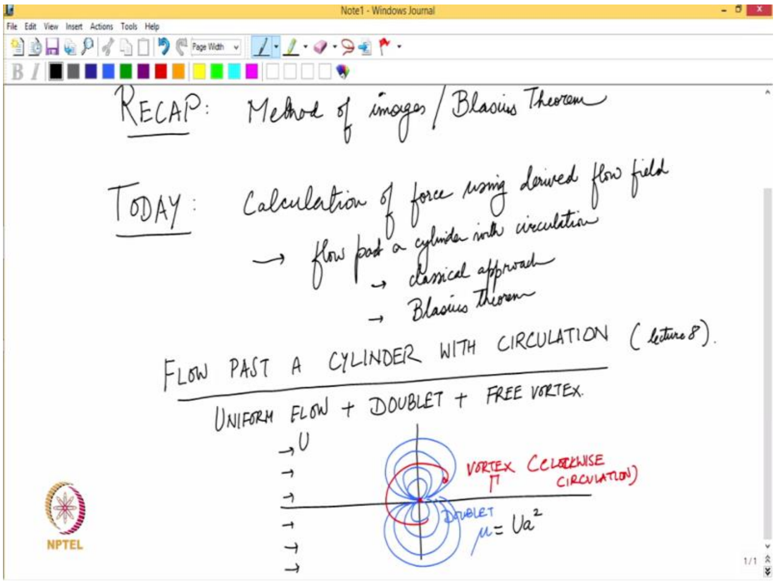

And we would show that how we can use this theorem for calculation of force over a body which is emerged in this flow and we will do this specifically for flow past a cylinder with circulation which would mean that gamma will not be 0 in this case and we will do it in two ways. One would be the more classical approach which is by calculating or integrating the pressure distribution over a surface of a cylinder.

The second approach which will be using the Blasius theorem and thus by the end of this lecture hopefully you will be able to understand how the Blasius theorem comes in very handy and you will understand the utility of the Blasius theorem how powerful it is in deriving these solutions or calculating these forces. So, let us start with the old problem now which is flow past a cylinder and this time as I said we are doing this with circulation.

So, we started this problem in Lecture 8 in case you want to refer back to the videos, but somewhere around lecture 8 we started this discussion and what we discussed this problem is that this is a problem which is formed by superposition of three elementary flows. We have uniform flow then we have a doublet and finally we have a free vortex. So, these three elementary flows when combined give us flow past a cylinder.

So, we had as I said uniform flow so U is the velocity, we had a doublet centered at a the origin. So, this is a doublet which had some kind of a flow feature that we have seen now and then we also had a free vortex, but this is a very special free vortex this is a free vortex which had a clockwise circulation. So, we had free vortex which is also centered at a origin, but with a clockwise circulation.

So, I can say this is clockwise circulation, the strength of it was gamma, doublet you could write as mu and of course U is given. So, when we added these three we got the flow past a cylinder. Now, you also might remember mu was U a square and a was the radius of the cylinder for which this flow is applicable.

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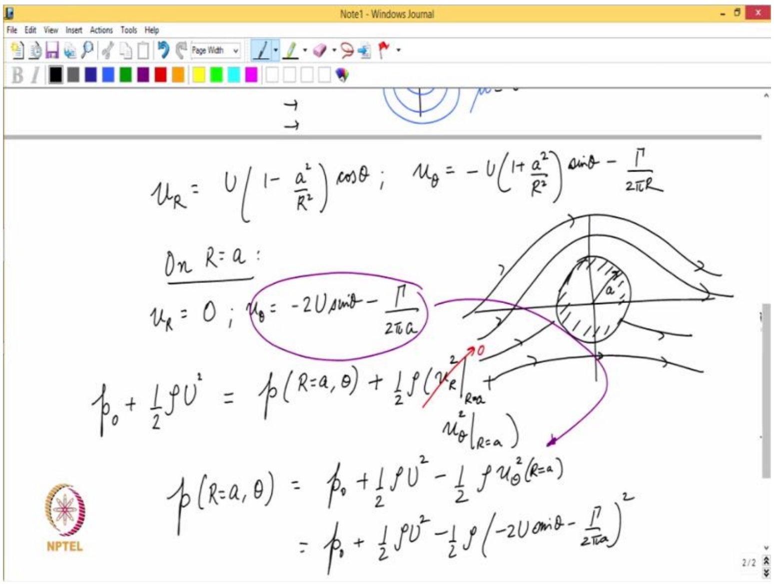

For this flow we found that the velocity component U R comes out as U 1 minus a square by R square cosine theta and u theta is minus u 1 plus a square by R square sin theta minus gamma by 2 pi R. These were the velocity components in general. Recall that the flow fields looks something of this type that we have stagnation point where the flow happens in this fashion.

But what I am saying is that the flow field the velocity vector around the cylinder will be given by these components U R and u theta. Now, to calculate force on this cylinder as I said we will do two methods one is integration of the pressure over this cylinder that would give us a force on this object. What I need to do is to calculate first the pressure distribution. So, to calculate the pressure distribution let us first derive the velocity variation on the surface of the cylinder.

Note that this cylinder has a radius a. So, on R equal to a which should become any point on the surface of the cylinder U R as you can see will be 0. If you put R equal to a in that function and u theta will come out to be minus 2 U sin theta minus gamma by 2 pi a that is the velocity or the two component of velocity on the surface of the cylinder. I can also apply Bernoulli's equation now to be able to write the pressure distribution from the information about the velocity.

What I mean by that is we can say that if p naught is a pressure far field and we already know that the velocity is capital U so half rho U square is a dynamic pressure. This should be pressure on R equal to a as a function of theta plus half rho say U R at R equal to a square plus we can say U theta square on R equal to a. I know that U R on R equal to a is 0.

So, I need not worry about this term, but basically I can use this now to derive what pressure would be on the surface of the cylinder which would be p naught plus half rho U square minus half rho u theta square on R equal to a and we can use that function there to write this as p naught plus half rho u square we can bring let us say this function here into here then we can write that this would be minus half rho we will have a minus 2 U sin theta minus gamma by 2 pi a square.

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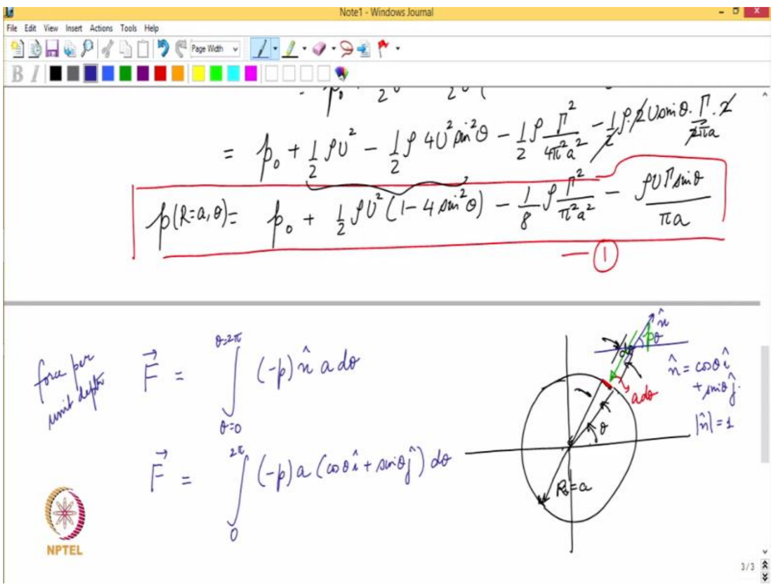

So, we can sort of simplify this a little bit open above brackets in maybe take like terms together. So, what we will get is p naught plus half rho U square. We will have a minus 1 by 2 rho 4 U square sin square theta then we will have minus half rho gamma square by 4 pi square a square and then we will have minus half rho times 2 U sin theta times gamma by 2 pi a into 2 which would give us well we can look at there is a rho U square term common here.

So, I could maybe combine this as p naught plus half rho U square 1 minus 4 sin square theta. We have an additional term which is minus 1 by 8 rho gamma square by pi square a square and then finally we have a 2 cancelling here and 2 maybe cancelling there. So, we will have rho U gamma sin theta by pi a. This is p, R a comma theta and this is an important result that we would need say this is equation 1 for now.

So, you can recall remember when we looked at pressure distribution over a circular cylinder without circulation we only had the first two terms and we even looked at how this pressure varies as a function of theta. Clearly, the solution that we have also make sense because if you put gamma to be 0 there was no circulation we would revert back to the same solution for pressure which we got for a cylinder without circulation.

So, it is a good way of verifying whether we are on the right track at least gamma 0 gives us the same pressure distribution. Now, before you move further let me talk about how we would get force on this cylinder. So, say we have this cylinder here of radius a. This is say R equal to a or let us put it here R as a this is the radius. Now, we need to integrate over the surface the only independent variable here is actually theta.

So, we can take at an angle theta a small segment of elemental distance d theta. So, this angular distance here is d theta. So, we take a small element which has the angular increment of d theta. So, the length of this element is actually this length which maybe I can do by a different color this length is actually a d theta because a is the radius of the circle or of the cylinder. More importantly the pressure actually acts on this surface inwards.

So, there is pressure acting inwards. The normal vector at this point on this surface is opposite which is the unit normal n cap it is acting outwards. What we need to do to calculate this force is to integrate or is to write F which is let me say force per unit depth of the cylinder. This will be the surface integral of minus p which is the pressure n cap which is where it is acting along that direction it is acting into the elemental distance or the length of the element which is a d theta.

And we have to integrate this over theta from 0 to 2 pi. So, we go all the way around in the circle. So, if you could somehow figure out this integral then we have our answer. Well we use one more fact here that since the element is at an angle of theta from the x axis clearly the normal vector would also have components given by, for example, normal vector will have component such as cosine theta i cap plus sin theta j cap knowing that the normal vector is a unit normal.

So, it has a modulus of 1. So, we could write this as theta going from 0 to 2 pi minus p a n cap is cosine theta i cap plus sin theta j cap d theta. This is the force per unit length acting on the cylinder. Now, we know the pressure distribution on the surface let us say write this again on a fresh page.

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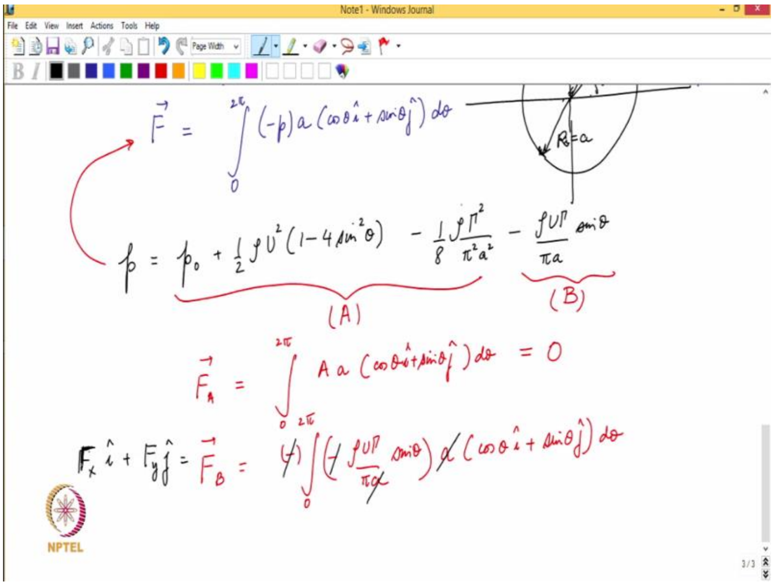

We know that the pressure distribution p on the surface is given as p naught plus half rho U square 1 minus 4 sin square theta minus 1 by 8 rho gamma square by pi square a square minus rho U gamma by pi a sin theta. Just to verify from the earlier expression this is what we have. Let me break this pressure distribution into two parts. Let me say this is part A I am just going to call this as A and maybe this is B.

So, we need to put this pressure into this expression and then evaluate the integral. Now, instead of doing it the whole thing in one shot I will do it in two parts that is the reasons I have written part A and part B here. Well, if you just look at part A and if you were to calculate the force let us say contribution let us say from part A. F A which should be integral 0 to 2 pi of let us say A contribution multiplied by a cosine theta i cap plus sin theta j cap d theta.

So, if you were to integrate this function over 0 to 2 pi you would actually get 0 and this I would like you to verify. This is a simple integral I am sure you can work this out. So, I do not want to spent time basically deriving this trivial result. I am leaving this to you to prove. What I want to do is to show you what the B part gives which is a second part of the problem. So, say we look at contribution on the force from the B segment of this expression which would be 0 to 2 pi.

We had a minus p so minus p would become minus of minus rho U gamma by pi a sin theta times say we will have a cosine theta i cap plus sin theta j cap d theta. So, just to make it a little simple this negative sign cancels with this one we have this a cancelling of with this one. What I will also do is I will split this. Remember this force is a vector force so this will be force, instance, along x.

So, we could say this would be the component along x i cap plus some component along y j cap. So, what I will do is I will split the integral on the right side into two parts where we just look at the x component and the y component.

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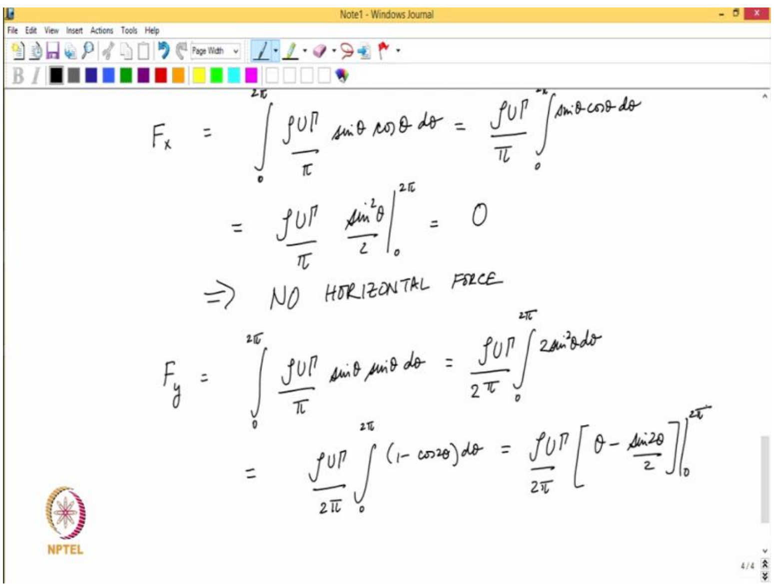

So, for instance, if I look at F x that would be given as integral rho U gamma by pi we will have a sin theta cosine theta d theta that would be because the x part would come from this contribution the coefficient multiplied by cosine theta i cap. So, we can evaluate this integral now very quickly this would be rho U gamma R constant. So, they come out with a integral we have pi anyway being a constant we have just integral 0 to 2 pi sin theta cosine theta d theta.

And I note now that if sin theta is my function then cosine theta d theta is its derivative. So, this integral is quite easy to evaluate we will just have rho U gamma by pi times sin square theta by 2 as a integral which is evaluated between 0 and 2 pi and now I note that sin theta at 2 pi is also 0 and sin theta at 0 is also 0. So, we get a 0 here for this part. What this means is that for the problem that we are dealing with there is no horizontal force for flow over a cylinder with circulation there is no horizontal force to be felt.

Now, what about the vertical component F y this will be rho U gamma by pi sin theta will have another sin theta d theta and what we do is we can pull out rho U gamma by pi we will have a sin square theta so I will just multiply and divided by (()) (19:27) in the numerator, denominator. So, we will have 2 sin square theta in the numerator. So, the reason for doing that is 2 sin square theta can be written in terms of the cosine function cosine 2 theta angle.

So, I would write this 0 to 2 pi 1 minus cosine 2 theta d theta which is what is 2 sin square theta and this is now easy to evaluate. So, we have rho U gamma by 2 pi the integral will become theta minus sin 2 theta by 2 evaluated at 0 and 2 pi.

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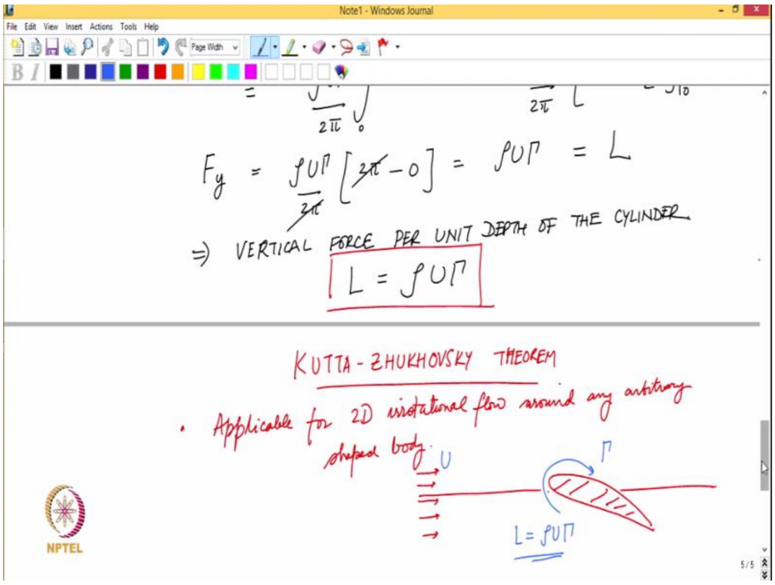

So, let us do this. So, we have rho U gamma by 2 pi theta will just become 2 pi and now note that sin 2 theta at 2 pi and 0 is 0. So, we will just have 2 pi minus 0 and this 2 pi will cancel with this 2 pi so we get rho U gamma that is the force per unit length along a vertical direction; vertical force which I can say per unit length or we want to put it depth it is also the same thing depth of the cylinder.

So, this is what I am going to call as left L. So, we have reached a very powerful result in fact. We have now derived that in this case the left force is rho U gamma there is a vertical force and it depends on the free stream velocity U or the far field velocity and the circulation around the cylinder. Clearly, it also means that if the circulation was 0 the left force would be 0 which is what we derived when we had just flow past a cylinder without any circulation.

So, this result where left is rho U gamma is called as a Kutta–Zhukhovsk theorem and it is a powerful theorem because it is not just valid for or it is not just the result with the outcome of the result of flow past a cylinder with circulation it is actually true for any irrotational flow and around any arbitrarily shaped body. So, this result is applicable for 2d irrotational flow around any arbitrary shaped body and that is why it is a theorem that it says that the left force or the left force generated by an object in a flow where there is some circulation would be given as rho U gamma.

It is also applicable, for instance, you had a situation of flow passed in airfoil which is where the main purpose of deriving this law classically or this theorem came in, if you had flow going over this airfoil. So, knowing the circulation around the airfoil which maybe I can denote by gamma and the distance velocity U knowing the circulation around the airfoil I could say that the left force would be rho U gamma that was the purpose or the utility of this theorem.

So, any arbitrary shaped body as long as there is some circulation the left force would be non zero. So, this is what we did using the more classical approach I would say. We derived the Kutta–Zhukhovsky theorem using the first principle approach by integrating the pressure distribution. We can also do this using the Blasius theorem.

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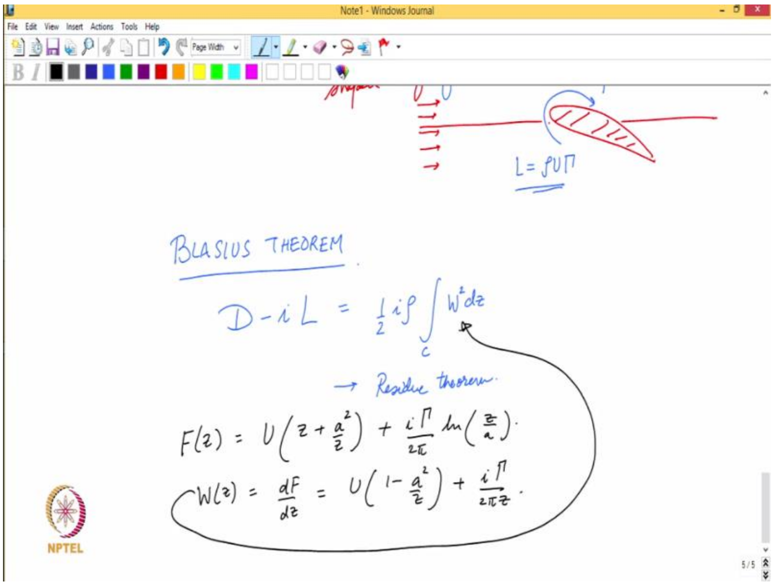

The same result can be derived using the Blasius theorem and I will do it for the case of flow past a cylinder with circulation. Now, the Blasius theorem says that D minus i L is given as half iota rho times W square d z where the integral is around the specific body that is under consideration. Now, in complex analysis there is a well-known theorem which is called as residue theorem which is what we will use to evaluate this integral for our problem.

If you are not familiar with this or you do not recall what it is I requested you to go and look at it. So, I will quickly sort of give a description of what this theorem is, but the details I am deliberately not considering as part of this lecture series. So, how do we use this Blasius theorem? Now recall that the complex potential in our problem for flow past a cylinder with circulation is U z plus a square by z plus iota gamma by 2 pi log of z by a.

So, the complex velocity is W which is d F d z is U 1 minus a square by z plus iota gamma by 2 pi z. What we need in the residue theorem is the square of this velocity so we just square W.

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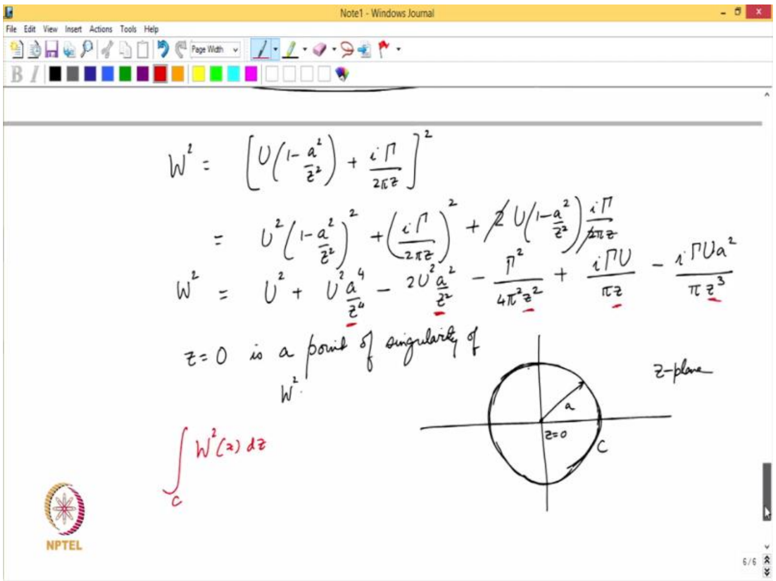

So, we say what is W square which should be U 1 minus a square by sorry this should be z square plus i gamma by 2 pi z this whole thing square. So, let us expand this we will get U square 1 minus a square by z square, square then we will have plus iota gamma by 2 pi z square plus 2 times U 1 minus a square by z square times iota gamma by 2 pi z. So, maybe 2 cancels here we can expand these terms a little further.

So, what we will say is we will get U square plus when you open up this square we will get U square plus U square into a 4 by z 4 minus 2 U square a square by z square. The second term here i gamma by 2 pi z square will give you iota square will become minus 1. So, we will have minus gamma square by 4 pi square z square and this last one will actually give you imaginary terms.

So, we will get an iota gamma U by pi z minus iota gamma minus U times a square by pi z cube this is W square. Now, let us see where do we have evaluate this integral. We have to evaluate that integral W square d z in the complex plane over the surface of this cylinder which has a radius a. So, we have trying to integrate (()) (28:04) in the complex plane and the radius is a and this is the contour C basically.

The surface of the cylinder is the contour. Now, it contains this point z equal to 0 which is the origin. Z equal to 0 is also point of singularity for this function W square. We say that singular because if you can see here now there are specific terms where z equal to 0 makes this function undefined. So, the function is not well defined. So, let us look at integral W square d z over this curve c which contains the singularity and this is where the residue theorem comes in.

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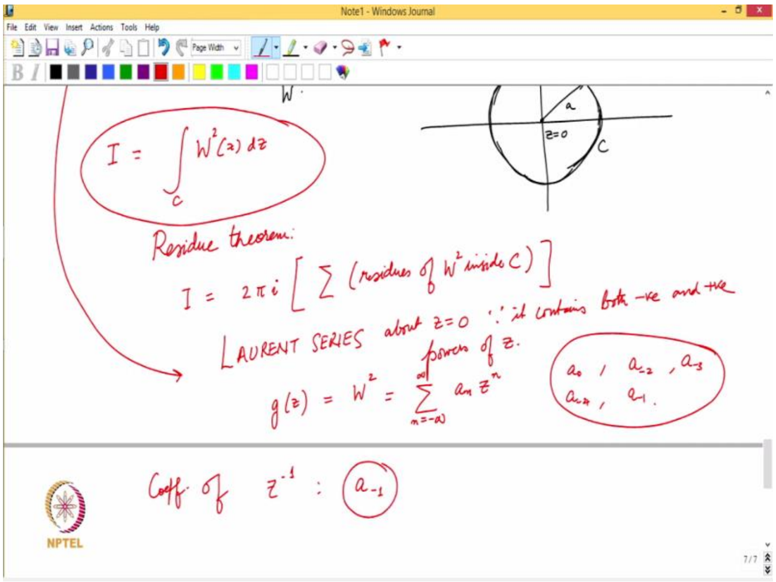

If we say this is integral I then the residual theorem says that this integral I could be written as 2 pi iota times summation of all residues of the function W square inside this curve C. It talks about taking residues of this function inside this curve C or whichever singularity exist inside curve C. Now, if we look at this function W square it has different powers of z. It has negative powers of z, it has also a power of z to the power 0 for instance.

So, I can say that this function W square is in the form of Laurent series about the point z equal to 0 because it contains powers of z which are both positive and negative because it contains both negative and positive powers of z. The general form of a Lorenz series which is about z equal to 0 would be for example some function I could say maybe g of z some function complex function which is in our case is W square.

The general way we could write this is in the form of as I said in the form of Laurent series which will have all possible powers both negative and positive and that would be a n say z to the power n. So, that would be the Laurent series typically that we could write. However, in our expression I mean it falls under the broad category of Laurent series except certain coefficients are only non-zero.

For instance we have a 0 to be non zero, we have a minus 4 to be non zero, we have a minus 2 to be non zero, a minus 1 to be non zero and a minus 3 to be non zero only these are the non zero coefficients in that series the rest are all 0 if you just compare the series with this which we can put it in this form. So, to compute this integral I the only term of significance for calculating this integral is the coefficient of the power z to the power minus 1. The coefficient of Z to the power minus 1 which we are saying is a minus 1 for instance this is the only number or only thing that we need to compute this integral I.

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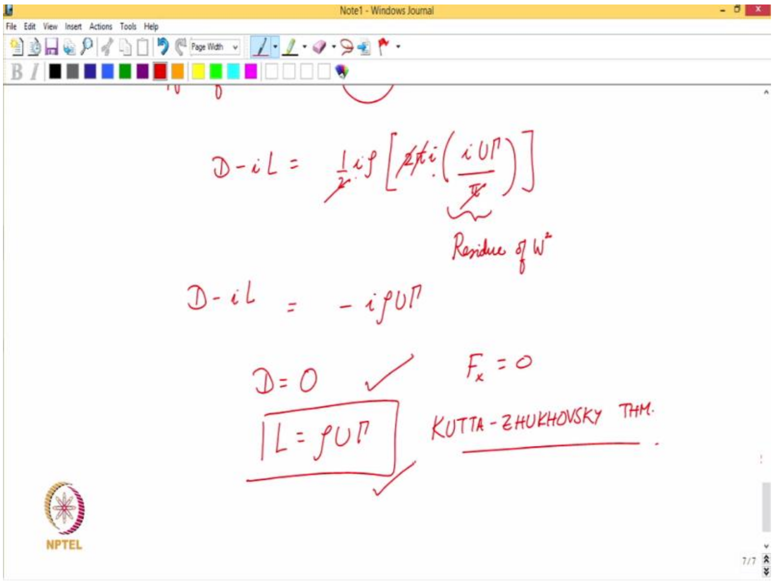

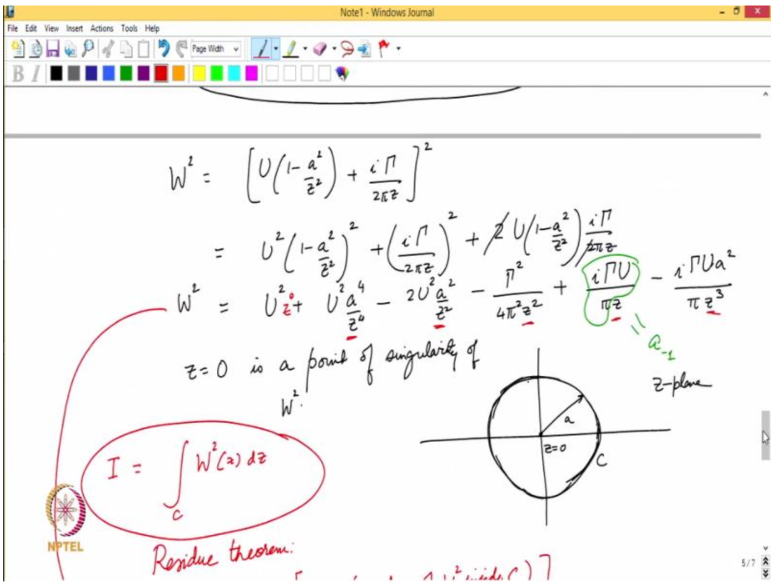

For our case we can write that D minus i L now which was a definition of the drag force in the Blasius theorem right here which is half i rho times the integral I; I is actually integral W square d z. So, that is this integral that we would put up here. So, we need the integral I; I as I said is 2 pi i times the residue and in this case we only look at coefficient a minus 1. So, coefficient a minus 1 is if you look at the expansion here.

What is the number or the coefficient of Z to the power minus 1 it is this number here which is the coefficient of Z to the power minus 1. So, this is a minus 1 so we bring it here. So, we say this should be 2 pi i times the residue which is i U gamma by pi. This is residue of W square. So, we have i pi i gamma U by pi which we put up here. Now, we can simplify this. So, this 2 cancels here there is a pi which should cancel and iota, iota will give you minus 1.

So, we will get minus iota rho U gamma as D minus i L. So, we can compare both sides knowing that D and L are real valued numbers. So, clearly D must be 0 which is right we have already shown that the x component of force is 0 and L is rho U gamma which is again the Kutta-Zhukhovsky theorem. So, we arrive back to the same result using a different approach which is a Blasius theorem and that is what I wanted to demonstrate by this method. Thank you.