Ideal Fluid Flows Using Complex Analysis
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Lecture 10
Method of Images, Forces on a Body, Blasius Theorem

So, in the previous lecture we looked at the superposition of a source and uniform flow and the kind of flow pattern that we derived and specifically the shape of the body for which that flow is applicable was actually what is called as the Half Rankine Solid. So, we saw how this flow pattern images out of this simple superposition and then we also looked at the pressure distribution.

How do we determine pressure distribution on the surface of the solid typically the psi 0 streamline. So, we will extend this discussion here today and look at three specific topics in that sense. We looked at the method of images and I will talk about what this method is.

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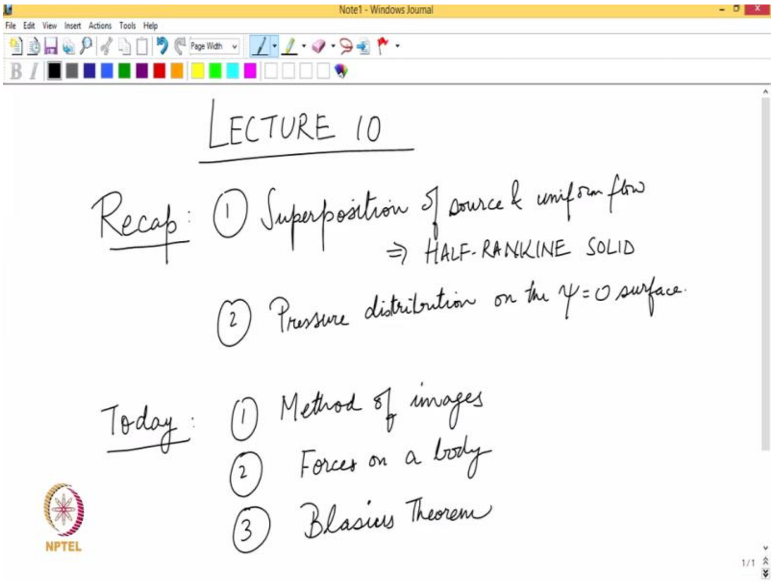

Then I will extent this idea of getting a pressure distribution and then applying it to calculate force on a solid object or a body that is emerged in a potential flow and then finally I will talk about as a consequence of the force of what the force distribution looks like. I will talk about the Blasius theorem. So, let us begin today's lecture.

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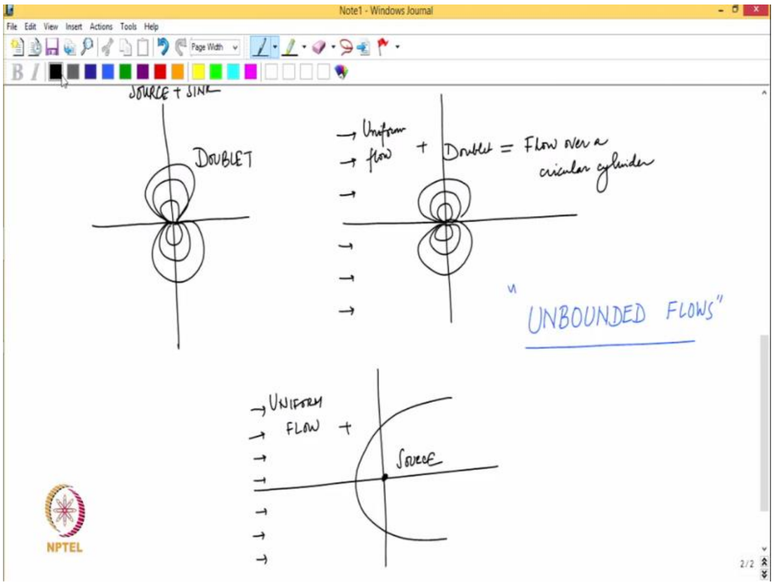

So far we have looked at superposition of elementary flows and kind of flows that we have looked at just to bring your attention to them again. We looked at how can we superpose a source in a sink which are very near each other and near the origin we could derive at doublet. So, we got a doublet of this superposition then we also saw that if we have a uniform flow and there is a doublet at the origin then we got flow over a cylinder.

So, this is plus a doublet give us flow over a circular cylinder and we also saw as I was mentioning in the last lecture we looked at what happens if you have uniform flow added with a source and we got a Rankine Half Solid. So, we looked at these three scenarios. Now there is something about these three and the thing that is common in these three situations is that these are actually flows which can be classified in the category of unbounded flows.

What I mean by that is that there is no physical boundary which is holding these flows or which is sort of confining these flows in a certain domain or a in a certain specific region. Now, in most practical problems that we encounter we typically have the presence of a rigid boundary which constraint the flow in a specific zone.

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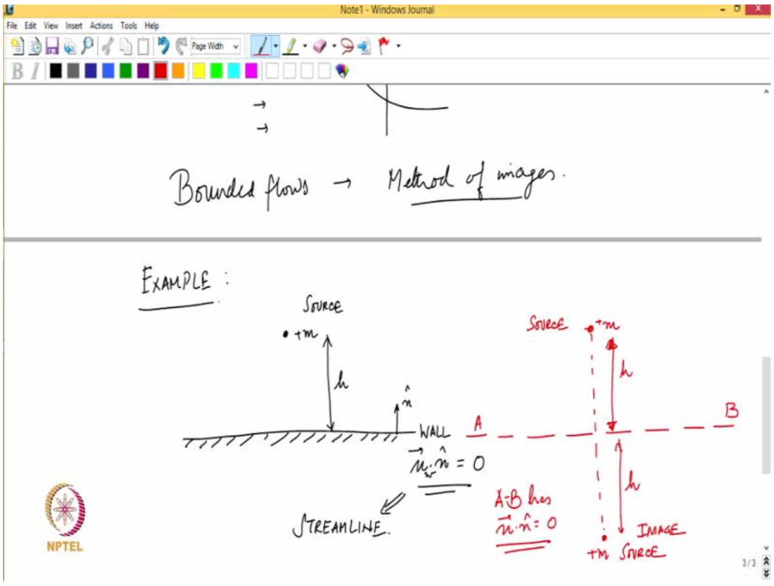

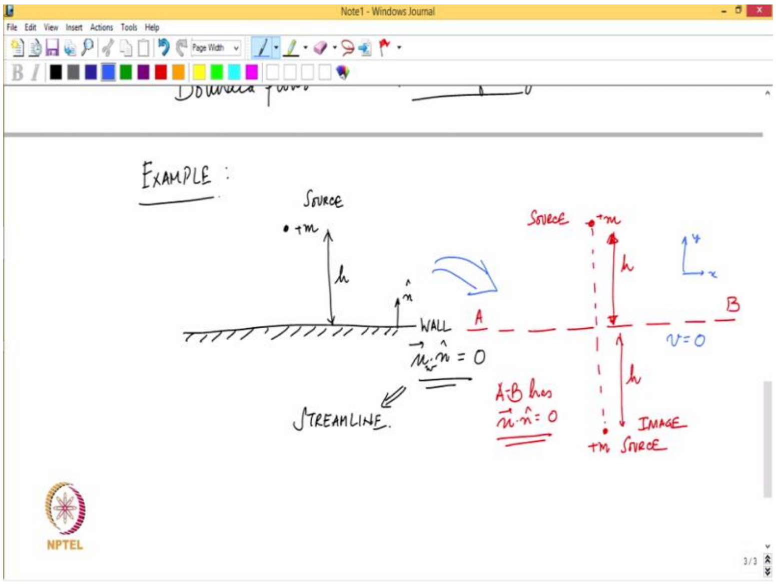

So, in those cases where the flow is not unbounded rather it is bounded we rely on this method of which I call as method of images and I will give you some illustrations of this method is going to work. So, let us take an example of the following type say we have a wall a rigid wall, a horizontally inclined rigid wall and I have a source let us say of strength m which is located at some distance from this wall.

The question that I am asking is what kind of flow pattern would we expect when we have this scenario. Now, notice that if this is single source the flow will flow out radially, but the only issue now is that if the flow if it goes out radially there is a wall which sort of constraints the flow it does not allow the flow to go through it is a physical boundary or physical barrier. So, how do we do this now, how do we model this?

So, the thing that we exploit in such situations is that the wall being a physical boundary and typically a station evolve it basically means that the normal velocity or the velocity component or velocity dotted with a normal component on the wall is 0 on this wall so if I say u w dot n cap this is 0 on the wall because there is no penetration through this wall and the normal vector is basically let us say this is the normal vector.

So, clearly if this is the case then I could say that this wall surface is a streamline after all that is how we define a streamline. It is a line across which there cannot be any fluid flow. So, the method of images exploits this idea and the way it exploits is with this illustration we can do it again. So, instead of putting a wall there let us say let me define this wall by a dotted line or a by a dash line.

So, let us say this surface A B and I define my source at the same distance which is say h from the wall, but what I do is at an equivalent opposite distance and using the wall as a mirror which means at a distance h on the bottom half I put another source. So, let us consider this as an example. Now, having two sources and one being an image. So, let me say this is an image source and that one is a original source.

So, having this situation or having these two sources at equal distances ensures that the entire wall or the surface A B I should say surface A B has u dot n cap to be 0. This you can work out just by superposition of these two sources now you can work out that the surface A B will always have this will always satisfy this condition as long as this sources are of equal strength and the distances from the surface are the same.

So, you see now that the wall could be replaced by this new scenario. So, we have replaced the original problem with a modified problem which is an unbounded problem by the way this is an unbounded flow, but we now know that the method that we followed is that we convert the bounded flow problem into an unbounded flow problem by using an image of the original elementary flow. So, we get the y component of velocity if I define x and y here in this fashion then v is 0 on this surface A B. Now, similarly we can do even for a vortex.

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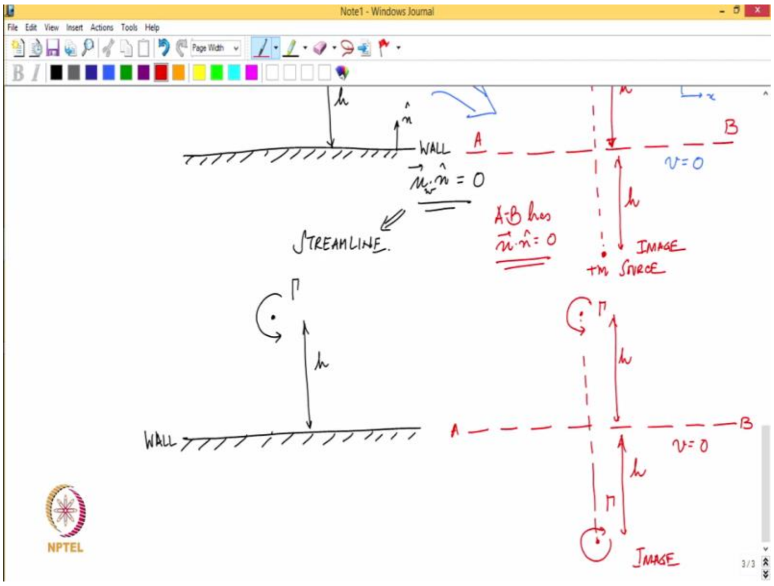

So, say we have a vortex near a wall. So, say we have a counter clockwise rotating vortex of strength gamma at some distance say h from the wall. So, this is the wall and I want to derive the resulting flow field. Well again this is a bounded flow program, but I can easily replace this with an equivalent problem by using the method of images by taking a surface again which is a mirror again maybe again A B placing the original vortex right in its original position which is gamma at same distance h.

But right underneath it that means under the surface A B and at a same distance h I can put another vortex to cancel out the effect of putting another vortex at the bottom and to get the surface A B as a streamline. It is easy to see here that even though in the original case I put a source in a source to get a 0 velocity on surface A B. Here we have a counter clockwise vortex.

So, the vortex that I have to put on the image point this is the image point, this is the image where I have to create it, we have to create another vortex I would put a vortex of equal, but opposite rotation. So, we will have to put a vortex of strength gamma, but in the clockwise direction that will ensure on the surface A B you do not have any y component of velocity. So, this is what the method of images broadly looks like that we counter, we create.

We remove the physical boundary, but to remove that boundary and to ensure that boundary remains a streamline we create additional elementary flows which give us a desired effect. So, this is all that I wanted to say about method of images and maybe in one of the problem solving session we will take up an example of how to work this out.

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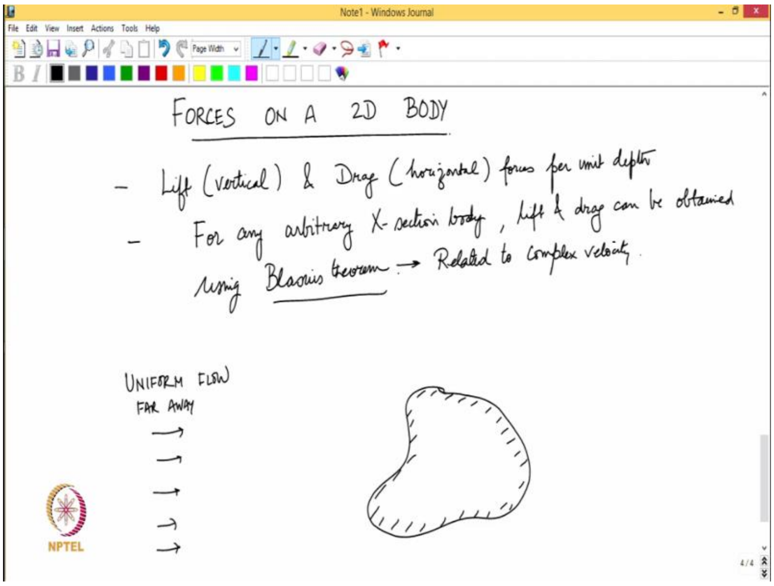

Now, let us look at forces on a 2 dimensional body. So, we have now derived the velocity field and the pressure distribution on any surface and you should now be able to say a lot about what kind of forces does a body feel or would a body experience. So, mainly we are looking at lift which is the vertical force and drag which is a horizontal force and these will be forces per unit depth as we are only dealing with a 2 D representation of any physical three dimensional object.

Now the thing that we are going to work out is that these lift and drag forces for any arbitrary shaped for any arbitrary cross section body these lift and drag can be obtained using the Blasius theorem which we will eventually now derive in this lecture and to give you some idea this Blasius theorem is a theorem by which the force is related to the velocity distribution.

So, the force is related to in our case as I will show you it would be related to complex velocity on the surface of the solid, so how do we determine this force and then eventually we will get to what is a Blasius theorem. So, let us take a stationary two dimensional solid with some arbitrary cross section. So, there is no particular preference for me, but let us say this is a solid far away from which we have uniform flow.

So, this is uniform flow which is far upstream and we could have any sort of combination of elementary flows along with uniform flow. So, this solid is stationary it is not moving and the part we are concerned about is what is the high dynamic force on this solid because of flow going pass this body.

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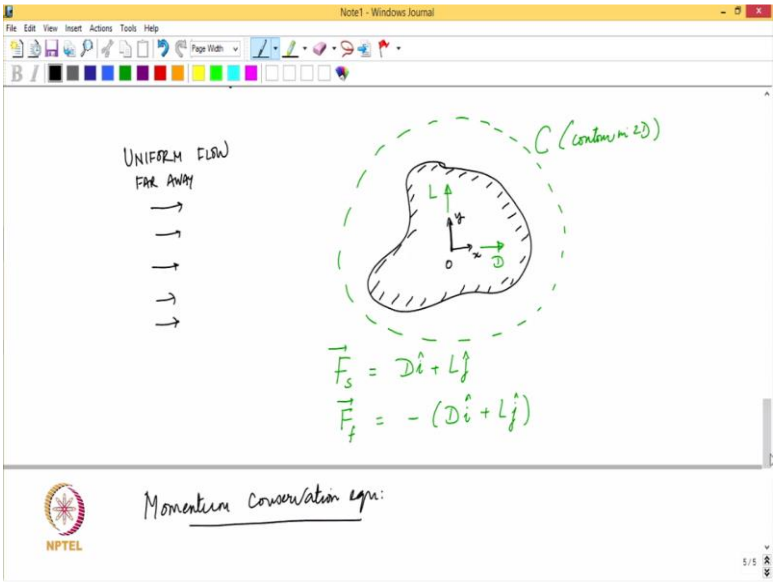

So, say we define our coordinate system we say this is y, this is x probably the origin we can keep inside does not matter. We can say that we are looking at calculation of the drag force D and maybe the lift force L on this solid because of flow going pass this solid. Now, if the force on the solid as I said if the lift forces L the drag is D. So, I can say the force on the solid the subscript s denoting solid is D i cap plus say L j cap that we can say is a force on the solid.

By Newton's Third Law clearly the force on the fluid let us say F f could be equivalent opposite. So, we can say force on the fluid would be minus D i cap plus L j cap. So, that makes it a little easier now. Now, that I have force on the fluid I can use some principles or some theorem from fluid mechanics to correlate the velocity field and pressure with this force and the way to do it is we use control volume analysis.

So, say I take a control volume for now any arbitrary shapes. So, this is the contour C which encloses this control volume. So, this is a contour in 2 D and we can then write the momentum conservation equation for the fluid in this control volume. So, we can write momentum conservation equation then we will write it in a general form.

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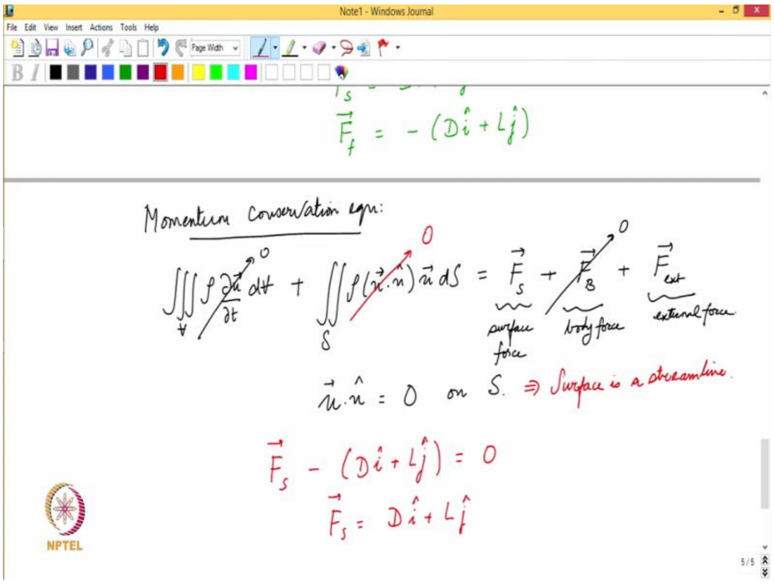

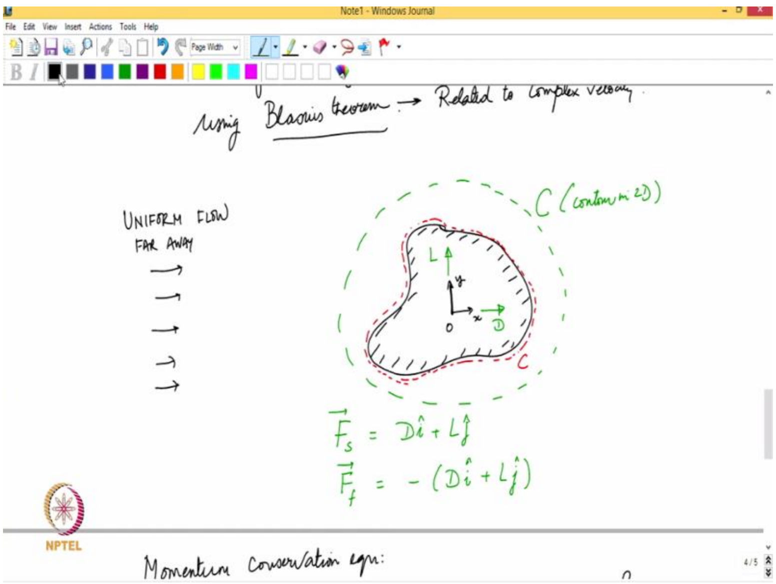

So, we can say this will be the volume integral of rho; rho being the density of the fluid d u d t where u being the velocity d V and I am going to use vectorial for a little while just because it will make it easier for you to understand. I am sure all of you have done some level of fluid mechanics at your undergraduate level and you would be familiar with the vectorial form of writing these equations.

So, we will say its rho du u d t d V plus the surface integral S of rho u dot n times vector u into d S this should be equal to the sum of the surface forces plus the body forces plus any external forces on the fluid. So, this is surface force on the control volume, these are body forces and then we have external force. Now, we will assume steady rotational flow. Well steady flow means that this will be 0 this integral will give you a 0.

We will also assume that there are no body forces on this system. So, say this will also go to 0 then. So, the momentum equation that we have would look like double integral of rho u dot n u d S which we typically call as the e flux term the net e flux of momentum. So, that should be equal to the surface plus the external force. Now, I did not say anything about the contour C as of now.

Well, contour C could be of any shape, any size, but I am also free to decide or to choose a control volume C that exactly matches with the shape of the solid. What I mean by that is that if I could shrink this control volume down to match the shape of the solid as a limiting case that should also be valid. I can shrink C down to match the solid shape. In that case if the contour can be shrunk or if I choose a control volume which is really essentially the shape of the body then the surface being stationary means that u dot n on the surface is 0.

And so this integral that we have here will also go to 0. It also means by the way u dot n is 0 it also means that the surface is a streamline we should never forget this and I will use this fact in a few minutes from now. So, the body is stationary u dot n is 0. So, we can finally write our momentum equation as F s plus F external. Now what was F external? F external is essentially the force on the fluid because of the presence of the solid which should be I can say minus D i cap plus L j cap is 0 that is all we are left with or we could say F s is D i cap plus L j cap. Now, what is F s?

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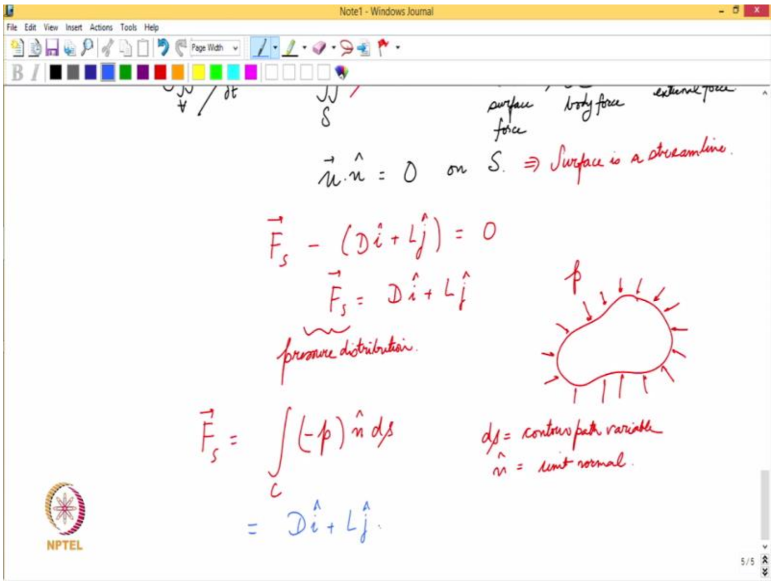

Now, F s is the force on the solid because of the presence of the fluid. In our case is from the pressure distribution this is manifested as pressure distribution. So, notice that when you have an arbitrary shaped solid you have pressure acting normally at all points the fluid pressure acting in that fashion so we have p on the solid. So, F is essentially integration of all this pressure distribution on the surface to get a net force.

So, what we can do now is to write the surface force F s as a pressure integral. Now pressure acts normally inwards into a solid. So F s is written as so in this we will have a path integral because we are dealing with forces per unit depth. So, we will have a integral over the contour c of minus p n cap d s where d s is the contour path variable and n cap is of course the unit normal at any given point.

And we are writing is as minus p n cap fully knowing that pressure is inwards. So, the effect the pressure forces basically minus p n cap. So, just to remind you again what is d s if you go back to the original sketch we can say d s is maybe an element of this type. We are going around this contour. So, d s on an exaggerated sense is this element, it is this small element which we will go around we will integrate over the entire contour to get this integral.

And this integral should be D i cap plus L j cap. Now, so referring to this figure again now that I have drawn here let us look at the infinitesimal elements d s and let us try and resolve it into its projections along x and y axis.

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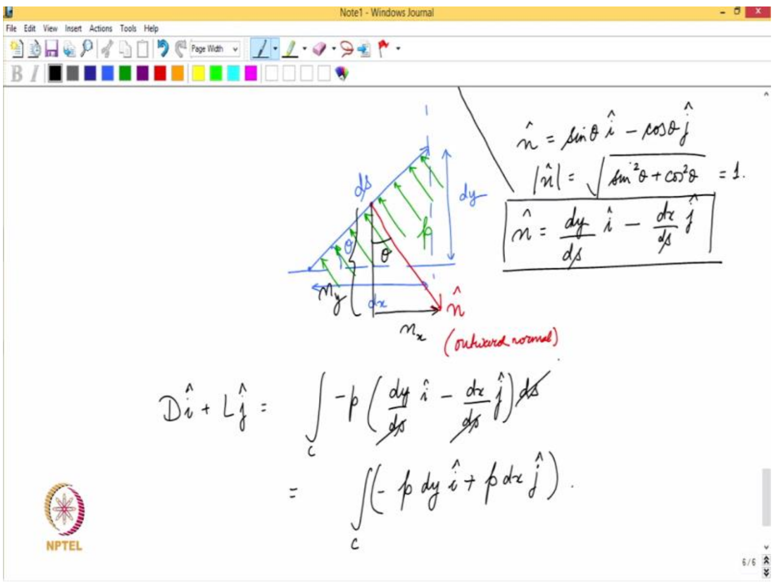

So, we have the element sort of like this is d s and I am going to try and resolve this into its projections along x and y. So, say this is the projection along y is d y, the projection along x axis is d x and say the slope of this d s with the x axis is theta. Now, on this element d s small element pressure acts normally inwards. So, let me choose a different color for pressure let us say this is the pressure distribution on this element infinitesimal element this is p.

The normal vector to the surface is pointing outwards this is n cap for this element the normal vector is in this form. It is an outward normal. More importantly normal vector as the term denotes normal vector is normal to the line d s itself. Now that comes in very handy for the reason that I can write what normal vector will be in terms of the angle theta. So, note that if this line d s makes an angle theta with the x axis.

Then the normal vector should make the same angle theta with the vertical axis with the y axis which is normal being perpendicular to or being orthogonal to d s. So, when we resolve the normal vector into its two component say n x and maybe n y into the two components. We would say normal vector n x i cap and minus n y j cap just in this case for instance n y is pointing downward or it is a projection on the negative y axis.

So, I would say it would be n x i cap minus n y j cap now n x for the reason that normal vector has a unit magnitude n x comes out to be as you can see already now hopefully n x will come out to be sin theta. So, we will have this as sin theta i cap as the x component and the y component will be cosine theta j cap. More importantly if I take the magnitude of n cap you can see it would be square root sin square theta plus cosine square theta which is 1.

So, it is a unit normal. Now, I can also exploit the fact the projections of this element d s along x and y are d x and d y. So, sin theta is basically d y by d s. So, this is d y by d s i cap minus cosine theta is d x by d s j cap. So, our normal vector to this element is given by this form. So, let us use this and substitute it here, let us put it here let us see what we get. So, when we substitute we will get D i cap plus L j cap to be minus p.

We will have d y d s i cap minus d x d s j cap d s which I could write as noting that it is a chain rule in d s as it appears. I can say this would be minus p d y i cap minus or plus p d x j cap. So, for getting rid of these terms and so now noting that if the equation has to be valid then the individual coefficients of this orthogonal vectors i cap and j cap must be equal.

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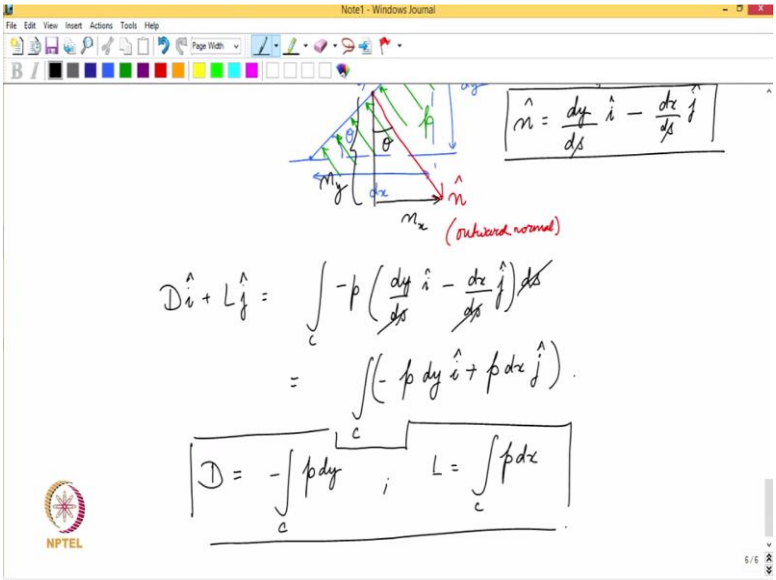

So, I can say D the drag force by unit depth must be integral minus p d y over the contour c and lift should be integral p d x and that is a very powerful tool that we have come up with now. So, in this in the solid body this is what the force should look like. So, far we are using vector notation now let me get to the idea of using complex potentials and the complex velocities. So, hopefully this is clear now so hopefully this part we have all been able to follow.

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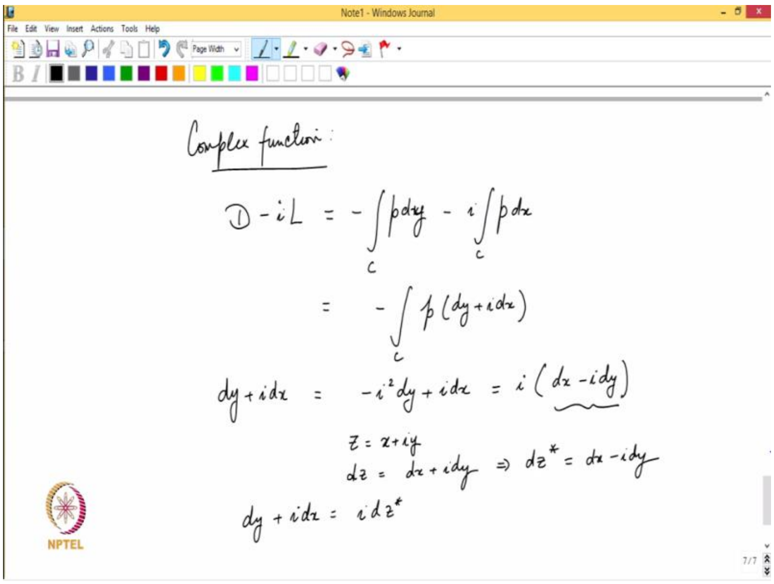

Now, we will do is we will construct a new complex function. So, say we construct a complex function given as the following. We construct a complex function which is D minus i L where D is the drag force, L is the left force. In terms of the integral this would be minus p d y minus iota p d x which I can also write as this is integral over the same contour. So, I can say this would be p d y plus i d x. Now, note something very special here that d y plus i d x which I can write this as minus iota square d y plus iota d x which could be written as if I pull the iota outside is a common factor.

We get d x minus i d y so if I say a complex number in the plane is z which is x plus i y then d z is d x plus i d y and so the complex conjugate of d z which is d z star the complex conjugate of d z would be d x minus i d y and this is exactly what this is. So, we can say d y plus i d x is i times d z star.

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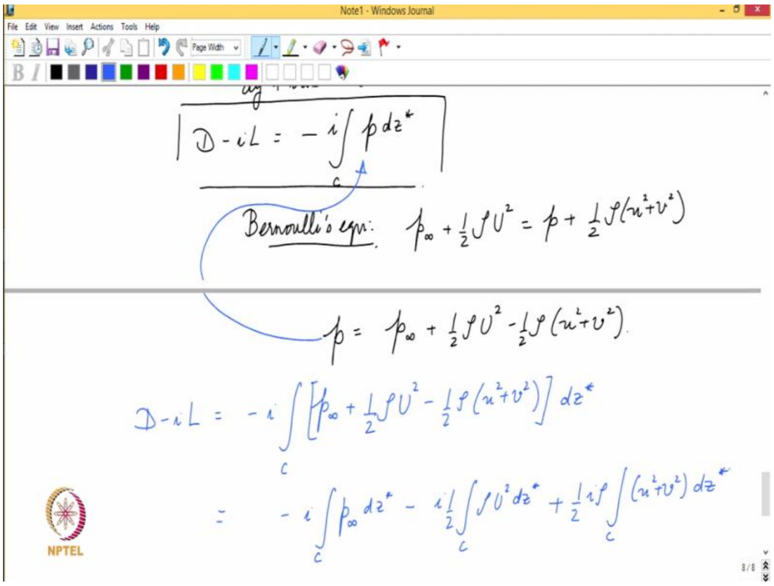

So, D minus i L is minus integral of p as I said d y plus i d x is i d z star so iota being a constant we can write that this equation if I have the pressure should give me the force on a solid. So, this is a form of the complex force in our example. So, finally we need to integrate the pressure distribution over this surface in a certain way and that would give us the forces. Now, to integrate pressure distribution we need to first derive the pressure distribution.

So, for this we use again Bernoulli's equation because the flow is rotational and we use the Bernoulli's equation to correlate pressure in terms of velocity on the surface of the solid. So, we can say that far away because pressure is some p infinity and the velocity is uniform which we say as capital U so half rho U rho. This must be the pressure on the solid plus half rho u square plus v square that should be the form of the kinetic energy.

And so clearly p should be equal to p infinity plus half rho U square minus half rho u square plus v square. So, we can substitute this pressure here and then let us do a few steps to see what we get? So, we get D minus i L to be minus iota integral over a close contour by the way c is a close contour which is a very important part of this derivation so this should be p infinity plus half rho U square minus half rho u square plus v square d z star.

And so what we get is minus iota p infinity d z star minus iota half rho U square. Well, I can maybe keep it inside just to make a point here let me say this is integral of rho U square d z star plus we can write the last term will be half iota rho u square plus v square d z star. Now, notice a few things the contour c is a closed contour. If you start at a certain point you will go around on the entire surface and reach back to the same point.

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$$= -i\oint_c \cancelto{0}{p_\infty\, dz^*} - i\frac{1}{2}\oint_c \cancelto{0}{\rho U^2\, dz^*} + \frac{1}{2} i\rho \oint_c (u^2+v^2)\, dz^*$$

$$u^2+v^2 = (u+iv)(u-iv)$$

$$= \frac{1}{2} i\rho \oint_c (u-iv)(u+iv)\, dz^*$$

$$u+iv = [u^2+v^2]^{1/2} e^{i\theta}$$

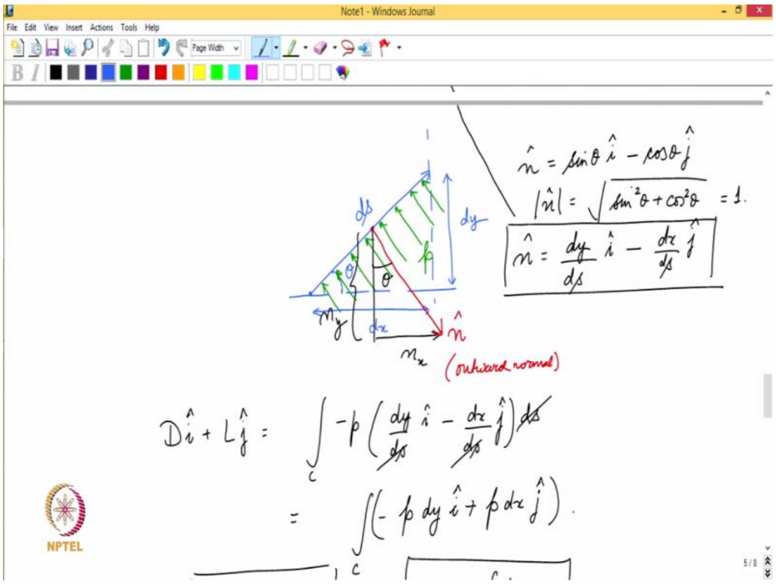
$$\theta = \tan^{-1}\left(\frac{v}{u}\right) \qquad \theta = \tan^{-1}\left(\frac{dy}{dx}\right)$$

$\vec{u}\cdot\hat{n} = 0$ on solid's surface

$$dz = dx + i\,dy = |dz|\, e^{i\theta}$$

$$dz^* = |dz|\, e^{-i\theta}$$

$$\left.\frac{dy}{dx}\right|_{\psi=c} = \frac{v}{u}$$



And also noting that p infinity is a constant rho U square is a constant. The integral over of d z star over the close contour will actually be 0. So, you basically you get to the same point so this function will be 0 similarly this function will also be 0 because rho U square is a constant so integral of d z star over the close contour is 0 you basically have the same initial and final points.

The last thing I want to tell you is that u square plus v square which is basically the square of the velocity magnitude could be written as u plus i v into u minus i v and we note that u minus i v is essentially are complex velocity. So, we will put this here so we can say that this will be a half iota rho u minus i v u plus i v d z star. Now, we go to the surface of a solid again.

And we look at this number u plus i v which could be written as well u square plus v square; square root which is a magnitude of this function into the argument given in terms of angle theta. So, if you think about what is this we would say theta should be tan inverse of v by u that is what it should be, but also note that since the surface is a streamline. It is a surface through no flow can cross then theta is also the same as tan inverse d by d x that is his definition.

So, this angle theta is the same as actually this angle that I had drawn here originally the angle theta in this picture which is parallel to the vector which is along the line which is tangential to the surface of the solid at a given point. What I mean by that is and this maybe little hard to absorb, but since I am using since u dot n is 0 on the surface then even if I write for example the d z function the small increment in complex plane which should be d x plus i d y.

This could also be written as modulus of d z into e to the power iota theta which further means that the complex conjugate of d z which is d z star would be modulus d z into e to the power minus iota theta. Noting that the complex conjugate does not change and magnitude of the complex function it only changes its polar angle. So, we are keeping a few things here we are noting that the surface of the solid is a stream line.

So, the velocity function which is u plus i v must be parallel to the element along the complex surface these two angles must be in the same direction noting that the surface is a stream line and for a streamline we always say that d y d x along psi equal to constant is given as the ratio of v by u. So, we use this property now to simplify our integral.

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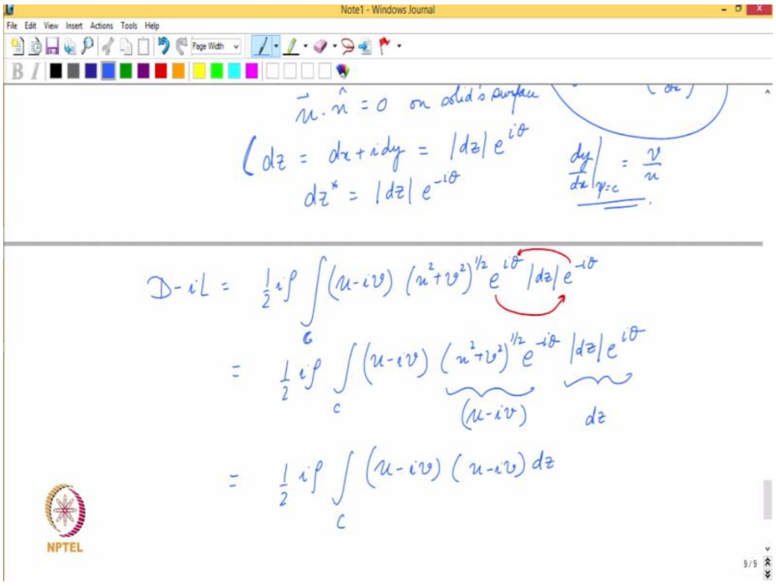

So, we write D minus i L which was half iota rho over contour c which we wrote as u minus i v, u plus i v which I can sort of write as or let us write it in this form we can write this as u square plus v square, square root into e to the power iota theta d z star is modulus d z e to the power minus iota theta. So, essentially this is the integral that we have to compute. What I can do is I can rearrange this by rearrangement what I mean is I can put this here.

And this here I can just move them from their places. So, I can say that this will be u minus i v u square plus v square, square root times e to the power minus iota theta times modulus d z e to the power iota theta. Now by doing this what I have done is if you look at this function alone modulus d z e i theta. I hope it reminds you of something modulus d z e i theta it is just a function d z.

The element on this curve in the complex plane u square plus v square, square root into e to the power minus iota theta becomes u minus i v. So, we can this as the integral as half iota rho over surface c u minus i v times u minus i v times d z which could be written as half iota rho u minus iota v square d z. This is D minus i L.

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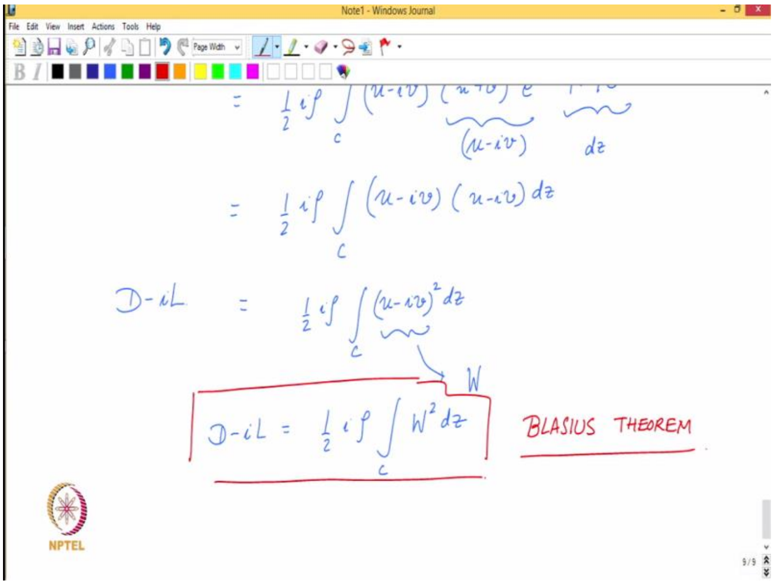

Now, note that u minus i v is the complex velocity W in the Cartesian system. So, D minus i L or the force on this solid would be given by the integral of W square the square of the complex velocity over the surface c into half iota rho and this is a very fundamental theorem that we derived which is also as I said is called as Blasius theorem.

Some books may actually call this as the first or the Blasius theorem. Now, that we have done all this complex analysis. So, for any flow if I know the velocity W on the surface of the solid I could theoretically calculate this integral and be able to derive the drag and the left forces experience by that solid quite easily and that is the beauty of this whole description in this whole method.

So, we will take an example of this in the next lecture and then we will actually close this course with the discussion on conformal transformation. So, hopefully this should have been clear and as I said we will definitely apply this for some special cases to show you that this actually gives you the right answer for the force. Thank you.