Ideal Fluid Flows Using Complex Analysis
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Lecture 10
Problem solving session 1

So far, we have looked at applying the principles of complex analysis to get new flows using superposition. What I am going to do in this session is to work out a simple numerical problem using the idea of superposition and then show you how we can obtain different hydrodynamic variables such as velocity or pressure at different locations in a specific 2D potential flow problem.

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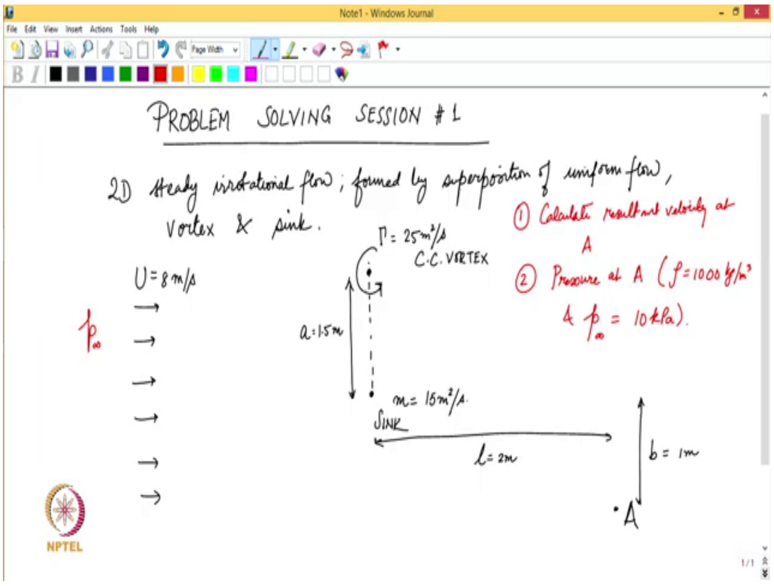

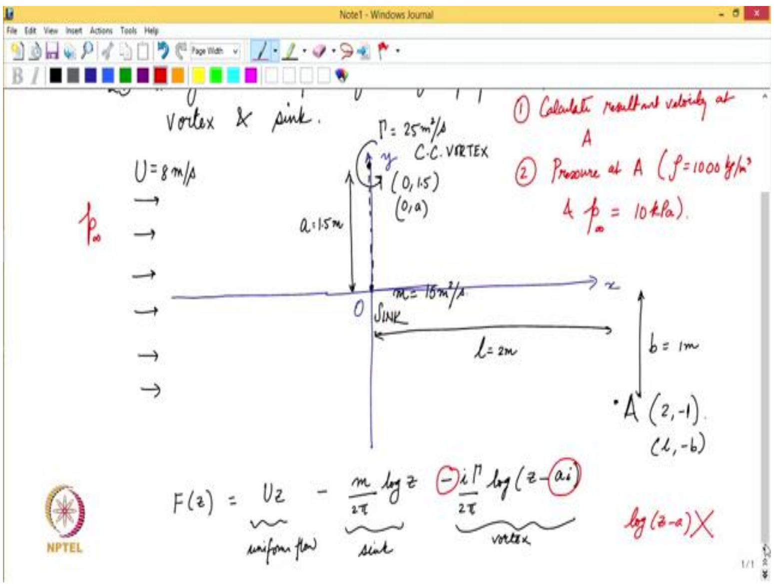

So, let us look at this problem which is about a 2D steady irrotational flow and this irrotational flow is formed by the superposition of say uniform flow, we have a vortex and then we have a sink and the vortex and the sink are placed at specific locations so say we have free stream uniform flow which has some velocity U the velocity is given say eight meter per second.

We have a sink somewhere so this is sink which has a strength of say 15 meter square per second and say we have just a little distance above the sink at some distance say let me label it as a which is given as say 1.5 meters, we have a vortex which is actually a counter clockwise vortex, so we have a counter clockwise vortex of strength gamma which is 25 meter square per second.

The problem requires us to calculate the velocity and the pressure at some point a which is at a certain distance from the sink, so this point A is at a distance of about L which is 2 meters horizontally and say distance b which is one meter vertically. So, the question is saying that we need to calculate first the velocity, the resultant velocity at A.

And the second part is calculate pressure at point A given that the fluid is water so the density is 1000 kilograms per meter cube and the pressure far upstream which I am going to now say is p infinity which would be pressure which is far far away from the sink and the vortex is given to be say 10 kilo pascal, given the fluid is water which has a density of 1000 kilogram per meter cube given the pressure for upstream is 10 kilo pascal we are supposed to calculate the pressure at point A in this problem.

So, I hope the question is clear now we have three elementary flows, we have a uniform flow, we have a vortex and then we have a sink and we are supposed to calculate velocity at some point, we could choose many more points but let us say at this in this case we are looking at point A. So, how do we solve this?

So, think about this if I want to apply the principles of complex analysis that we have been developing so far, we need to first define convenient coordinate system. So, I will define my coordinate system in such a way that the sink coincides with my origin. So, maybe I will take it in this way that coordinates are axis are in this in the following, way we have x axis here, y axis there and the origin is here so that the sink happens to be at the origin.

You can choose it any which way I could even put the origin at the location where the vortex is located the answer will not change it is just the matter of convenience because putting the origin at the sink ensures that the distance give from of point A which is given with reference to the sink is easily turned into coordinates of point A about the origin.

So, what I am saying is that in this case now point A is at coordinates of 2 meters, so I will say 2, comma minus 1 because it is one meter below the origin. Similarly, the vortex has coordinates of 0, comma 1.5 which is b basically I can say this is 0 comma b, 0 comma a and A becomes l comma say minus b.

Now, we need to work out the velocities and also the pressure. So, the first thing we should do is look at the velocity because you know now that irritational flows if I know the velocity I can use the Bernoulli equation to calculate pressure. So, clearly the way to begin is to calculate velocity.

So, we will first write the complex potential for this case which is F of Z which would be the resultant of superposition of these three elementary flows. So, we can say this will be U Z from uniform flow, this is from uniform flow. Then we have a sink in this problem, so its potential will be minus m by 2 pi log of Z just because the sink is located at the origin.

And then we have a counter clockwise vortex. So, this is important, if you recall if you have seen the lecture so far you would know that the potential for counter clockwise vortex is minus I gamma by 2 pi log Z minus the position where the vortex is located. So, in our case the vortex is located at a point a times i, that is a point a along the y axis imaginary axis so its coordinate must be a i.

So, this is from the vortex, this is from the sink, the contribution of the sink. So, important points as I said here is are that there is a negative sign here even though the vortex is in the counter clockwise direction and the second important point is that the location is a i not a, a would just mean that it is along the real axis not the imaginary axis, so we cannot use this as log Z minus a, this would be wrong.

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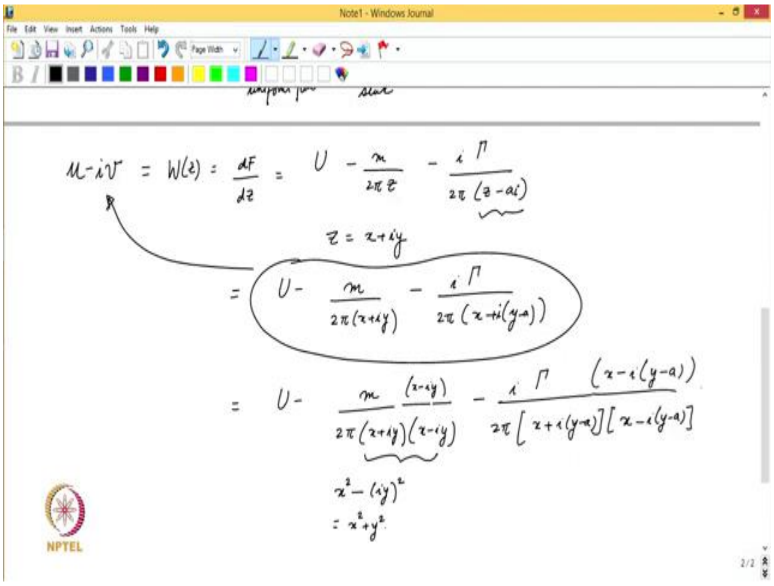

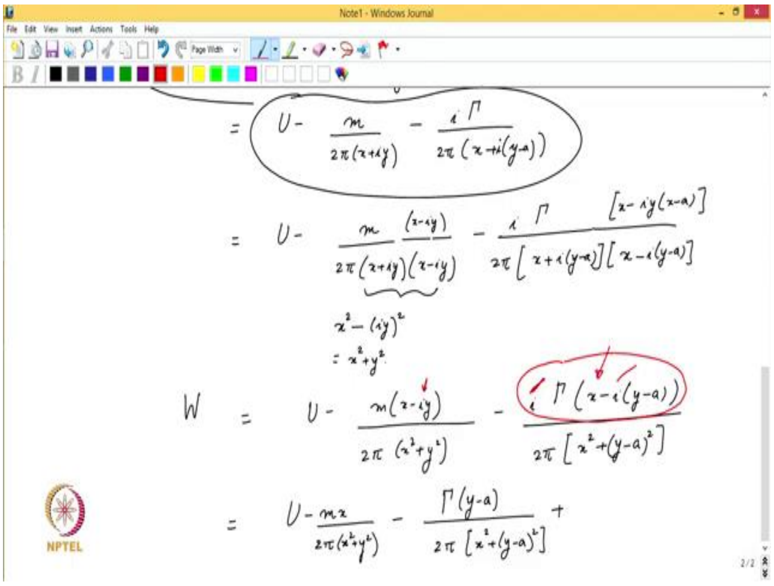

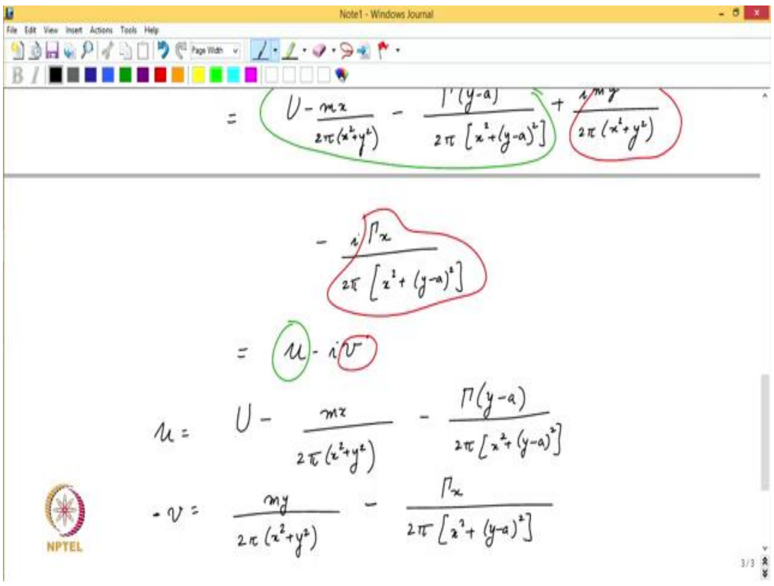

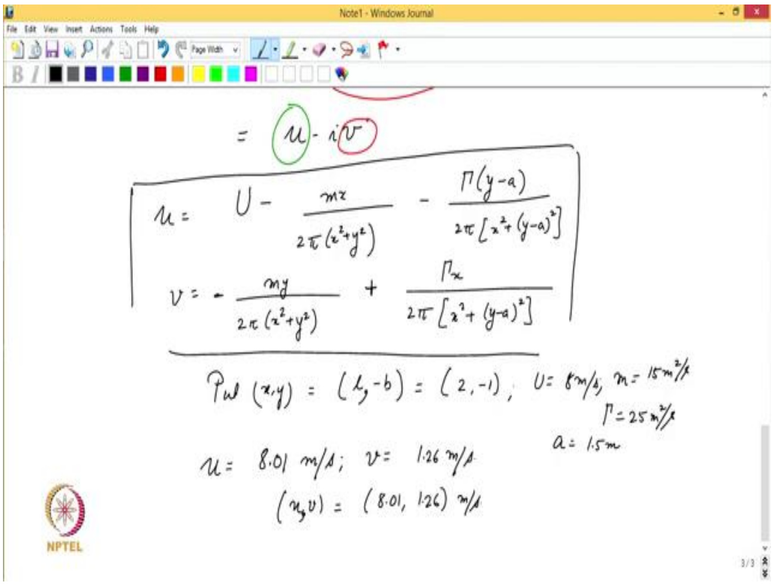

Now, let us go further now that we have written the complex potential for this problem we know that velocity could be derived as the derivative of the complex potential so that would be U minus m by 2 pi Z minus i gamma by 2 pi Z minus a i, that is just a simple derivative of the three individual expressions, there are various ways to go from here, what we have been doing till now is to use, is to exploit cylindrical coordinates, let me show you how we can do this using cartesian coordinates.

The reason I would rely on cartesian coordinates is that I have this Z minus a i so even if I put Z to be r e i theta it would be difficult for me to simplify that part, it can be done but it would be slightly challenging, let us use Cartesian coordinates so that we can write Z to be x plus i y in the complex plane.

So, we will get U minus m 2 pi x plus i y minus i gamma 2 pi x plus will have y minus a times iota, so this is what the complex velocity should look like. Now, the way I am going to solve this and I, the way I expect you to work out these kinds of problem is to look at what are we heading towards, we are heading towards calculating W in terms of two components U and V and they are separated by the fact that the y component goes with the iota and the so the real part of the complex velocity will be the x component of velocity, the negative of the imaginary part will be the y component of velocity.

So, I need to somehow bring this right-hand side, this side in this form, I need to somehow take it from here to there so that I can then separate out the real and imaginary parts. Well I cannot do it in this form as of now just because iota appears in the denominator I need to bring the imaginary iota in the numerator to be able to write it as u minus i v.

So, the simplest way to do this is we can multiply the numerator and denominator by complex conjugate of the complex number that appears in the denominator. So, for instance if I multiply this by x minus i y, the first part which appears because of the source by x minus i y I can then bring iota in the numerator because x plus i y times x minus i y just this product, this will give me x square minus iota y square, so let us, as we do a square minus b square and that we can see would be x square plus y square.

So, we have, we now have a real number in the denominator. So, that is what we are going to do with the second part as well, so we will write this as 2 pi, we can say this as x plus i y minus a multiply that by x minus i y minus a and the same thing would be in the numerator. And then we can write this as U minus m x minus i y 2 pi will have x square plus y square and here we will have iota gamma x minus i y minus a by 2 pi then we will have x square plus y minus a square, this is W.

Now, that iota is in the numerator only I can now separate the real and imaginary parts. So, we can separate the real and imaginary parts so that we can get this as U minus m x by 2 pi x square plus y square if we were just writing the real part minus now if we notice from this second expression here what we see here the real part will come when iota gets multiplied by iota that will make it minus 1, so that will give me the real part.

So, I would write this as minus gamma y minus a, note that minus and minus will make it plus 1 and then you have iota square so that has to be a minus sign there. So, we will have gamma y minus a by 2 pi x square plus y minus a square that is the real part of our solution and then we can write the imaginary part as well, the imaginary part will come from this iota here and then this iota multiplying by the x part here.

So, we will get this as plus iota m y 2 pi x square plus y square and then we will get minus iota gamma x by 2 pi x square plus y minus a square. So, this now we can write as u minus i v and we can note that u is given by this part and v or I would say minus v will be given by this part plus or rather minus this part.

So, we can say u will be capital U minus m x by 2 pi x square plus y square minus gamma y minus a 2 pi x square plus y minus a whole square and we will get v to be or rather I should say minus v because it is a negative sign on u minus i v, so minus v would be m y by 2 pi x square plus y square plus gamma x by 2 pi or rather I should, I have a correction here so this should be minus v and there is a minus sign here, so this should be minus gamma x 2 pi x square plus y minus a square or if you want to remove the minus sign from both sides we could just multiply both sides are minus 1 so this becomes minus here and this becomes plus that gives us v.

So, this is the final velocity that we were trying to derive in this superposition problem. Now, it is fairly easy we know the coordinates of point a so all we need to do is put x and y for what the coordinates of point a are. So, we can write, now we can put x, comma y to be l, comma minus b which is going to be l is 2 b was minus 1 and we can also put U to be 8 meter per second, we can put m which was I think 15-meter square per second and gamma which was 25-meter square per second.

So, if we substitute these values we can calculate the values of u and v. And that you can work out you can prove to yourself that u will come out to be about 8.01 meter per second and v will be about 1.26 meter per second. So, at this point the two components u comma v are 8.01, 1.26 meter per second, very importantly both pointing in the positive direction, that is just by substituting these numbers now and also using of course a which I believe was about I think 1.5 meters. So, that is for you to then work out.

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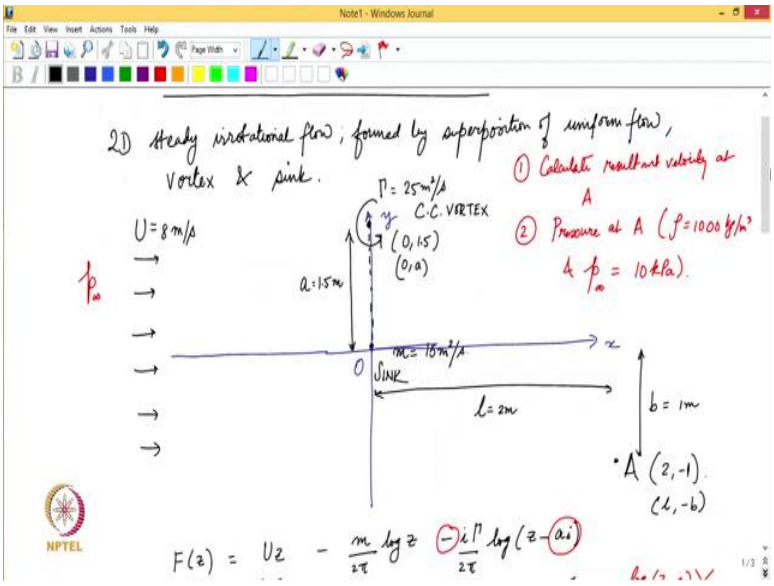

Now, that is the first part we were to calculate the velocity at this point. Now, how about calculating pressure. Now the question says if you if I take you back to the question statement it says that far upstream the pressure is 10 kilo pascal and the density of the fluid is 1000 kg per meter cube. So, I can apply Bernoulli equation now.

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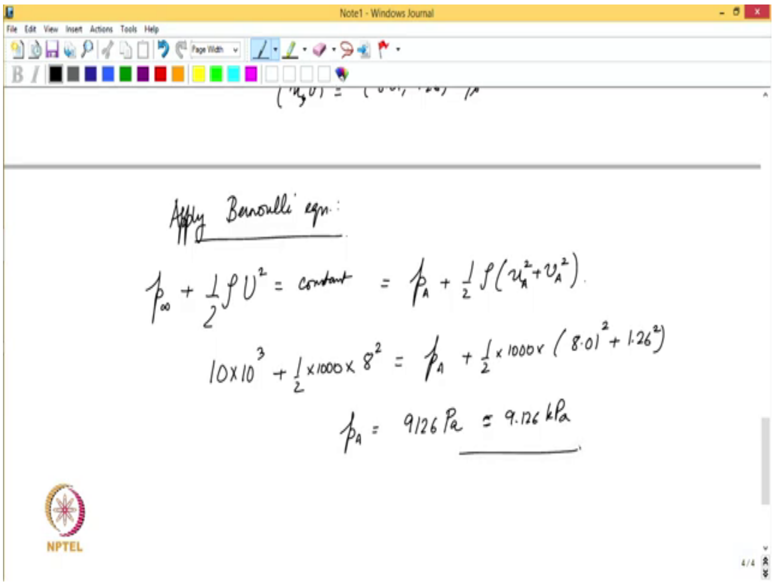

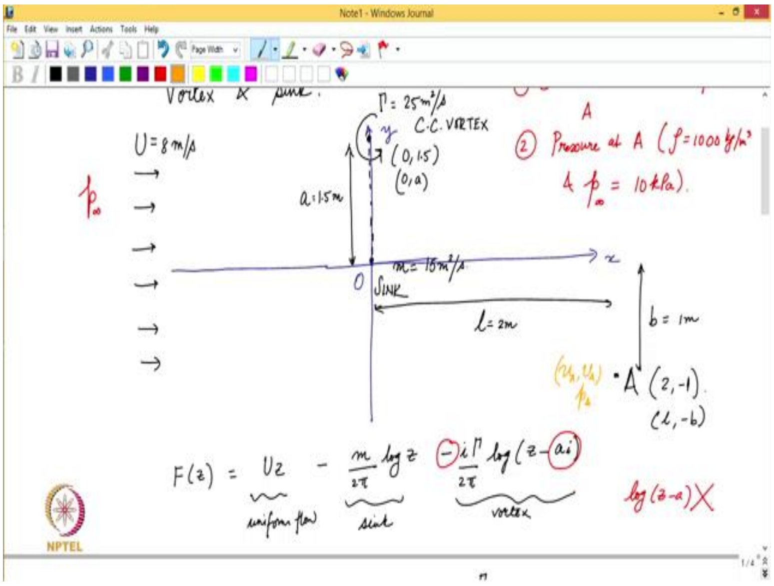

We can apply Bernoulli equation so we know that far away from the sink and the vortex pressure is p infinity and the velocity there is u, so we have p plus half rho u square to be constant in the flow ignoring all potential energy contributions. So, this should also be the same as p a plus half rho u a square plus v a square which is at point a, p a becomes a pressure at point a, u a b a are the components of velocity at point a.

So, we can substitute, we can say pressure is 10 kilo pascal so 10 10 to power 3 plus half density is given as 1000 into u is 8, so 8 square is p a plus half into 1000 into 8.01 square plus 1.26 square and so p a you can work out will come out to be about 9126 pascal or we can say this is 9.126 kPa. So, that is a pressure at point a in this problem. So, we have now computed both velocity and pressure, so we now have u a, v a, and p a at this point and that completes this problem.