Ideal Fluid Flows Using Complex Analysis

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Lecture 11

Problem solving session 2

So, so far, we have looked at superposition of different elementary flows and we were able to derive the velocity and the pressure at certain selected points. Let me now take you into a much more analytically oriented problem where we would look at superposition of certain elementary flows and how we derive the velocity potential and the stream function using the approach that we have done and how we can actually separate out from the complex potential which looks a little tedious, how we can separate out the real and imaginary parts to obtain a neat and clean version of phi and psi.

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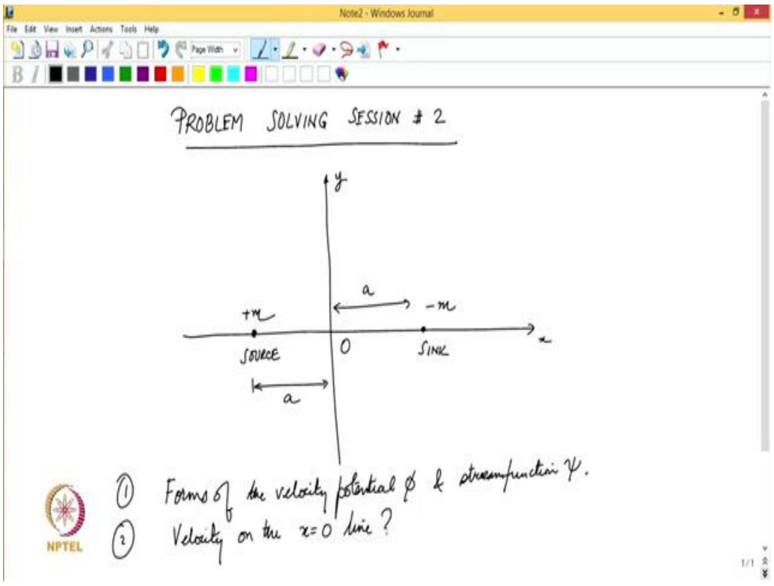

So, the problem that I am choosing is something like this, say we have in our coordinate system we have a source of strength plus m which is located at some distance say a from the origin and we also have a sink of equal strength and magnitude which is also located at some distance a from the origin but to the right. So, we have a source and sink located either side of the origin.

The problem that we are going to work out is we will derive the form of the velocity potential which is phi and stream function. So, we will first derive what are the forms or the velocity potential and the screen function. In the second part we will derive the velocity in this resulting flow on the x equal to 0 line which means basically the vertical line which passes through the origin. So, the problem may seem slightly easy for us to take up but there are reasons why I think we should spend some time on this.

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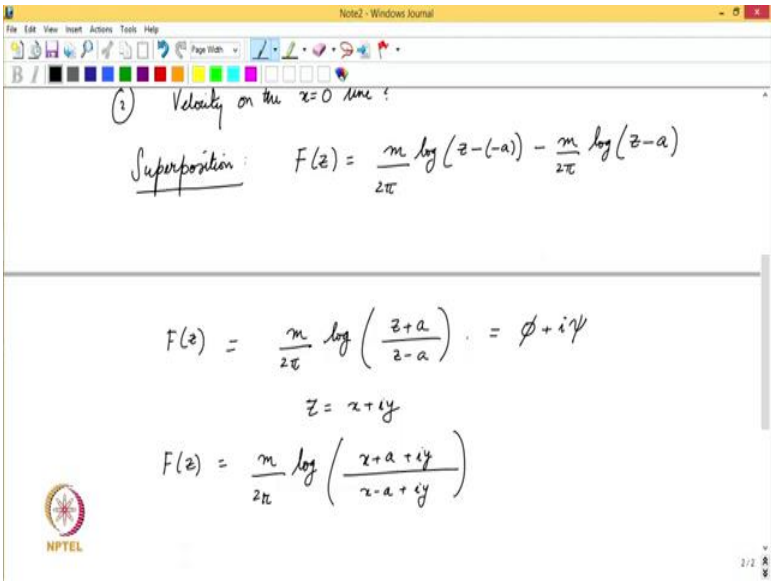

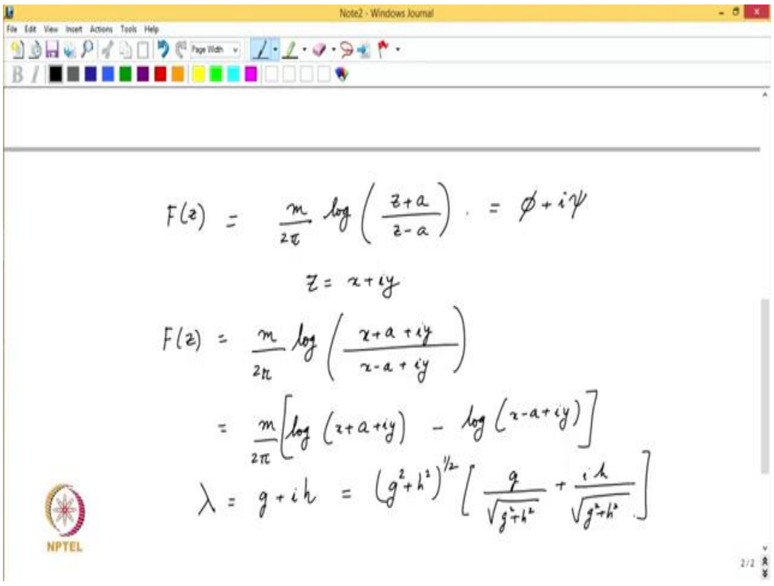

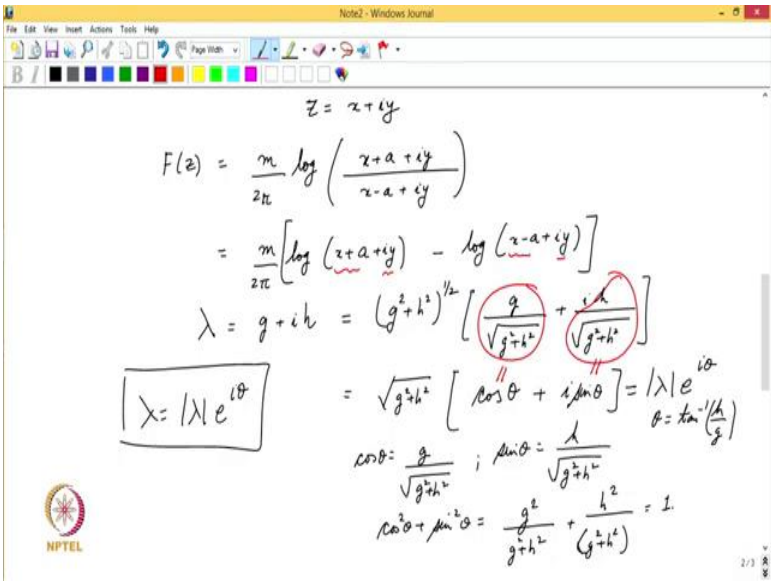

So, let us start with this superposition problem. So, when we superpose these two elementary flows we will write the resulting complex potential in this case will be m by 2 pi say log Z minus minus a noting that this source is located to the left of the origin at a coordinate minus a comma 0 and we have a sink which is located at a coordinate a comma 0 so we will have Z minus a, so this is I can say this coordinate here is a comma 0 and this is minus a comma 0.

Well we can write F as then m by 2 pi log of Z plus a by Z minus a this is f of Z in this scenario. Now, note that I said we have to derive the forms of the velocity potential and the stream function which ideally should be easy to derive out from F of Z. So, noting that if F of Z is a complex number the real part of it represents the velocity potential, the imaginary part represents the stream function.

But now there is a challenge, we see that in the current form the way the function is written, the way F of Z is written it is not possible for me to separate out the real and imaginary parts, not in the conventional sense, we need to do something else. Now, let me talk about that technique. So, what I, what I will do is I will again work in Cartesian coordinates, I will again write Z to be x plus i y and then try and somehow split out the real and imaginary parts of F of Z, so F of Z which would be m by 2 pi log of, so in the numerator it would be x plus a plus iota y, in the denominator will have x minus a plus iota y.

And for certain convenience that is as I will show you I will just write this as m by 2 pi and I will write them as two log terms. So, we will say this is log of x plus a plus i y minus log of x minus a plus i y. Now, the method that I would use to simplify and to separate out these log of these complex numbers into real and imaginary parts is the following.

We know that if we have a complex number lambda which is say g plus maybe iota h but g and h are real numbers then I can write this complex number lambda in the following way I could write this as g square plus h square square root which is basically the magnitude of lambda into g by square root of g square plus h square plus iota times h by square root of g square plus h square.

Now, why did I do this? Well by doing this I am ensuring that these two parts that I have here are definitely less than 1 more importantly I could write this as square root of g square plus h square times cosine of some angle say theta plus iota sin theta noting that cosine theta is say g by g square plus h square square root, sin theta is h by g square plus h square square root such that cosine square theta plus sin square theta if you just do it here will be g square by g square plus h square plus h square by g square plus h square which is equal to 1.

So, clearly, I can write the two factors g here as this or the real part as this the imaginary part in that fashion. By doing this I can then say that lambda could be written as modulus of lambda which is g square plus h square square root times e to the power iota theta because cosine theta plus sin theta could be written as e to the power i theta. And this here theta I could write as tan inverse of h by g. So, the same principle of the complex number could be written in this form lambda could be written as modulus lambda times e to the power i theta I will try and exploit to simplify the complex potential F of Z.

So, what we will do now is we will write, we will note that if you look at the first expression here this here this is the real part this is the imaginary part so to calculate the modulus I would say the modulus will be x plus a whole square plus y square the whole thing under root and on the other side we will have x minus a whole square plus y square and the whole thing under root. So, that would be the modulus of those two complex numbers.

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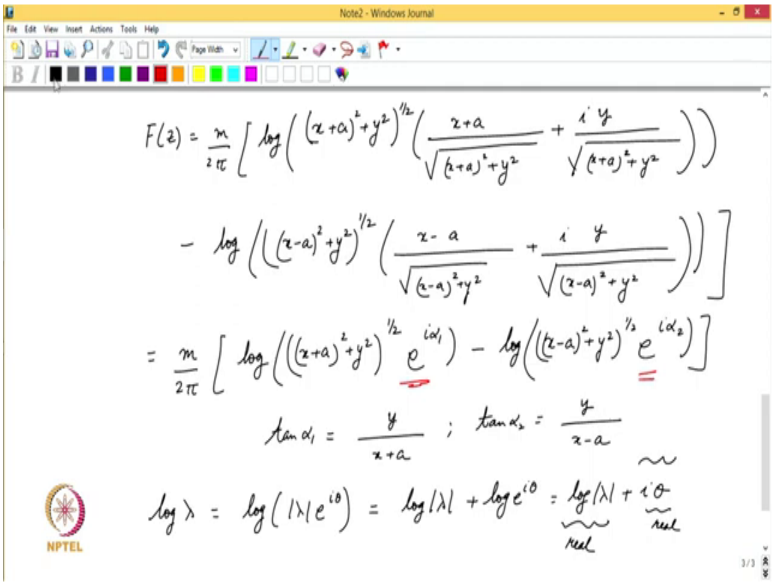

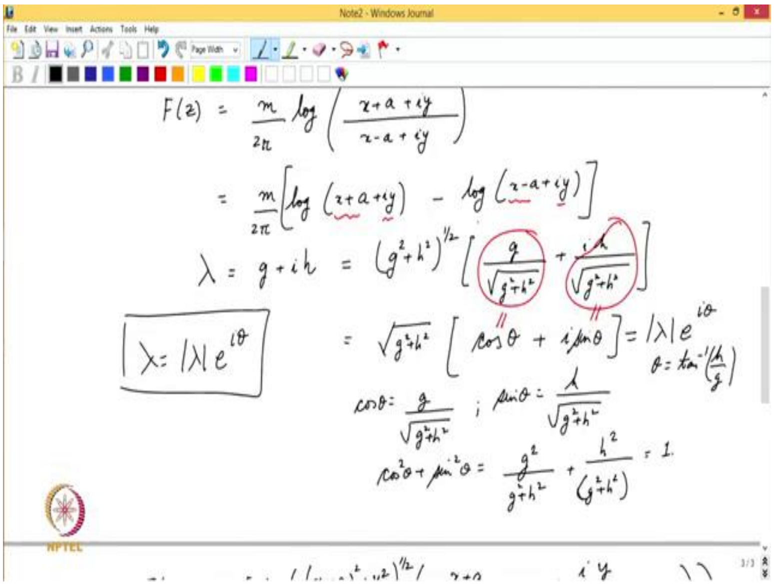

So, F of Z would be m by 2 pi times log of can I now jump over and just say or let us just do it in completeness we can say this would be x plus a square plus y square the whole thing under root multiplied by x plus a by x plus a square plus y square square root plus iota y by x plus a whole square plus y square so that would be this.

Log, and then we will have minus log will have x minus a square plus y square square root that is because we looking at the second part here so log x minus a plus i y so this is x minus a square plus y square into x minus a by x minus a square plus y square square root plus iota y x minus a square plus y square this is what we should get.

Now, we can take this even further and I could write this as m by 2 pi log of this number which is x plus a square plus y square square root times sum e to the power i alpha 1 where alpha 1 becomes the argument of this complex number and we will also have minus log of the second expression which is x minus a square plus y square, square root times e to the power i alpha 2 where let me now define alpha 1 alpha 2.

We have tan of alpha 1 will be y by x plus a and tan of alpha 2 will be y by x minus a that is how those two functions are defined. Now, what did I achieve out of this? Well what I have achieved now is that when I have log of for example lambda and I write this as log of modulus of lambda times e to the power i theta when I take the log now I can say this is going to be log of modulus of lambda plus log of e to the power i theta which would be log of modulus lambda plus i theta.

Now, I know that this is a real number and theta is also a real number but because it is multiplied by iota we get this as the complex part or the imaginary part. So, noting this now, keeping this in mind I can now say that by doing what I have done here by writing it in terms of the exponential parts I can now separate the real and imaginary parts from this complex potential.

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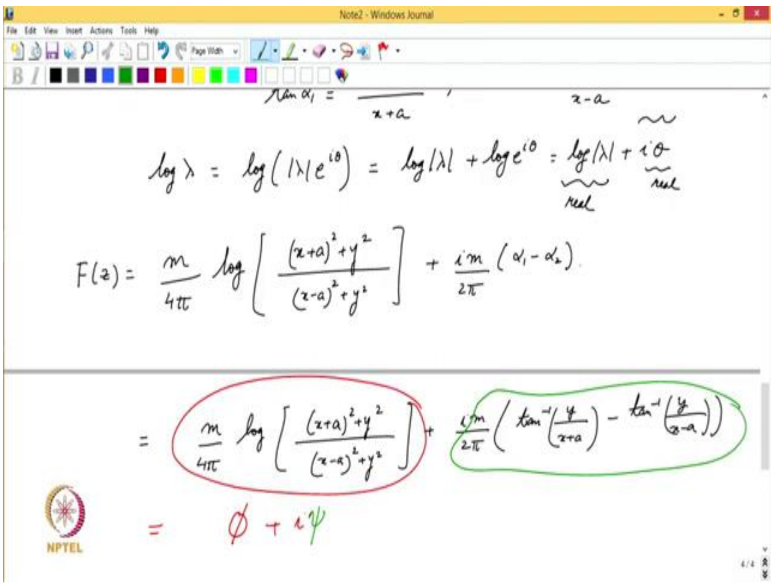

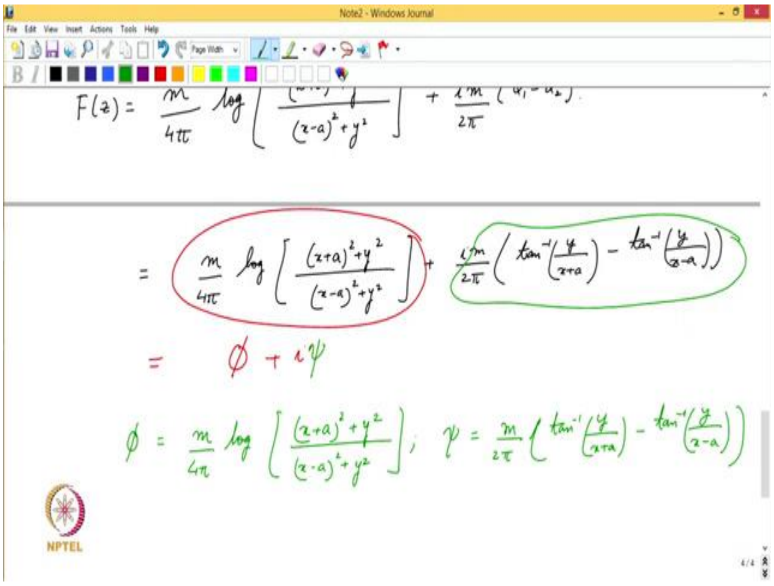

So, we can write F of Z and I am going to jump maybe a couple of steps here I think you can work out that this will be what it would come out to be, m by 4 pi log of x plus a square plus y square by x minus a square plus y square and then we will have an iota m by 2 pi times alpha 1 minus alpha 2.

So, we will get alpha 1 because we will have log of e alpha 1 here, we have log of e alpha 1, so that will give me iota alpha 1 times m by 2 pi and then we have a negative of log of e alpha 2 that will give me minus alpha 2 here and that multiplied by m by 2 pi will give me this number, so this is what the expansion would be.

And now it is easy to see that if you want to write it purely in terms of Cartesian coordinates that can also be done we can write this as log of x plus a square plus y square x minus a square plus y square plus iota m by 2 pi I can write alpha 1 as tan inverse of y by x plus a which is what we define alpha 1 as and then we can write alpha 2 as tan inverse y by x minus a.

And this should be what we were trying to derive in the first-place phi plus i psi so clearly maybe if I can use a different colour just for convenience here to show you by colour scheme, clearly the value of phi should be this value, this is phi and psi is whatever we have here except the iota.

So, we now know that phi is m by 4 pi log of x plus a square plus y square by x minus a square plus y square and psi is m by 2 pi tan inverse y by x plus a minus tan inverse y by x minus a. So, now we have derived the two functions that were asked of us, that what is a velocity potential, what is the stream function we have them now. Now, how do we determine the velocity at say x equal to 0? So, let us take a break here let us say how do we determine velocity.

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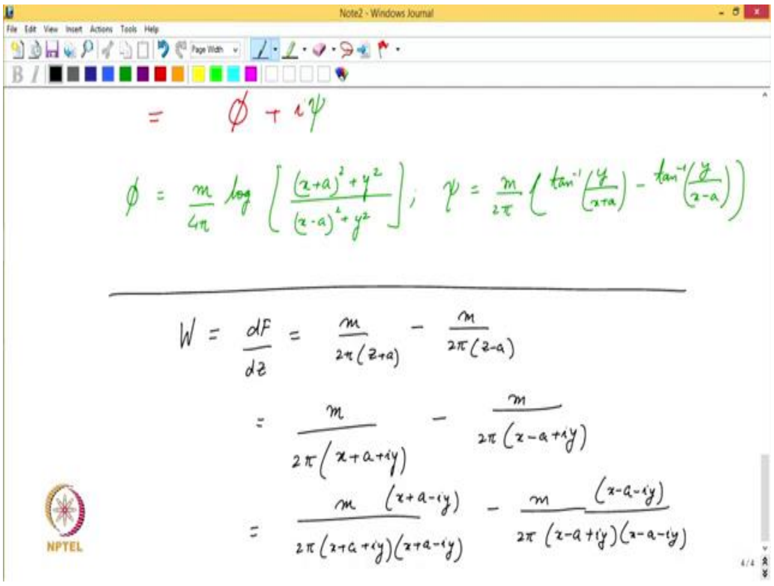

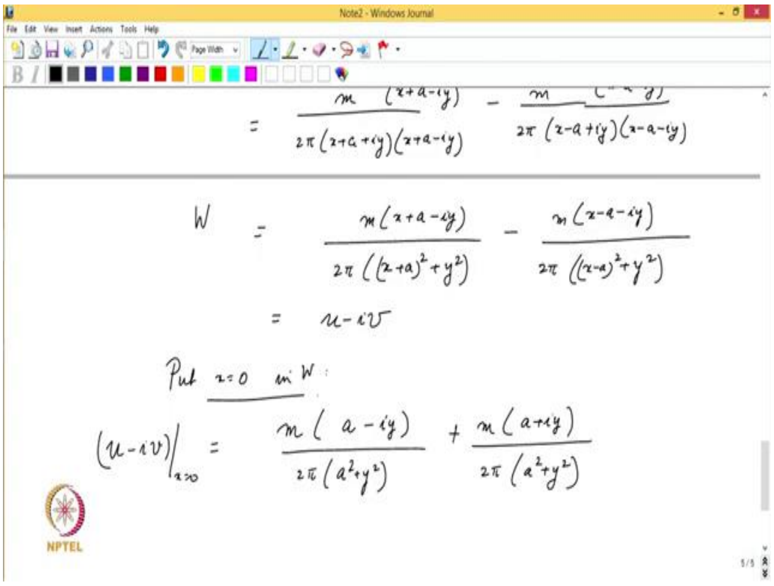

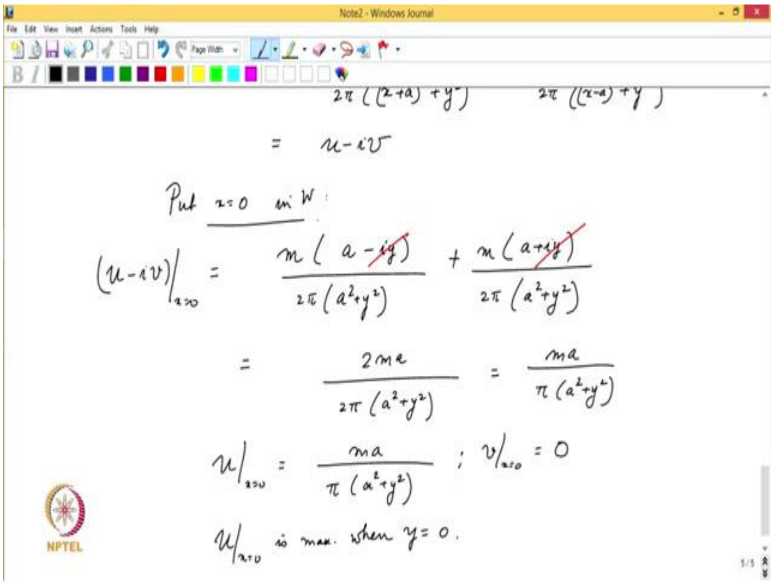

We need W essentially and W is dF dz, so let us look at the definition of F. So, F was m by 2 pi log of Z plus a so when you take the derivative that will give me m by 2 pi Z plus a in the denominator and then we had a minus m by 2 pi log Z minus a which will give me Z minus a in the denominator. And the same thing we do here, we put, we use Cartesian coordinate so we can write Z as x plus i y that will give me x plus a plus i y minus m by 2 pi x minus a plus i y. And as usual to bring the imaginary part in the numerator I need to multiply by the complex conjugate of this number.

So, I will just do m by 2 pi x plus a plus i y times x plus a minus i y both in the numerator and denominator, and the second term also will do the same thing, we will say this will be x minus a plus i y x minus a minus i y x minus a minus i y. So, to rationalize and to be able to ensure that we get the complex number in the numerator.

So, what we will get is m x plus a minus i y in the numerator will have a 2 pi you can now see we will get x plus a square plus y square here and we will have minus m x minus a minus i y by 2 pi x minus a square plus y square, this is W. So, we could either write this as u minus i v and then separate out the real and imaginary parts which can be easily done here.

But just because we have to calculate velocity at x 0 let us just do that. So, let us put x equal to 0 in W. So, what we will get is say u minus i v let me say at x 0 will be m we can put x 0 so we will get m that will have a and then we can actually get a minus i y in the numerator and then we will have 2 pi a square plus y square and then we will have minus m, well, we have minus a minus i y so let me just add this, this will be a plus i y by 2 pi a square plus y square.

So, can you see that we have the same numbers I mean in the imaginary parts these two will cancel everything else the coefficients are the same so there will be no imaginary part that would remain. What we will get is 2 m a by 2 pi a square plus y square which I could also write as m a by pi a square plus y square.

So, clearly u at x equal to 0 is this number what remains on the right side and v on x equal to 0 is 0, because there is no imaginary part that remains. So, we now have the velocity on the x equal to 0 line. More importantly if you notice this velocity will be maximum so u on x equal to 0 is maximum when y is 0 when we are at the origin, that is the location where this velocity will be the maximum. Now, that is also quite intuitive because if you think about this problem now what do we expect in terms of the flow field?

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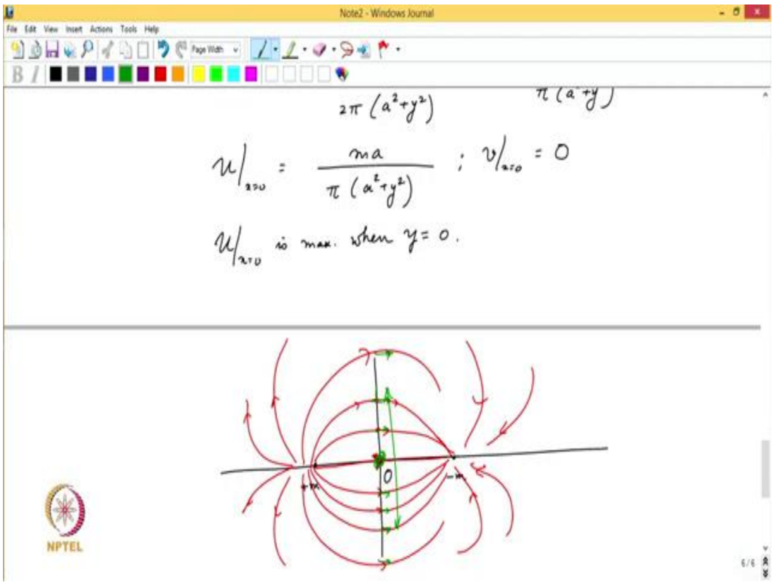

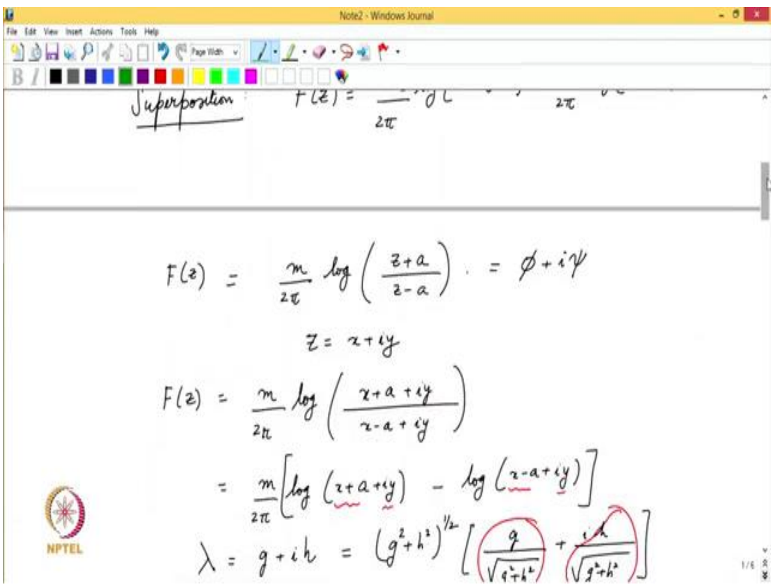

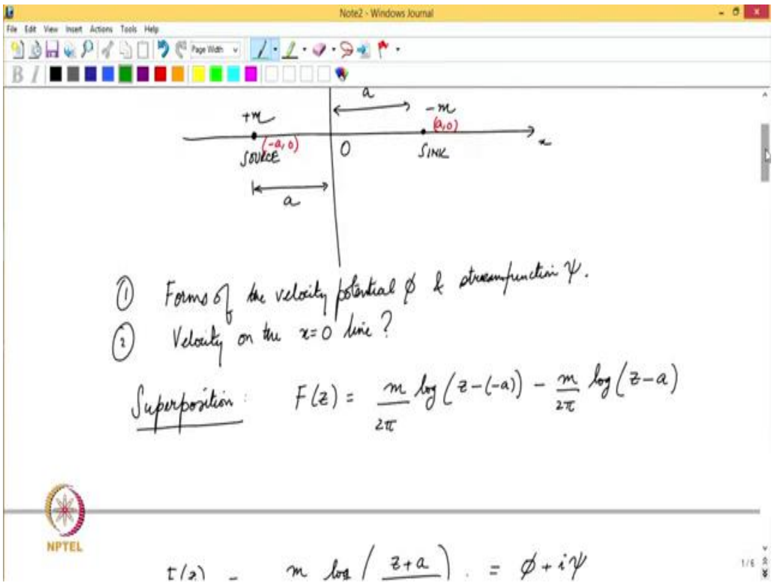

We said the problem is a source and the sink located either side of the origin what you expect are flow field lines which would look like this and they would be such that the velocity right at x equal to 0 is 0 or the stream lines would be normal to this line so you will get this kind of flow field.

So, radiating out in all ways and it would be entering from all directions. Very similar to in electrostatics we have a system where we have a positive negative charges placed in close proximity to each other so that same thing happens there if you look at the electric field lines they look in the same fashion, so we get a flow pattern of the same type so that is what we get and we notice that the velocity is maximum at the origin when you go over the x equal to 0 line.

So, if you are traversing this line either way maybe either way the maximum velocity is at the origin that is what we have found from the solution and that the y component of velocity which is vertically this is all 0 so all velocities are pointing in that direction. So, this is one more example of superposition but more importantly I wanted to show you how we can use complex numbers to be able to separate out the phi and psi values or psi, phi and psi functions even though the function itself may not seem very amenable to do so. So, thank you.