Ideal Fluid Flows Using Complex Analysis
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Lecture 1
Ideal fluids, Velocity potential, Potential flows

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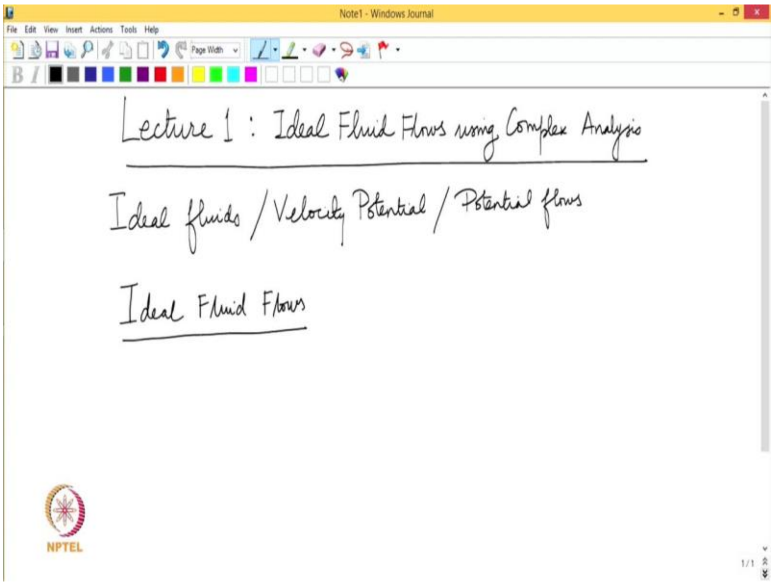

Hello and welcome to this first lecture on ideal fluid flows using complex analysis. So, in this first lecture, I will be discussing three concepts, one is on ideal fluids, because we are dealing with the course, which is named ideal fluid flows, I will then be introducing the concept of velocity potential, and then we will talk about potential flows. So, let us begin our journey in this course, let us first talk about ideal fluid flows and what constitutes an ideal fluid flow.

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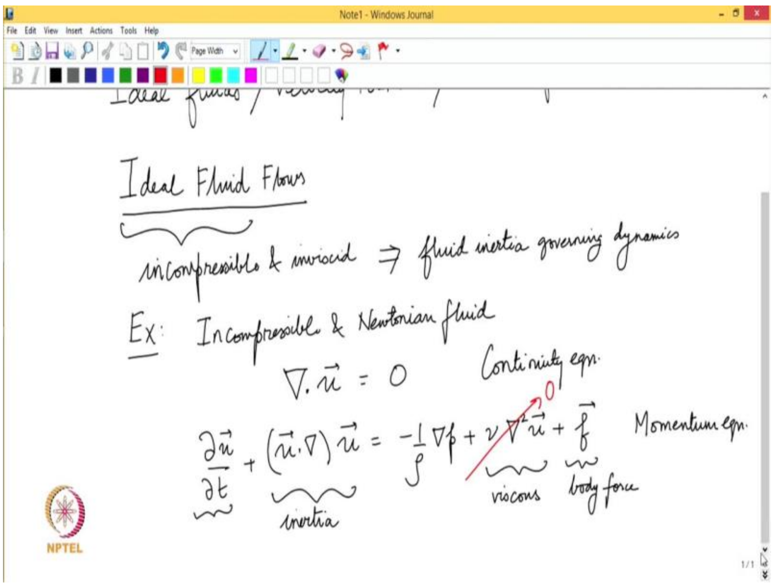

Now, we talk about an ideal fluid as I mentioned here. So, when we talk about an ideal fluid, we are dealing with a scenario or a particular fluid which has or which is incompressible and inviscid what it means is that, the fluid density cannot change and also the fluid viscosity is zero or there are no viscous effects. So, our study in this course is limited to these kinds of fluids.

Now, because viscous effects are negligible or are absent the resulting fluid dynamics is primarily due to fluid inertia. So, fluid inertia will be governing the dynamics of such a fluid now, let us, as an example, to make this point of what the fluid inertia really means, and what happens to how do we model an ideal fluid? Let us look at the governing equation for an incompressible and Newtonian fluid.

So, as an example, let us take an incompressible and Newtonian fluid. Now, the equations that govern the dynamics of an incompressible and Newtonian fluid are the mass conservation in the momentum conservation equations, better known as the continuity equation. So, what we will do is we can write del dot with u is 0, where del is a gradient operator u is the velocity vector. So, this is continuity equation and the momentum equation is the well-known Navier stokes which we can write as in this way so, let me explain what I have written in the momentum equation.

We have two contributions on the left hand side one is the rate of change of velocity at a given point. Then we have the inertia term or the advection term as it is better known as we have the gradient of pressure that you see, and then we have the viscous term and then the last term represents any body force any external body force on the fluid.

Now, for inviscid flows that we are working with here, since viscous effects are going to be negligible or are going to be absent, we can consider that this term will actually be 0. So, we will not have the viscous effect coming into the picture. But now, as you see, the resulting equation would be without this term in the overall scheme of things.

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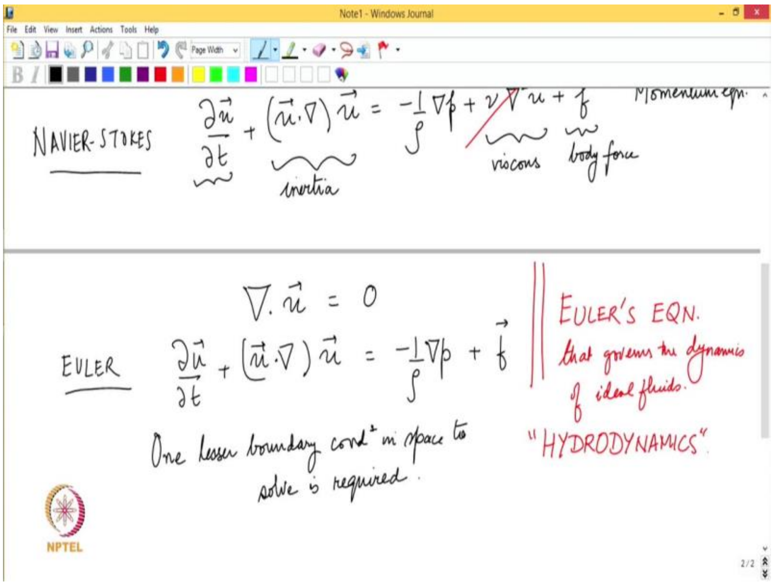

Now, an important point that I want you to take away from here is that if we now look at the resulting equations, which would be del dot u is 0 and d u dt plus u dot del u is minus 1 by rho gradient p plus somebody force these equations are independent of temperature or any variation in temperature. The reason I say that is that there is only one fluid property which can depend on temperature which is density, but we are already saying the fluid is incompressible and there is no viscosity in this equation or in these sets of equations because of which the temperature cannot have any direct influence on these equations.

Now, that is what actually makes it fairly interesting that these equations can be solved independent of any temperature variation in the fluid. Moreover, these equations are called as Euler's equations and this is the equation in fact that governs the dynamics of ideal fluids if you look at traditionally the study of ideal fluid is also called as hydrodynamics you may pick up some classical books and people used to refer to study of ideal fluids by the term of or by the name hydrodynamics.

Now, note that if you compare the Navier Stokes equation which is this part here with the Euler equation if you just compare these two equations, you should see one fairly interesting characteristic and the characteristic is that the Euler equation is one order lower than the Navier Stokes equation.

The Navier stokes is a differential equation which is first order in time and second order in space first order in time because there is a d u dt term that you see and second order in space because you see the new del square u term. So, that is the second derivative with respect to space or the highest derivative a second order in space, if you look at the Euler equation, it still has a first derivative with respect to time, but it only has a first derivative with respect to space. So, clearly we would need one lesser boundary condition in space to solve the Euler equation so that becomes a key characteristic or key feature that we will exploit.

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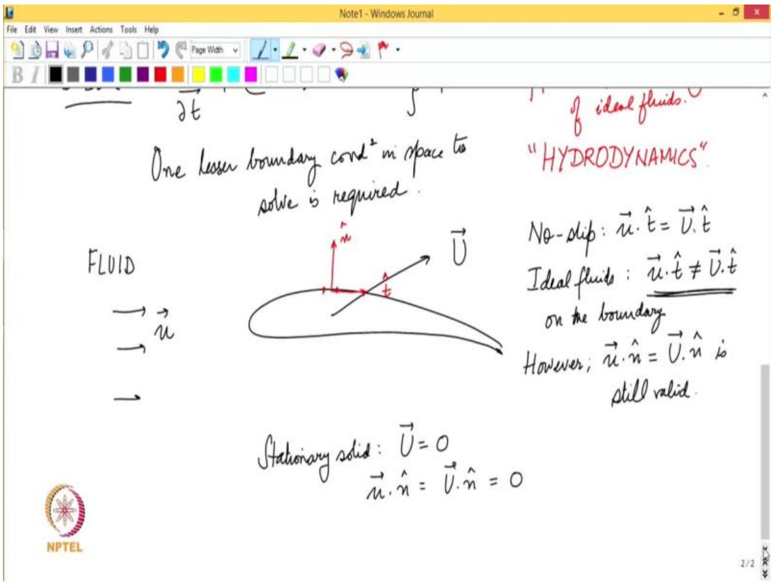

Now, since the viscous effects are absent, another key aspect that comes out is the fact that the no slip condition on a surface or at the interface between a solid and a fluid can no longer be maintained. So, consider this scenario that we have an airfoil across which there is a fluid that flows which has let us say the fluid has some velocity u vector, which is varying with space. Let us say the airfoil is moving with some velocity capital U in some direction. So, if I take locally a small element on the airfoil which has let us say a normal vector, a unit normal end cap and let us say unit tangential vector t cap.

Then the boundary condition that we typically use in the case of viscous flows are the following. We would say that no slip gives us that the tangential component of the velocity of the fluid must match the tangential component of the solid. So, it is u dot t cap is also capital U dot t cap. But here in this case, as I said since viscous effects are absent, there is no external friction to be able to bring the fluid to a rest or bring the fluid to the same velocity as the solid.

This condition now, for an ideal fluid is no longer valid. So, we cannot say u dot t cap will be u dot t cap that is no longer valid on the boundary of the solid. However, the other condition that is also true in the case of viscous flows, which would be u dot n cap to be u dot n cap is still valid. So, physically what does this mean when we say u dot t cap is not the same as u dot t cap, the tangential component of the fluid velocity can no longer be specified.

Moreover, if we talk about a stationary solid and we have fluid flowing over a stationary solid, so a stationary solid would mean that its velocity is 0. So, we can say u vector is 0 in which case, the second boundary condition will give us u dot n cap which would be u dot n cap to be 0. And that becomes a very important feature of these flows, which I am going to now use to take us further.

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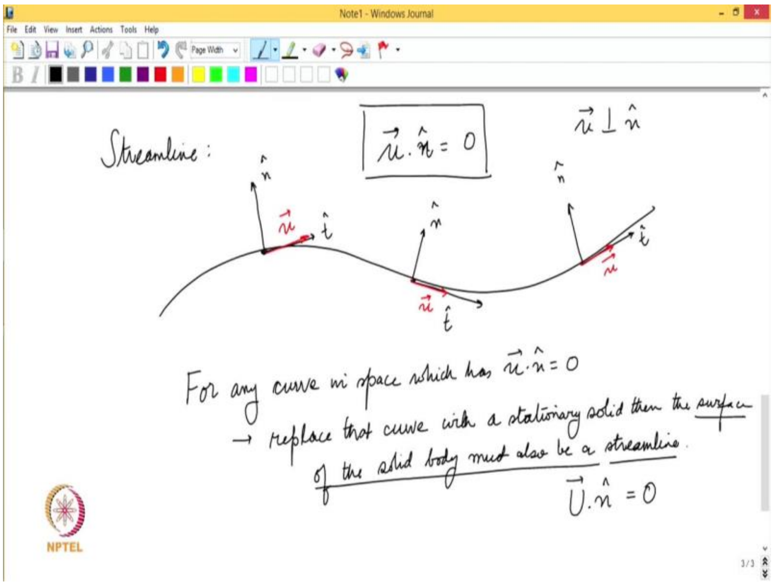

Now, to show you how this condition comes in really handy, let me remind you of what is a flow streamline. So, you may remember from your introductory course on fluid mechanics that is streamline is an imaginary curve in a fluid the tangent to which at each point gives us the direction of the velocity vector. So, if I have a streamline of a certain type, for instance, this curvilinear line in space is a stream line of fluid then if I draw the tangent at any point, which is let us say here is t cap and there is a normal vector n cap then our velocity vector at this point must be pointing along t cap.

Similarly, we can take some other point. So, say this point there is a t cap vector here, and same thing will happen that will have u vector along the direction. And we can maybe take one more scenario, let us say we take t cap here and the velocity vector would be pointing in the direction.

What it means is that since the velocity vector is in the direction of the tangential component, or the tangent of that curve, clearly, you dotted with n cap is 0. So, the dot product of the velocity vector with the normal vector at that point must be 0 on a streamline or we can say u vector is orthogonal to the normal vector.

Now, we will use this condition along with what we have talked about, for a stationary solid where u got n cap was also 0. Now, what I am saying here is that, since, the dot product to the velocity vector with the normal component or with the normal vector is 0. So, if there is any curvilinear surface in space, where this dot product happens to be 0, then we can say that or we can hypothesize that if I replace that curvilinear line or a curvilinear surface with a solid object, which is stationary. And when I say it is stationary, it means it has 0 velocity it would mean that surface of the stationary solid would also be a streamline.

So, for any curve in space which has u dot n to be 0 if I was to replace that curve with a stationary solid then the surface of that solid body must also be your streamline and this follows from the fact that u dot n or capital U dot n for that stationary solid is 0 so, we will use this idea very soon in the forthcoming lectures.

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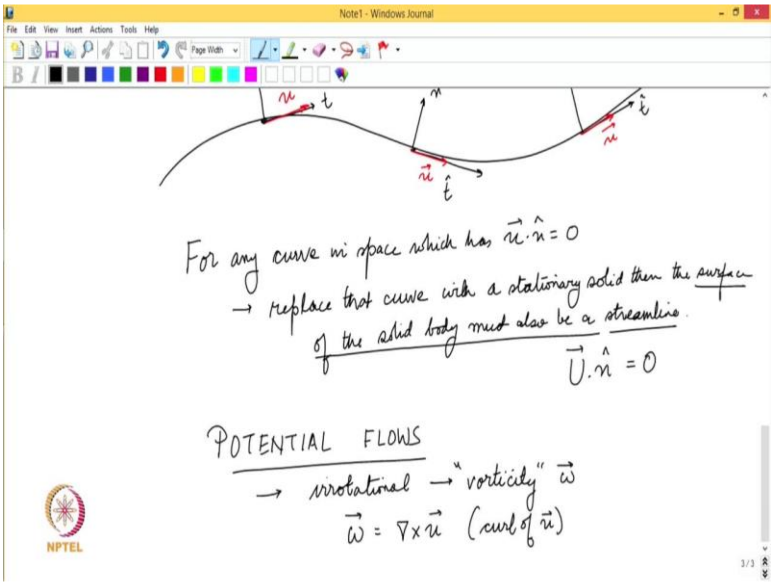

So, now, let us look at defining what is a potential flow? Now, we call a flow to the potential flow or any flow we can call this potential flow if it is irrotational and the property the fluid property that characterizes whether a flow is rotational or irrotational is its vorticity. Now, we denote in fluid mechanics typically vorticity by omega vector so, omega denotes the vorticity it is a vectorial quantity, you may also remember that omega is given as the curl of u. So, del cross u is what is given as vorticity.

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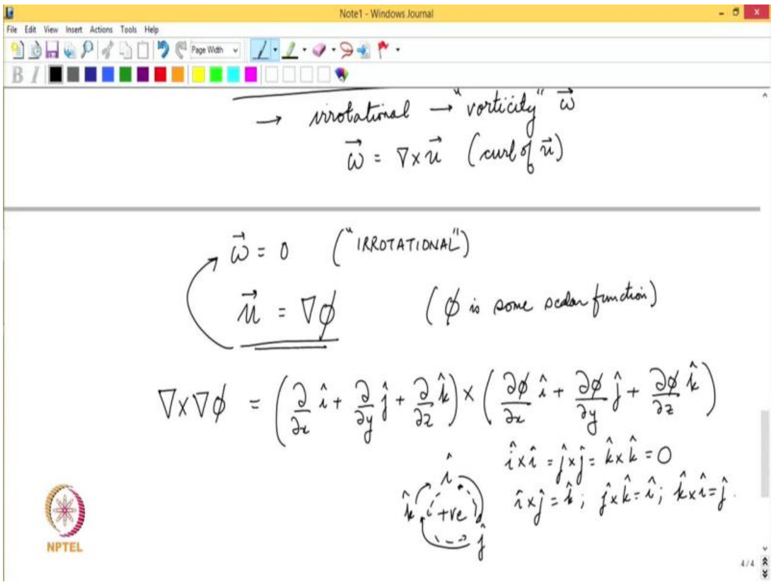

So, what I am saying here is that if the flow is supposed to be irrotational, if we say the flow is irrotational, then omega is 0 and that flow would be called as irrotational. Now, let us say that if omega is 0, let us define a velocity vector as gradient of some scalar phi, phi is some scalar function which could have space dependence.

Now, what I want to prove to you is that this choice of velocity vector which is gradient of phi naturally satisfies the condition that the vorticity is 0. So, we can prove this by just taking the curl of gradient of phi. So, in Cartesian system or in Cartesian coordinates let us try and work this out what this gives us.

So, we can write this as d by dx, i cap d by dy j cap plus d by dz k cap crossed with gradient of phi would be d phi dx, i cap d phi dy j cap plus d phi dz k cap. Now, remember the rules of vector cross product in this case since we have 3 bases vectors i, j, k which are orthogonal to each other, you may remember that we go from i to j or if we do i cross j that gives us k. If we go from j to k, it gives us i and k to i it is j gives us j.

So, we can say this is the positive sense of the cross product. If you do anything against this, then that would be the negative sense. So, what it means is first of all, i cross i is the same as j cross j is the same as k cross k which is 0 and i cross j is k, you can say j cross k is i, and k cross i is j. So, we can use this idea to simplify our cross product.

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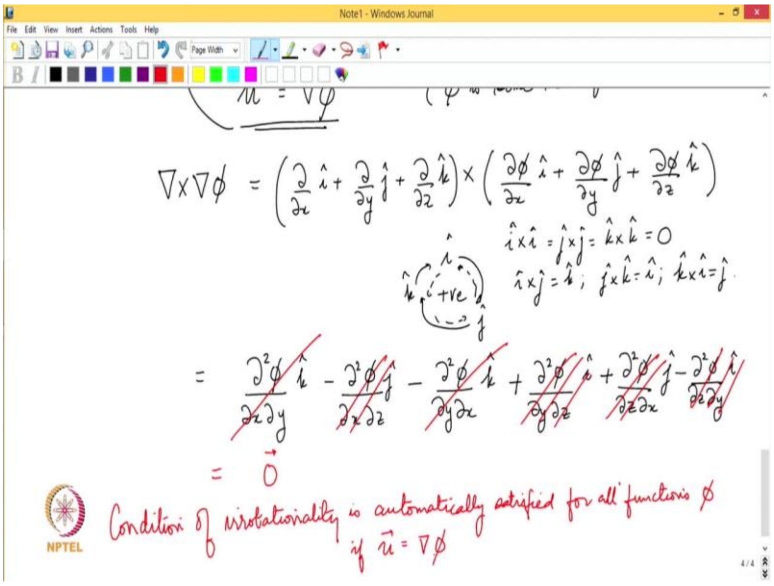

So, let us do that. So, what do we get? We will get d 2 phi dx dy k cap minus d 2 phi dx dz j cap minus d 2 phi dy dx k cap plus d 2 phi dy dz i cap plus d 2 phi dz dx j cap minus d 2 phi dz dy i cap. So, we can now look at the common terms or start canceling them. So, we can see that d 2 phi dx dy k cap cancels with this number here, we can see this number cancels with this one and finally, this I can say cancels with this one to give us 0 vector.

So, the important point to now take away from here is that the condition of irrationality is automatically satisfied for all functions phi if we could write the velocity vector u as a gradient of phi.

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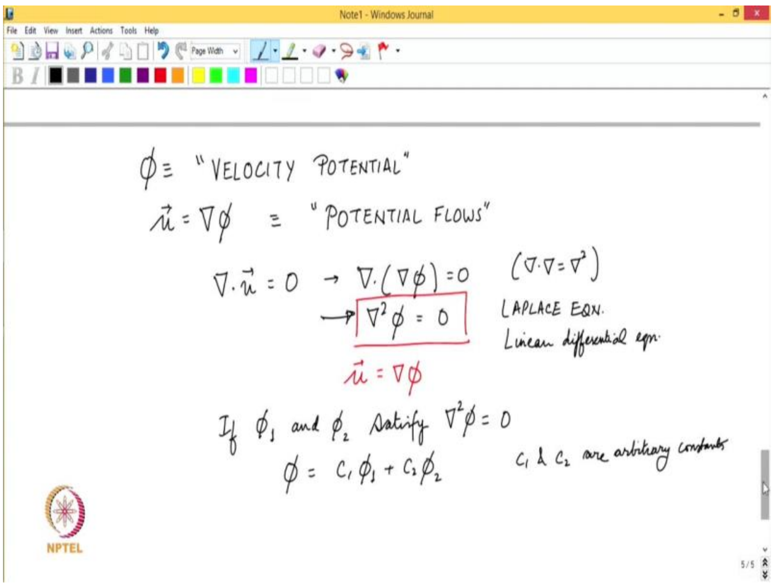

Now, this function phi is what we will call as the velocity potential that is a name that we will use to call this function phi and all flows where we could write u vector to be gradient of phi or where this is satisfied. All these flows are called as potential flows. Now, recall that the continuity equation for our ideal fluid was del dot u to be 0 so, if I substitute u vector to be gradient of phi here we will get del dot del of phi to be 0.

Now, we know that del dot del is del square which is also called as a Laplacian operator. So, what we get is del square of phi is 0. This is the Laplace equation which needs to be solved to determine the function phi.

So, the key aspect that I am now coming down to is that is that if we could solve this equation with given boundary conditions for phi and then calculate u vector as gradient of phi we would have solved or we would have recovered our flow field without actually solving the Euler's equation, which was a partial differential equation in both space and time. So, that is a key aspect of an ideal fluid that we can avoid solving the momentum equation, but rather rely on this formulation to get solutions.

But there is another key feature that now comes out, because of the way this equation is set up. So this is a linear differential equation and it is, even though it is a partial differential equation, it is a linear differential equation. And it has a very special feature, which I want to now show to you. Say, if phi 1 and phi 2 satisfy this equation, there are two solutions, let us say. So, these two equations or these two solutions satisfy del square phi to be 0 then let me say that I construct a new function phi, which is c 1, phi 1 plus c 2 phi 2. So, it is a linear combination of phi 1 and phi 2 where c 1 and c 2 are arbitrary constants.

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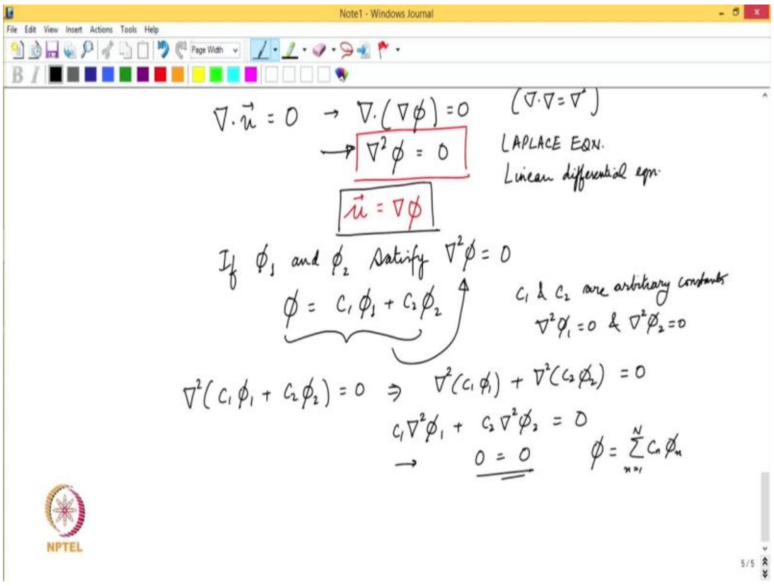

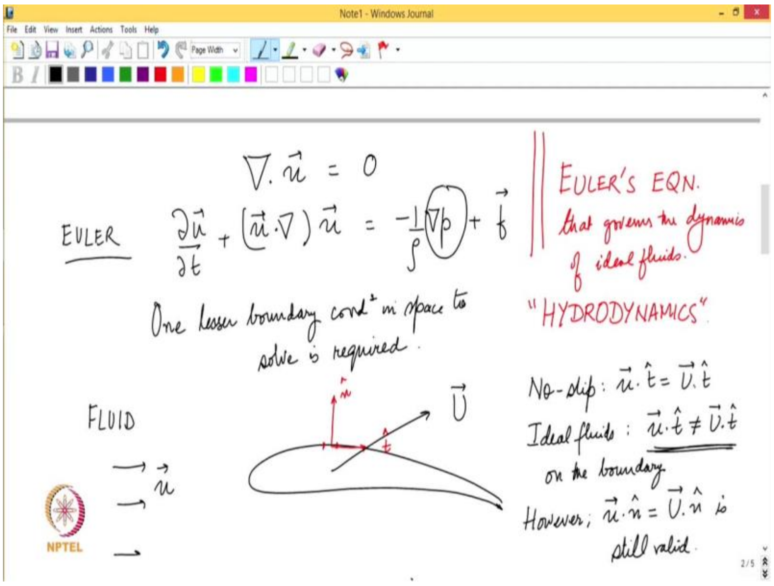

Then, we can plug in this solution or this form into this equation. And let us see what we get, we will get del square of c 1 phi 1 plus c 2 phi 2 which is 0 this could be written as del square of c 1 phi 1 plus del square of c 2 phi 2 note that c 1 and c 2 are constants. So, I can write this as c 1 del square phi 1 plus c 2 del square phi 2 is 0.

Now, because phi 1 and phi 2 satisfied the original equation, which basically meant that del square phi 1 was 0 and del square phi 2 was 0 you can now see that this gives us 0 is equal to 0 which means that for any arbitrary constant c1 c 2 I can keep constructing newer function phi and that could always satisfy my governing equation which is del square phi is 0.

Thus, we can construct as many solutions as possible and not just limited to two functions phi 1 and phi 2 we could say this is for n functions, we could say phi could be a summation of n such functions and still be able to get a new solution. So, you see that is the key feature here and that is a feature that is going to come in very handy in this course, which is that if we are solving our function or our governing equation becomes this function or this equation del square phi is 0 then the linearity of this differential equation will help us to keep constructing new and new solutions.

But, the problem is not solved yet the problem is not solved, because if you notice, although we have recovered the flow field we are using the value or the solution of phi we can derive the velocity vector and get the flow field in the domain. But we also need as is true for any fluid mechanics exercise, we also need the pressure the pressure is not known yet. So, how do we determine pressure in this problem?

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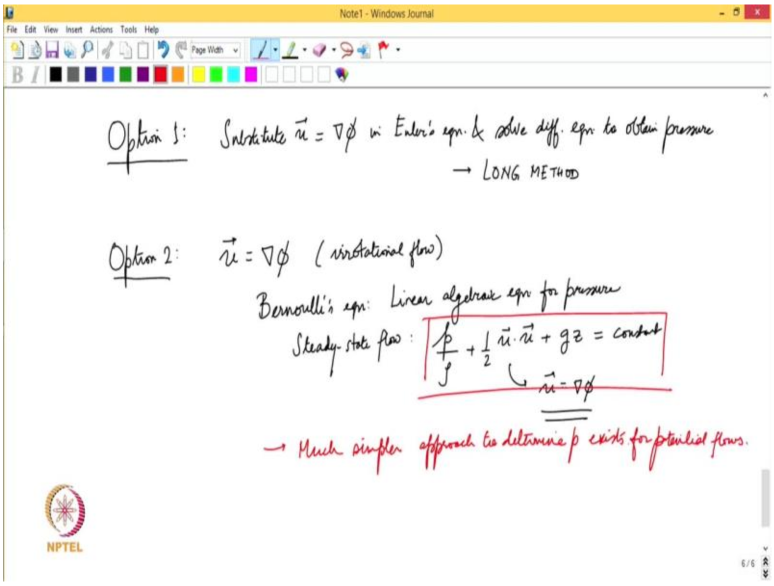

Now, there are two options available to us. So, the first option which is the more difficult option is that we can go back to the Euler’s equation and substitute u vector to be gradient of phi in that equation and then solve the resulting differential equation for pressure. Now, this will be the long method. Even though this is there is nothing wrong with this, this will be a long approach a fairly long way to get to the solution.

The second option, which is the more elegant way would be to use the fact that we are working with an ideal fluid and it is irrotational. So, we knew that u is gradient of phi and the flow is irrotational. So, we could use another powerful equation in fluid mechanics to obtain the pressure field.

Now, you may recall that if the flow is irrotational and if it is an ideal fluid, we can use the Bernoulli equation and say in this scenario now, the Bernoulli's equation is a linear algebraic equation for pressure. For example, if you are dealing with a steady state flow we could write along the streamline that p by rho plus half u dot u plus g z is a constant where u vector is gradient of phi.

So, you can see now that if the velocity field is predetermined which is calculated which has been figured out from the solving the Laplace equation for the velocity potential, we can plug that into this equation and obtain the pressure using a linear equation instead of solving a differential equation, which is the first option. So, a much simpler approach to determine the pressure exists in these potential flows.

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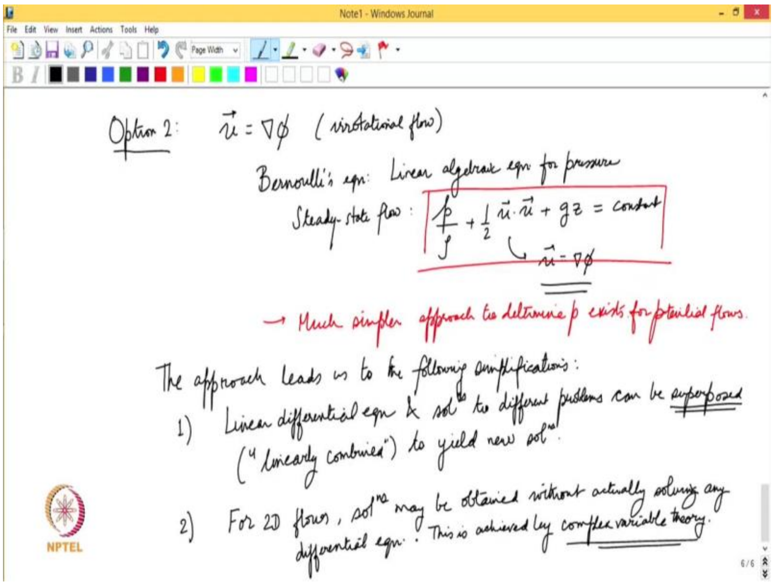

So, to summarize what you have done today we can say that the approach that we have discussed this approach leads us to the following simplifications. The first simplification that we would see or that we have described today is that the governing equation is a linear differential equation instead of the partial differential equation that is the Euler's equation and solutions to different problems or different scenarios can be superposed by superposed now, I mean linearly combined as we had done in today's class, which is we combined two functions phi 1 and phi 2 in a linear fashion c 1 phi 1 and c 2 phi 2 can be linearly combined to yield new solutions that is a very powerful aspect that we will be using.

The second is an even simpler is a second simplification that even though I have not described today, I will be covering in the forthcoming lectures which is that when we are dealing with 2D flows we will see that we can obtain solutions without even having to solve the linear differential equation or to the Laplace equation.

For 2D flows, so we can obtain solutions without actually solving any differential equation and this is going to be achieved by complex variable theory so, you see that even though I have not covered complex variable theory, but I will bring it in the next couple of lectures.

The point that I am making is that this approach will not even or will give us the advantage of not even having to solve the linear differential equation, which is del square phi 0. Rather, we can get solutions much more easily by complex variables and then go ahead and superposed them with each other to obtain new and new solutions. So, I will end this lecture here today, and see you in the next class. Thank you.