Introduction to Uncertainty Analysis and Experimentation
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Module - 06
Uncertainty in a Result
Lecture - 22
Examples of result uncertainly - 1

Welcome to the course Introduction to Uncertainty Analysis and Experimentation. Today, we will take examples of how to calculate Uncertainty in a Result. The first example is about a univariate result formula and how can we use the analytical technique to calculate the uncertainty.

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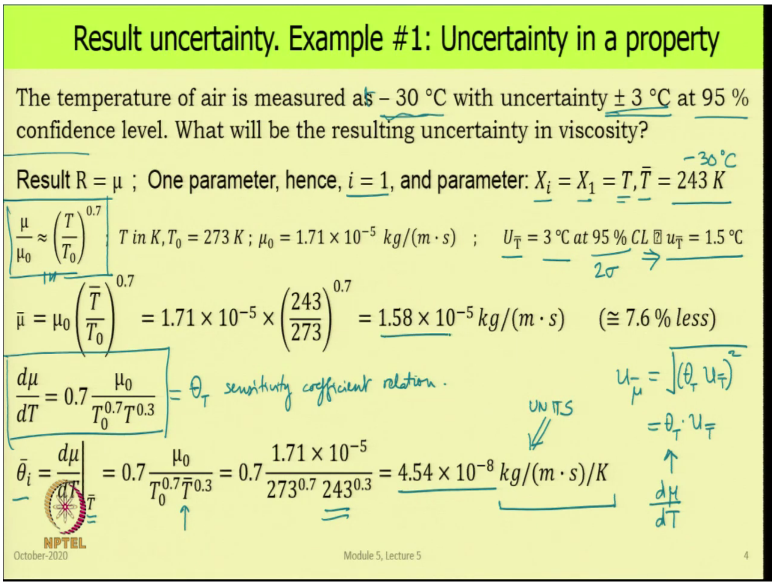

So, the statement of the problem is like this the temperature of air is measured at minus 30 degrees Celsius with uncertainty of plus minus 3 degrees Celsius at 95 percent confidence level. What will be the resulting uncertainty in viscosity? So, here is how we analyze this.

Our result is viscosity mu it is dependent on temperature, so we have i equal to 1 which means that X i is X 1 which is T. From the given information here, we know that the mean value of the measurand which is T bar is minus 30 degrees Celsius which is 243 Kelvin. The result relation that we will use is given here mu upon mu naught is almost equal to T upon T 0 rest to the power 0.7. T is in Kelvin T 0 is 273 Kelvin mu 0 is 1.71 into 10 to power minus 5 kg per meter per second.

Also from the given data we can say that plus minus 3 degrees C is expanded uncertainty at 95 percent confidence level. So, capital U of T is 3 degrees C at 95 percent confidence level; which implies that the standard uncertainty in temperature is 1.5 degrees Celsius 95 percent confidence level is 2 sigma and so we divide this by 2 and get 1.5.

Now, we follow our classical procedure that we have only one function. So, we can say that u of mu bar is equal to square root of theta T times u T bar square. Since there is only one and this can be written as theta T times u T bar. So, we want to calculate now the sensitivity coefficient theta.

So, we can differentiate it we know that theta this is equal to d mu by dT is a proper differential because there is only one variable in the expression and if we differentiate this relation this is what we get 0.7 mu naught upon T naught to the power 0.7 T to the power 0.3. So, this is our sensitivity relation. We want to calculate the value of the sensitivity coefficient at the mean value of the temperature. So, that is theta i with a bar on top is d mu dT evaluated at T bar.

So, we do all of that we substitute 243 K for T and the answer is 4.54 into 10 to power minus 8 kg per meter per second per Kelvin. This is important that when we calculate sensitivity coefficient we must always maintain the units at every step.

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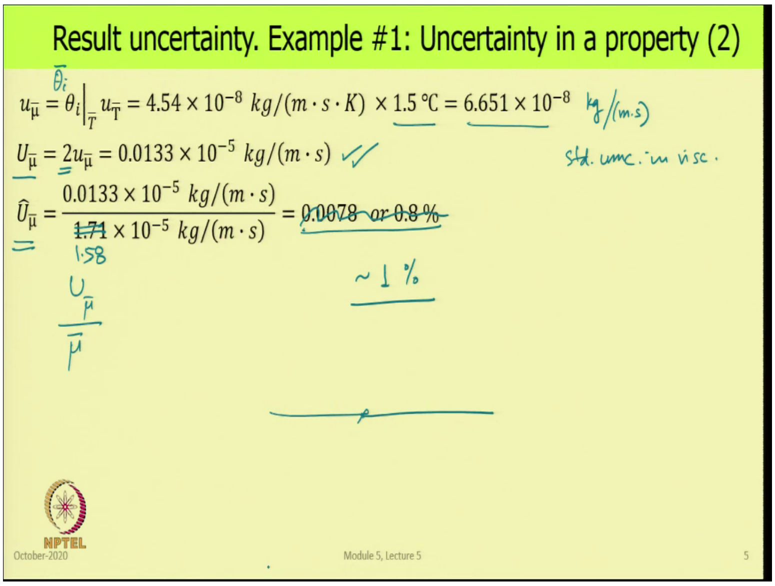

So, we calculate the value of the sensitivity coefficient and now we can calculate you u mu bar as theta i T or this is nothing but theta i bar times u T bar and this is the value we got there plus u t bar which we got from the statement of the problem this comes out to be this much kg per meter second.

So, this is the combined standard uncertainty of viscosity or you can just say this is the standard uncertainty in viscosity. The expanded uncertainty is capital U mu bar which had 95 percent confidence will be twice this value. So, we multiply it and we get this answer 0.0133 into 10 to power minus 5 kg per meter per second.

So, that is our value of the uncertainty of the viscosity. And you want to see how big a change this was we can calculate the relative expanded uncertainty which is U mu bar divided by the mean value at which we are calculating it. So, when we do that 1 minute. So, that will be mu bar which is 1.58 into 10 to the power minus 5.

So, this will not be 1.71 this is 1.58 into 10 to the power minus 5 and that you can calculate and get the number over there. So, what is it in here is not correct, where is of the order of about 1 percent ok. So, that completes what the requirement was and that is the end of this problem.

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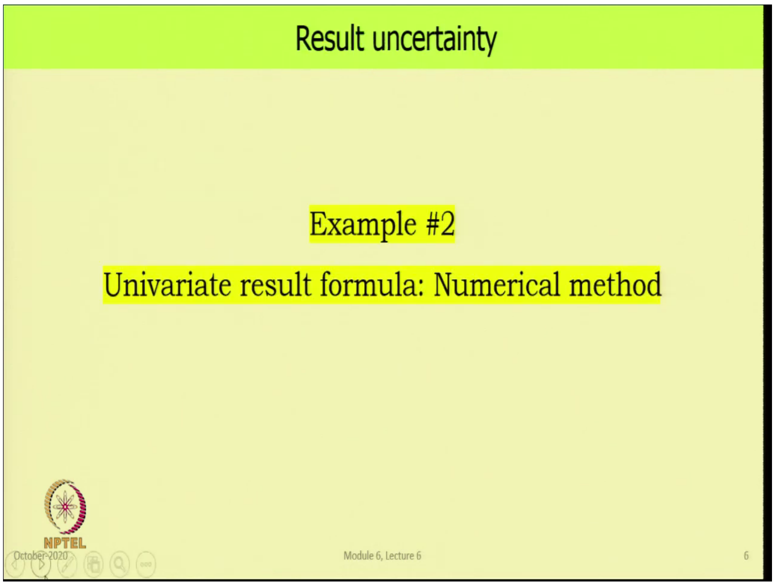

The next problem that we will take is of a univariate result formula, but with a numerical method being used. So, let us see what is this.

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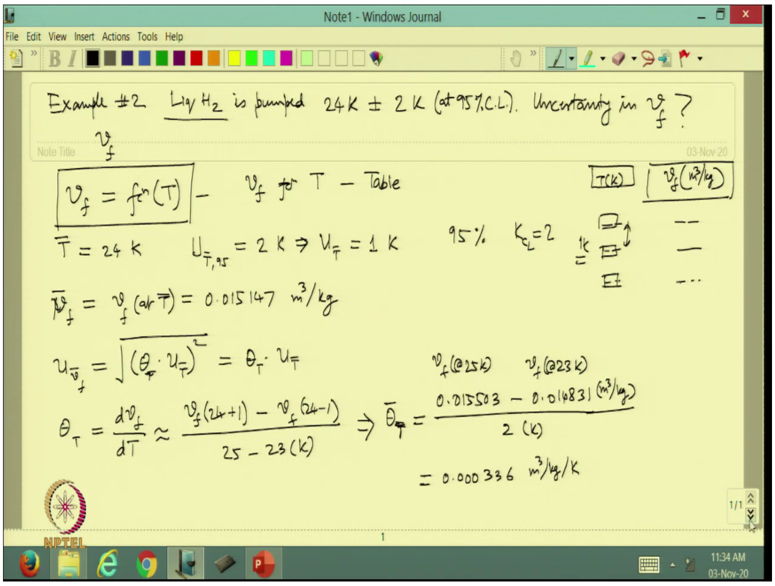

So, what we will do here is we look at the fact that we have a case where liquid hydrogen is being pumped at a temperature of 24 Kelvin its uncertainty is plus minus 2 Kelvin at 95 percent confidence levels. And we want to know what is the uncertainty in the mass of hydrogen that is pumped; the mass of hydrogen will be density multiplied by the volume the volume is fixed, we want to know what is the uncertainty in the density.

So, I if we recall thermodynamics we will have to work with what is the uncertainty in the specific volume v f that is way it is liquid hydrogen and we say that the specific volume of liquid hydrogen does not change significantly with pressure. So, we will take v f at any temperature around this as a representative value. So, the question is what is the uncertainty in v f? 1 upon v f will be the density.

Now, we do not have a nice simple expression or result formula for v f as a function of temperature for this is what the saturated liquid state. So, what we will do is we do have the data v f as a function of temperature at discrete values of temperature in a tabular form. So, as a table goes like this, this side is temperature in Kelvin and this side is v f in meter cube per kg. And so for this temperature there is something, next temperature there is something, next like that and these are discrete values.

So, what do you have to first do is see what we have been given, we are given that the mean temperature about we want the result is 24 Kelvin. The 2 Kelvin is the uncertainty which means capital u in T bar at 95 percent confidence level this is 2 Kelvin which implies that the standard uncertainty in T bar is half of this value which is 1 Kelvin, remember at 95 percent confidence level this is a factor of 2. So, we got these two.

Now, our first thing is to get the specific volume the mean specific volume which is v f at 24 Kelvin which is if you look up the tables we get this data as 0.015147 meter cube per kg. So, that the specific volume at 24 Kelvin. Now, we go back and say well what does our result formula have and we have that u v f this will be square root of theta T times u T bar square under the square root sign because there is only one variable. So, this becomes nothing but theta T the absolute value times u T bar.

Now, we want the sensitivity coefficient theta T. Since we do not have an analytical expression for v f as a function of T. If we had that we could have that differentiated it and then gone had like we did in the earlier problem we do not have it. So, what we do is we say that theta T which is dv f by dT, this we will approximate this as a finite difference which in our case we will say we will take v f at some higher temperature say instead of 24 we will take 24 plus 1.

1 is the intervals for interpolation we are taking 1 is not coming from here this is not this one it is the fact that this table is given at 1 Kelvin interval. So, we have taken 1 we could have taken 2, we could have taken 3 and we got are used many formula for this minus v f at 24 minus 1 Kelvin divided by 24 plus 1 is 25 minus this which is 23 Kelvin.

So, we are basically calculating the slope of this curve at this particular point. So, this is if you do the calculation then this becomes theta T bar or theta T with mean value that is evaluated at T bar this is distinct. So, this is v f at 25 Kelvin and if we look up the table this will become 0.015503, this is v f at 25 Kelvin minus 0.014831, this is v f at 23 Kelvin all of this divided by 2 Kelvin the units here are meter cube per kg.

So, if we solve this we get the sensitivity coefficient value as 0.000336 meter cube per kg per Kelvin.

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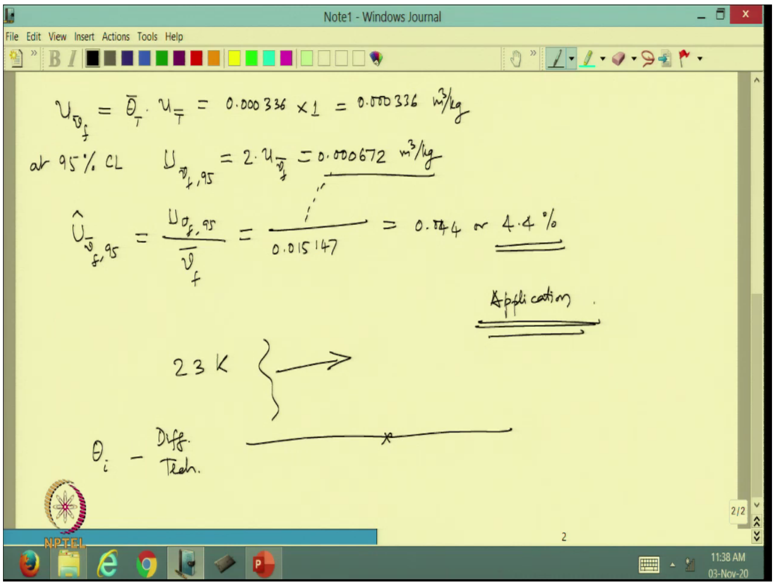

So, we got the sensitivity coefficient. Now, we can say that u in v f is sensitivity coefficient that we just calculated theta T bar multiplied by u T bar u T bar was 1. So, we substitute that and we get 0.000336 multiplied by 1 which is 0.000336 meter cube per kg at 95 percent confidence level.

The uncertainty in v f this will be 2 times the standard uncertainty in v f this is 0.000672 meter cube per kg. So, that becomes the answer we were looking for and if you want to see how big it is compared to what we are working with we can do one more calculation. Calculate the relative uncertainty u v f bar at 95 percent confidence level this is u v f bar at 95 divided by v f bar.

So, it will be this value in the numerator divided by what we had earlier the mean value 0.015147 and you calculate this, this turns out to be 0.044 or 4.4 percent is the uncertainty. Then we can make a decision is this big, is it acceptable, is it too small and for that you have to look at what is the application for all of this and in say in the case of a cryogenic engine where you are filling liquid hydrogen this uncertainty may be too large.

Because if you fill more if you say that if the filling was done at 23 degrees 23 Kelvin the density is higher we fill more mass in 25 we will fill less mass. In the first case, initial mass of the rocket to be lifted up is more, so it may not be the orbit. In the second case, the total mass of fuel in the rocket is less it may not again reach the orbit.

So, those are the type of implications that you have based on what application we are looking at ok. So, that is an example of how we can calculate the sensitivity coefficient from a finite difference technique or use the central difference formula you could have use any other formula which is there in numerical techniques.

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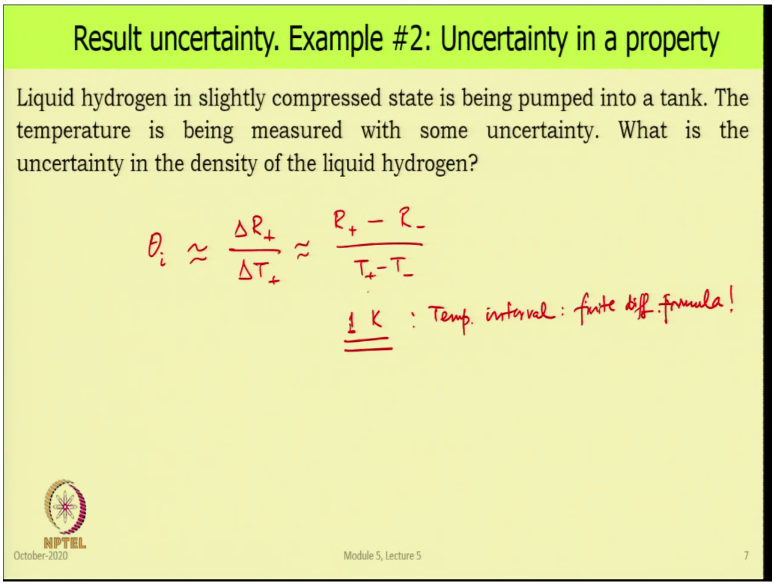

Now, so that was the liquid hydrogen case where are where we approximate a theta i as delta R plus upon delta T plus which was value at some higher value this at minus and the difference of the temperatures. Now, remember these temperatures that we took as 1 Kelvin in that example these had nothing to do with the uncertainty in the temperature, this was just taken as the temperature interval for the finite difference formula.

The rest of the method follow the same technique. So that is that problem. Now, we take an extension or more involved situation of the first earlier case but instead of a univariate result relation.

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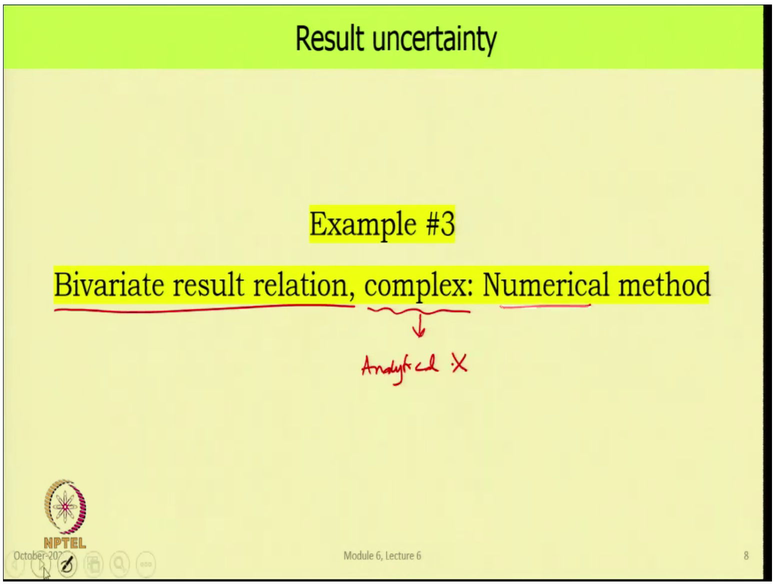

Let us look at a Bivariate result relation which is too complicated. So, we do not have analytical relation and we will use a numerical method.

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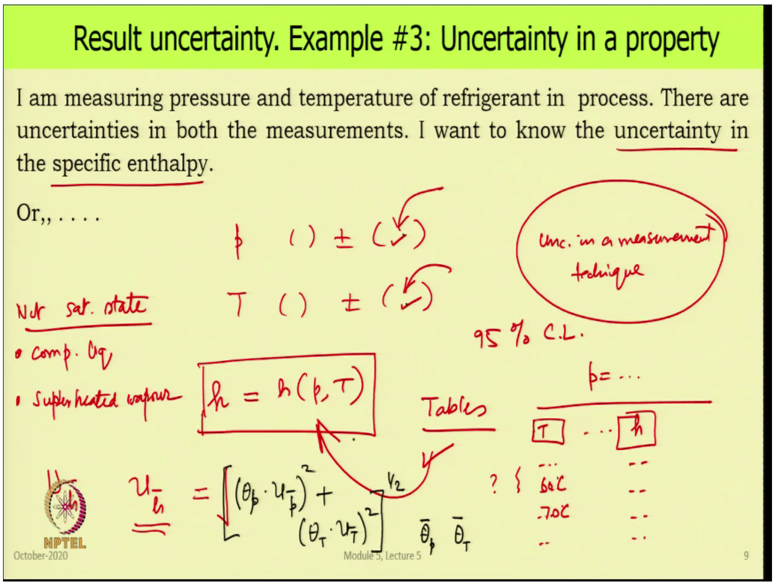

A problem statement goes like this I am measuring pressure and temperature of a refrigerant in a process this could be a refrigeration plant a chiller unit or whatever. There are uncertainties in both the measurements. So, I have measured pressure, I have measured this as something plus minus something temperature something plus minus something. And we are assuming that these uncertainties have been calculated using the same method that we learnt in uncertainty in a measurement.

So, we are assuming that we went through this entire exercise of calculating the uncertainty in pressure, uncertainty in temperature. And now we have those values over here with us which are this and this nothing else if it is mentioned we assume this is all being reported as 95 percent confidence level. And the question is I want to know the uncertainty in the specific enthalpy.

We need specific enthalpy because for all cycle calculations and to look at performance of a refrigeration cycle or air conditioning chiller plant we need the value of h. So, specific enthalpy is a function of pressure and temperature and we are taking a situation where the pressure and temperature are such that the refrigerant is not a saturated state. So, either it is compressed liquid or it is a superheated vapour.

In either of these cases there is somewhere out there an analytical expression for h as a function of p and T. It will have many constants it would have a very complex form or it could be a limited type of correlation which is not the best thing to work with, but it is easy and convenient. But we will not access that at this point and we will say that I will get I have the only information I have about this is property tables.

And in the tables for each pressure there is a chart we say that if this is a temperature then what is the specific enthalpy? So, for this temperature it is here for this one it is here and like that these are distinct values of temperature discrete in nature. So, in between them there is a gap. So, if there this could be 60 degrees Celsius this could be 70 degrees Celsius and not 61, 62 or anything finer than that.

So, this is the data that we have for this type of a relation. And using this data we want to calculate uncertainty which is capital U h bar for that we have to first calculate the standard uncertainty which is small u h bar. So, that is our objective and this will come about by saying that this is square root u p. So, this is theta i which will be theta p times u p square plus theta T u T square all of that raised to the power half.

Now, this is just a two part formula where we had that summation sign and our objective now is to calculate theta p which at the mean value will be theta p bar and theta T bar that is what we need to calculate. Once we do that we already would have the values of this and this and we can calculate this. So, let us do this calculation.

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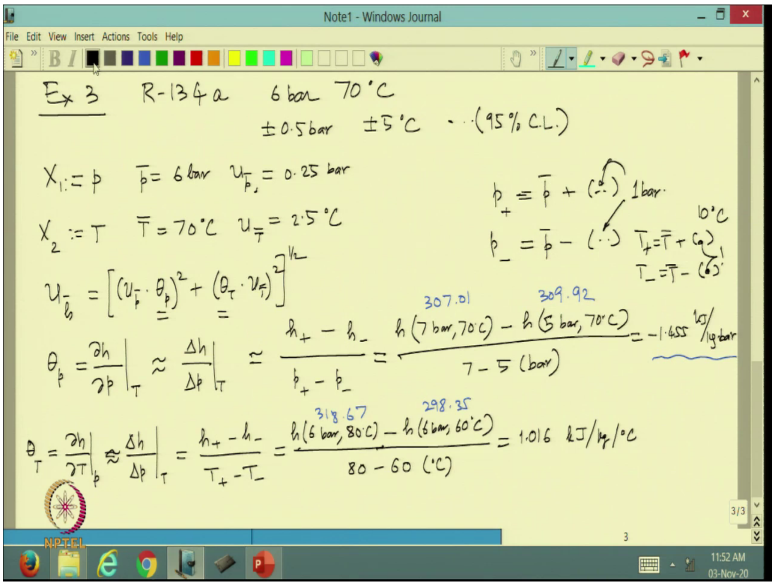

So, we are given that do you have refrigerant or 134 a at 6 bar 70 degrees Celsius and that uncertainty in pressure is plus minus 0.5 bar and uncertainty in temperature is plus minus 5 degrees Celsius in the absence of any other information we assume this is 95 percent confidence level.

So, here we have we solve this our first variable x 1 will be corresponding to p and we have p bar is equal to 6 bar and u p bar is equal to half of 0.5 bar which is 0.25 bar. X 2 corresponds to temperature the mean value of temperature is 70 degrees Celsius u T bar is equal to half of this value which is 2.5 degrees Celsius.

And the relation that we have said we are looking for is u h bar is equal to u p bar times theta p square or theta T times u T bar square all of this raised to the power half. So, we start calculating individual terms there. So, let us take first theta p this is now dh by dp at constant temperature dh by dp at constant temperature which we will now approximate as a finite difference relation and make it delta h by delta p at constant temperature about the point T bar.

So, this becomes equal to let us say that the interval that I will take for calculating this. So, this has to be h plus minus h minus upon p plus minus p minus where p plus is p bar which is our nominal mean pressure plus some value about that pressure p minus is p bar minus the same value, we can take 1 bar 2 bar 0.5 bar point 1 bar that is our discretion here it has nothing to do with this 0.5 bar. So, in the example I have taken this to be 1 bar.

So, what will this expression will become is that we will calculate h at the same temperature which is 70 degrees C 6 plus 1 becomes 7 bar 70 degrees Celsius minus h 6 minus 1 5 bar 70 degree Celsius divided by 7 minus 5 bar. So, we carry out this calculation and the number we will get is minus 1.455 kilo Joule per kg per bar. So, these values in which I have used in calculating over here.

They came from the tables and depending on the tables there could be slight difference in the values that you get. The tables that I was using this gives 307.01 this is 309.92 and this has a minus sign; other tables may give slightly different numbers for both of them. But the difference will be the same and the final number that you get will be identical to this.

Similarly, we do for theta T this is dh by dT at constant pressure and this we will approximate the finite difference which is delta h by delta p at that temperature which as before this will again we may h plus minus h minus upon T plus minus T minus. And now again we are doing the same thing here that T plus is T bar plus some value T minus is T bar minus some value these values we can select and the example I have given I have taken this to be 10 degrees Celsius.

So, this expression will now become h at the pressure has to be the same 6 bar temperature we have increased by 10. So, instead of 70 this becomes 80 degrees Celsius minus h at 6 bar 60 degrees Celsius divided by 80 minus 60. And if you do this whole calculation this is all degrees Celsius you will get 1.016 kilo Joules per kg per degree Celsius and the values that I had was 318.67 for this and 298.35 for this. So, we got both are theta p and theta T.

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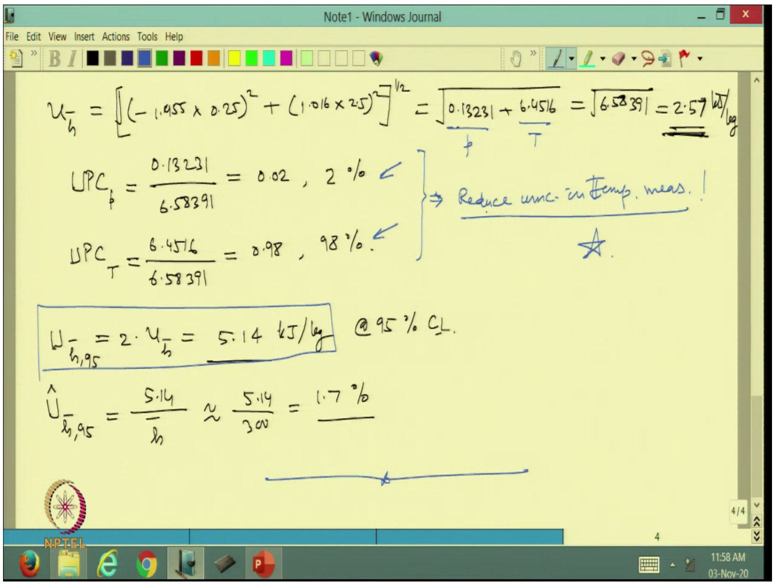

And now we go to the next step and we write that u h bar is square root of the two product 1.455 into 0.25 this is the pressure term. And this is 1.016 into 2.5 this is the temperature term. And if we do this whole calculation and I am going to keep these two numbers separate because we will use them later on 0.13231 plus 6.4516 and if you take this sum this is 6.58391 which is equal to 2.57 kilo Joules per kg is 2.57.

I will deliberately kept these things, but in a calculation one can skip it, but we will see in a minute why this helps us. So, that was the answer we were looking for this is the value of the uncertainty in the specific enthalpy and by looking at these numbers we can do one more thing you can calculate the uncertainty percentage contribution of pressure this; this is a term that came from pressure and this came from temperature. So, UPC for pressure will be 0.13231 upon the total 6.58391 this is equal to 0.02 or 2 percent.

We can do the same calculation UPC for T this will be the second value 6.4516 divided by the same denominator 58391 and this is equal to 0.98 or 98 percent. So, here we have we got the answer we wanted to know what is uncertainty in the specific enthalpy this is 3 2.57 and we can then even calculate if we want the expanded uncertainty which is U h bar which is equal to two times small u h bar, which is we multiply that this becomes 4 1 and 5 kilo Joules per kg at 95 percent confidence level.

And the last thing one can do is also look at you had the relative uncertainty that is there which will be this value 5.14 divided by the mean value of the specific enthalpy which in this case we can get that value there and this is of the order of the numbers that we were looking at in the previous page.

So, this is of the order of this some somewhere here 309, 310 somewhere like that. So, this is roughly 5 by 300 which is multiplied by 100 and you will get this as 1 point say 7 percent. Now, you are looking at about a 2 percent uncertainty in specific enthalpy and that is the answer; we not only got the uncertainty in the of enthalpy.

We also learned on the way that the uncertainty percentage contribution from pressure is 2 percent and temperature is 98 percent and this is valuable information where these two together they tell us that if you want to reduce this uncertainty in specific enthalpy from 1.7 percent wherever you want to go you focus on reducing uncertainty in temperature measurement; do not worry about the pressure this is where we should focus on.

This is now very valuable information as to that if I have to redo the experiment or plan it in a different way what is should be my strategy that is what this calculation told us. So, this completes this example of a bivariate result formula using numerical techniques.

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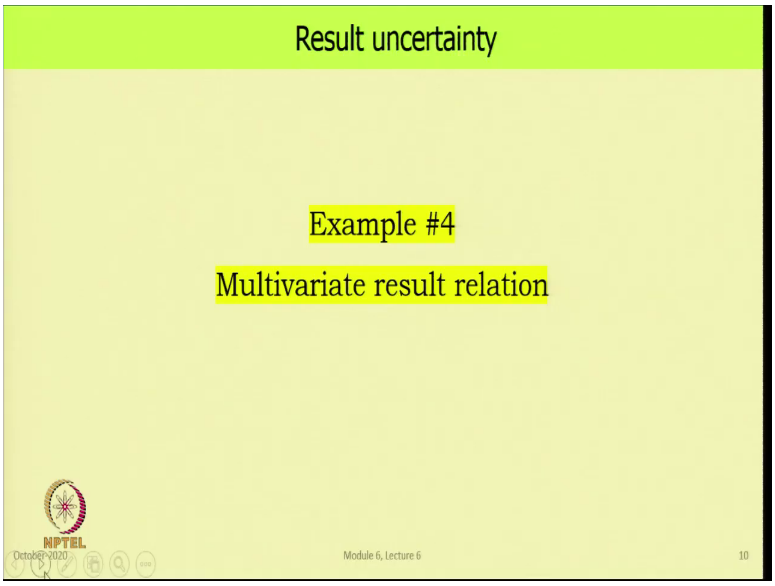

Now, we go to the next example Multivariate result relation.

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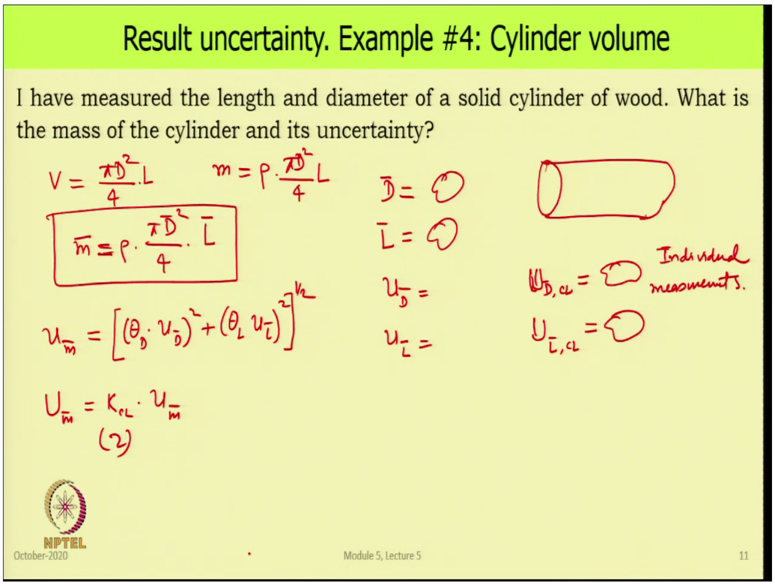

So, you have an analytical expression and we want to calculate uncertainty, though here is a typical statement of this type that I have measured the length and diameter of a solid cylinder of wood, what is the mass of the cylinder and its uncertainty? So, that is being told to us. So, there is a cylinder or rather a log of wood and we have the volume as pi d squared by 4 multiplied by L and if you want to get the mass this is rho times pi d square upon 4 into L.

And let us see what we have been. So, we have what information we required to calculate the mass of the cylinder and its uncertainty m bar will be rho times pi d bar square upon 4 multiplied by L bar. So, we need to have the nominal values of the diameter and the length which came from the individual measurements. So, that is one thing we want.

And we said that to calculate uncertainty in a mass we assume that uncertainty in density is negligible compared to the others, then this will become theta D u D bar square plus theta L u

L bar square all of this raised to the power half. So, we need to have u D bar which can be obtained from capital u D bar at some confidence level which is what will be coated and we also need the standard uncertainty in the length which we can get from the expanded uncertainty in the length at a certain confidence level.

So, these four numbers this one, this one, this one and this one they came from the experiment or individual measurements. And all of that will follow the same procedure that we have learned in how to do uncertainty of a measurement. Finally, we will calculate U m bar as the multiplication factor K cL which could be 2 if it is 95 percent confidence level into u m bar. So, this is our result formula.

This is a clear simple expression we can differentiate it and we can get D and L. So, let us do that. So, I will put some values there that the mean value of the diameter from the experiment is 0.5 meters and with its uncertainty capital U D bar at 95 percent confidence level is 0.002 meters.

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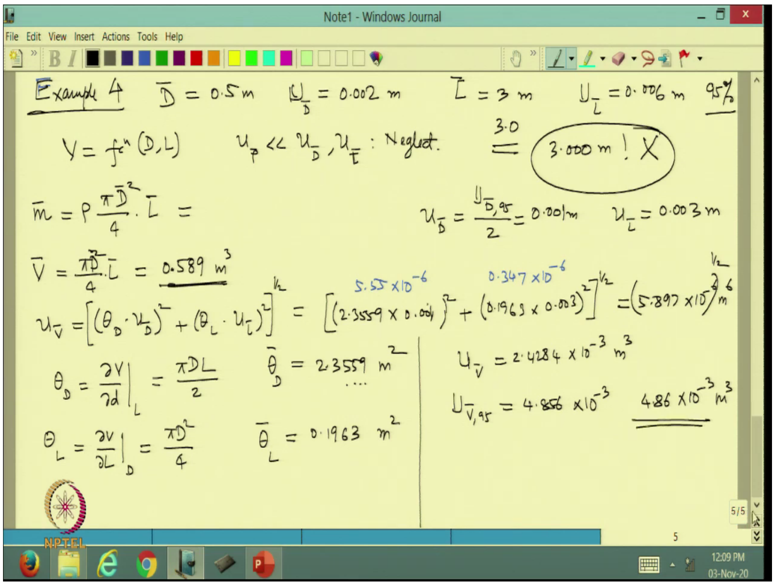

And the mean value of the length this is 3 meters and U L bar is 0.006 meters all of this at 95 percent confidence level. Now, here we said 0.5; the question is whether we should write 3 or we should write 3.0 meters? But this also tells us that it is we are not sure of the next decimal place this plus minus one more of that in this case it tells us that we were able to measure it up 2.1 meters. So, it is justifiable to write the length as 3.0 meters and not as 3 and definitely not as 3.000 meters, this would be wrong.

So, let us start off we have our result formula V as a function of D and L; we are assuming that uncertainty in row is very very small compared to uncertainties is in D and uncertainty in T in L and so we are neglecting it. But density is a measured parameter we are taking the value from somebody else’s experiment and we if we want we could return it becomes a 3 parameter relation. So, when we have this.

So, the mean value of the mass this is nothing but the result formula which is row pi pi D bar square upon 4 into L bar. And if you do all this calculation we have D bar here L bar here, we can take the density for the example that I am giving I am not calculating the density, but I am working with volume which is pi D bar square upon 4 multiplied by L bar and this turns out to be 0.589 meter cube.

Then we have our uncertainty expression u v bar is equal to theta D times u D bar square plus theta L times u L bar square all of this raised to the power half and when you do all this for this calculation, we now need theta D. So, theta D is dv by dd at constant L we can differentiate this and this becomes pi DL upon 2 and we evaluate the value and we get theta bar D by putting D bar and L bar in this expression and this becomes 2.3559 meter square.

How many decimals to keep? You can think about that the best thing is that if you have to do roundoff we do roundoff at the last step theta L. Similarly, is dv by dL at constant D which is pi D square upon 4 evaluated at L bar D bar and so theta L bar is upto D bar into this and we get the value as 0.1963 this is all this is now volume by length this is also meter square.

Now, we substitute in this expression. So, we have this theta D value coming from here 2.3559 into u D bar we need that. So, what we have to do is this is given that 95 percent confidence level. So, u d bar is U D bar at 95 percent confidence level divided by 2 which is equal to 0.001 meter and similarly u L bar is 0.003 meters. So, here we put 0.001 square 1 the whole thing square plus 0.1963 into 0.003 whole square raised to the power half.

Again we can keep the individual squares if we do that the numbers we will get I will write down in blue here, this will be 5.55 into 10 to the power minus 6 and this is 0.347 into 10 to the power minus 6 we will use this again later on to calculate the UPC's and. So, if you do this complete calculation this will be 5.897 into 10 to the power minus 6 meters to the power 6 and taking the square root of this we get u v bar this is equal to 2.4284 into 10 to the power minus 3 meter cube.

And the expanded uncertainty at 95 percent confidence level this will be doubles this value and we can put this as 4.856 into 10 to the power minus 3 or 4.86 into 10 to the power minus 3 meter cube. So, that is the answer we were looking for.

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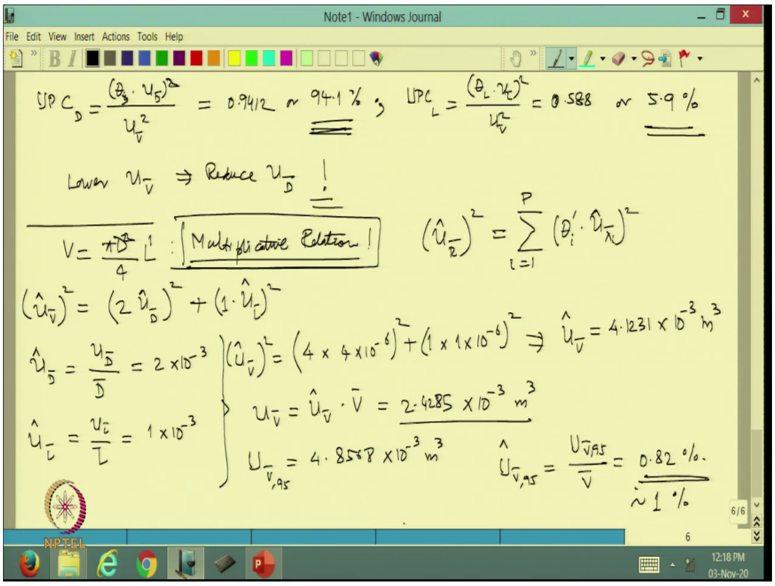

Now, we can look at a few more things on this one we got the volume there, we got the uncertainty over there. So, the answer is complete. Now, we can look up the individual terms that we had just written there and we will find that UPC the Uncertainty Percentage Contribution due to diameter.

Now, this will be theta i theta D times u D bar square upon u v bar square. So, if you do this calculation the number comes out to be 0.9412 or 94 percent. And we do same calculation and say what is UPC in length it will be the same formula theta L u L bar square upon u v bar square and this calculation will come out to be 0.58 and if you run this off this we can say is about so this is 94.1 percent and this will be 5.9 percent.

So, it tells us that the contribution the uncertainty in the volume the biggest contributor is uncertainty in the diameter, length is pretty much in control. So, if you want to improve the uncertainty you want a lower uncertainty in the volume, then reduce uncertainty in the diameter. So, that is another important decision that comes up do not worry about the length too much the diameter is what is getting at here.

So, that is one more information that we got from this analysis besides the uncertainty value but there is. Another way which we would have done and done much more simpler by recognizing that when v is equal to pi D square upon 4 into L we realize that this is a multiplicative relation.

So, there is a shortcut that we can use and we know that the exponents of D and L will come in the way the uncertainty propagates and the formula that we had from notes is that u hat R bar square; that means a relative standard uncertainty in the result this is a summation of theta i hat into u hat x i bar square for all the parameters in the result.

And when we do it for a multiplicative relation we saw that this expression becomes very elegant with theta i prime being replaced just by the exponents of this. So, we can without going through the long details say that u hat v bar square is equal to u hat D bar square and the one before this is the exponent of D which is 2; so this comes here plus u hat L bars holding square and this exponent here is 1. So, this 1 comes over here.

So, we straightaway got the final relation we did not have to go through calculating each one of the theta i's we can calculate u hat D bar which is u D bar upon D bar and if you do this calculation the answer is 2 into 10 to the power minus 3 we can calculate u L bar hat this is u L bar upon L bar which is 1 into 10 to the power minus 3.

And when we put it in this relation using these two we will get u hat v bar square is equal to this whole thing square which becomes 4 multiplied by this i square which is 4 into 10 to the power minus 6 plus 1 square there into this also happens to be 1 into 10 to the power minus 6 whole square.

And when we do this calculation you will find that u v bar hat this will be 4.1231 into 10 to the power minus 3 meter cube. And from there we can get u v bar which is u hat v bar multiplied by v bar; we do this calculation and we get 2.4285 into 10 to the power minus 3 meter cube which is the same answer we got by this other method.

So, this from example illustrates that if you have a multiplicative relation you can use a much simpler technique without going through the details and then work backwards and get the UPCs and UMFIs we got U v bar there also v hat at 95 percent confidence level the uncertainty will be twice this value which will be 4.8568 into 10 to the power minus 3 meter cube.

And then we can calculate U hat v bar this is at 95 percent confidence level u hat 95 percent confidence level this will be U v bar 95 percent confidence level divided by v bar and if you do this calculation this is turns out to be 0.82 percent. So, we have a plus minus 1 percent uncertainty in the volume which will also mean plus minus 1 percent uncertainty in the mass and it could be that this is a piece of wood that is being sold from somebody to somebody on a mass basis.

So, both have to agree that uncertainty of the order of 1 percent is ok and then you just build them on the mean value of the volume otherwise we have to end up using something else. So, that completes this example. So, we got we saw one more example over there and we will conclude at this point and pick up more examples later on.

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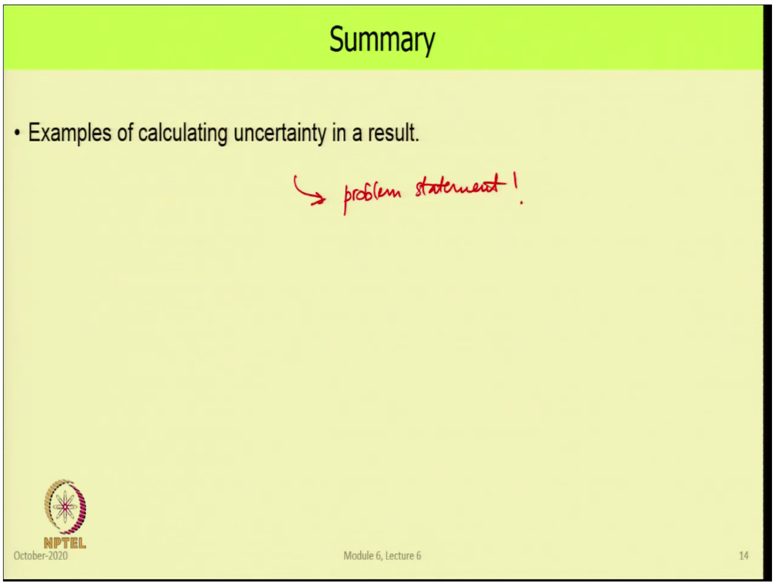

So, with that example we will stop at this point, we will take more examples later on. And to summarize what we have done we have seen various ways by which we can calculate the uncertainty in the result it depends on what type of a situation we are dealing with or what is the type of problem statement that we have.

So, we will stop at this point. So, this has been some examples of how to calculate uncertainty in the result we stop here.

Thank you.