Introduction to Uncertainty Analysis and Experimentation
Prof. Sunil R. Kale
Department of Mechanical Engineering
Indian Institute of Technology, Delhi

Module - 06
Uncertainty in a Result
Lecture - 19
Sensitivity Coefficient Result Uncertainty from TSM

Welcome to the course Introduction to Uncertainty Analysis and Experimentation. We are on module 6 which is uncertainty in a measurement and in this lecture we will look at Sensitivity Coefficient Result Uncertainty from TSM.

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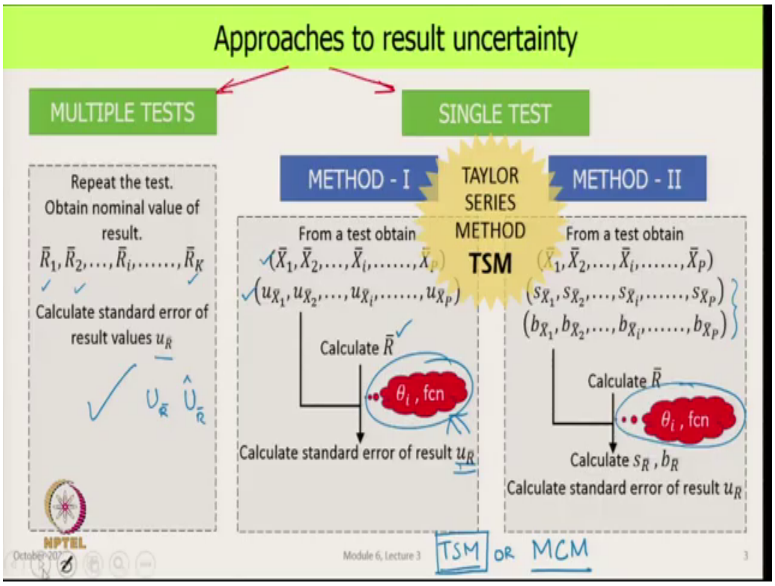

In the previous lecture we had seen what are the approaches to result uncertainty? We said we could follow the Multiple Test route or the Single Test route. So, we have two options there yesterday we saw in detail how the multiple test technique works and we saw that all we needed was to make measurements get the mean value of the measurements and from that calculate the mean value of the result.

Then we do the same thing over and over again to get many values of that result, which are these and then using these as a sample a statistical sample we calculate the standard uncertainty in the result. Once we have that then calculating the expanded uncertainty and the relative uncertainty this can be done with the assumption of a confidence level. Then we looked at single test method and we saw that we have two approaches that we could either use the Taylor series method or the Monte Carlo method and we saw that this is a discrete probabilistic model.

It is a very powerful simulation tool, but we will leave it not for an advanced course. In this course our attention will be on uncertainty analysis using the Taylor series method. And here we have two methods, we will look at method 1 develop the analysis and after that a small modification to that and we will get the required information for method 2 and that completes our mathematical treatment.

In both cases, our common objective is that we want a function which is shown here, the same function in both cases, which links measurement uncertainties mean values of the measurement and result to uncertainty in the result; this is the same story in both cases. So, this is what we set out to look at last previous lecture.

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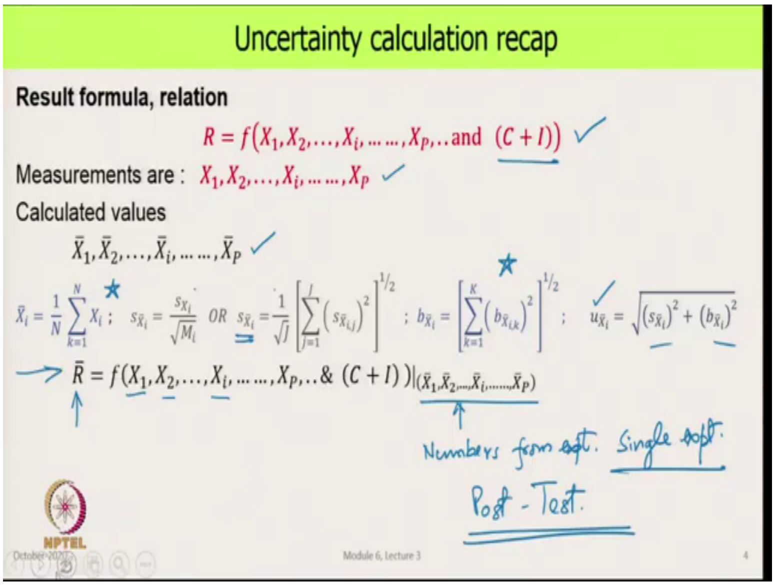

And we said that we what is the basic mathematics of the Taylor series method and from there we came to a nice relation. So, to recap what we have done so far is that we have a result formula which is a combination of parameters and some constants, we make measurements of these parameters multiple measurements from that we calculate the mean values of each parameter.

And then so for that we use this first relation over here and then with the data that we have either this method or this method we can calculate the random standard uncertainty, from this method we get the systematic standard uncertainty and then by combining these two we get the standard uncertainty in the measurement.

So, this was all about the measurement and then for the result we start with the result formula and in that formula we substitute all these parameters with their mean values that we got from the experiment. So, this is the numbers and these numbers came from experiment and this is a single experiment.

That means, we did the experiment once got all the data we want and now we are doing the analysis. So, in that sense we are in a way looking at post-test uncertainty analysis. So, we can easily get the mean value of the result formula and that is what you probably have done in your school and college experiments.

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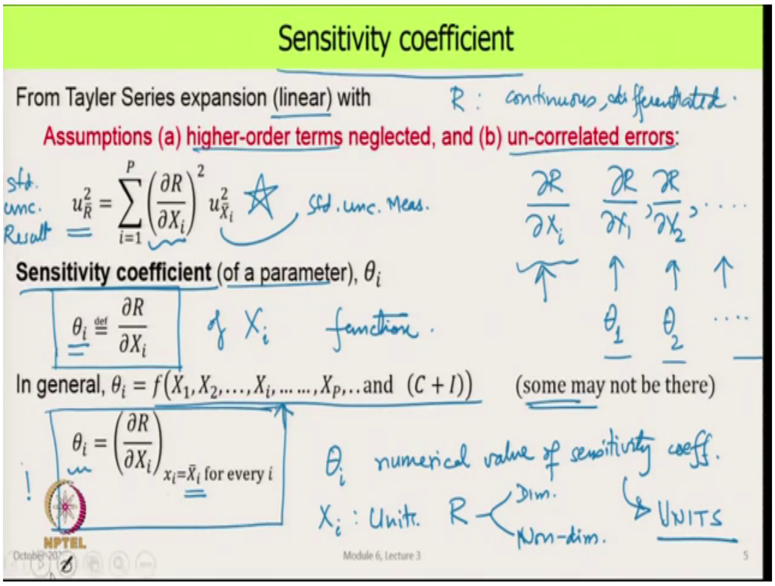

Now, we go beyond that and we will define a new term which is called the sensitivity coefficient and the reason for defining this is that when we looked at the Taylor series expansion. We saw that in the expansion process if we neglect the second and higher order terms, the term that stays is a disrobe dR by dX i, the partial derivative of the result with respect to each of the variables.

So, we will have one function which will be dR by d X 1 another function which will be dR by dX 2 and so on. So, we generate that many functions. So, this was something a form which was coming as on the right side of that equation and coming everywhere. So, these are lot of implications and this is where the, we define it at the sensitivity coefficient.

So, from what we did with the Taylor series expansion under the assumptions that we are looking at the linear expansion second and higher order terms and neglected and errors are uncorrelated, then we got this formula for the result uncertainty. So, this is standard uncertainty of the result, this is standard uncertainty of each measurement and this was the differential that we have been talking off.

So, now we define this differential either sensitivity coefficient of a parameter theta i. So, this is the sensitivity coefficient of parameter X i. So, this is now dR by d X 1 this will get the nomenclature theta 1 this will get theta 2 and so on, so this is the function. And in general since we are just taken the result formula and differentiated it in general theta i will be some other function of the variables and the constants.

And it is quite possible that some of these variables and constants may not appear in the sensitivity coefficient function. So, we get a function and in that we put values that is the mean value of every parameter and when you do this calculation by putting this in this function we get the theta i, which is the numerical value of the sensitivity coefficient.

Now, recall that X i is they are all dimensional they are measurements, so they have units the result expression, the result formula may or may not have units. So, it could be dimensional or non dimensional. So, when we look at theta i we can see that X i has dimensions R may or may not have dimensions. So, theta i would generally have dimensions may be some cases where theta itself could be non dimensional. So, in general this has to be accompanied by the units that came out of the formula.

So, that is our basic definition of the function called the sensitivity coefficient and this is the formula for calculating its value. So, we will see if we can both side theta i is coming, but this is a function this is a number maybe we could give it a slight modification of this symbol.

So, that we differentiate theta i here and theta i over there. So, what we have seen is that by this definition as many measurements are there in the experiment or in the result formula sorry that many sensitivity coefficients we will get. And that is all assuming that the result formula that we had in the first case was continuous and everywhere it could be differentiated.

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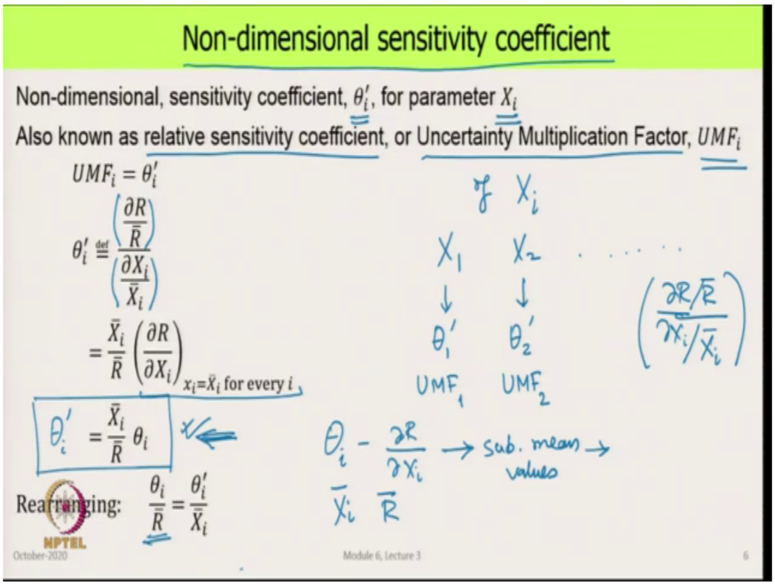

Now, we look at a modification of the sensitivity coefficient and develop the relation for the non-dimensional sensitivity coefficient. The symbol for that is theta i prime and this is theta i prime is this non dimensional sensitivity coefficient for the parameter X i theta i prime is also known as the relative sensitivity coefficient.

And in some texts you will find this mentioned as Uncertainty Multiplication Factor with the symbol UMF subscript i. So, this is the uncertainty multiplication factor of parameter X i. So, for X 1 we will get theta 1 prime or we can even call this as UMF 1 for variable X 2 we will get theta 2 prime or UMF 2 and so on. And the definition of this the non dimensional sensitivity coefficient is dR by R over dX i by Xi, so what we have done is normalized.

What we had earlier as dR by dX i divided this by R made it a ratio non-dimensional divided this by X i becomes a ratio in the denominator becomes non-dimensional. This entire function has no units, that is our definition and we are calculating this at the mean or the nominal value of both the x parameter as well as the result.

And if you rearrange it that becomes X i bar upon R bar and the differential comes over there which is like before and we can immediately recognize that this is the way we defined theta i. So, we have the relation that theta i prime the non dimensional sensitivity coefficient is X i bar upon R bar into theta i.

So, that is what we get and that is our relation that we have been looking for this definition. We could rearrange the terms and make it this way that theta i upon R bar is equal to theta i prime upon x bar that this yet another way of writing this. So, this is a another nice thing to have the uncertainty multiplication factor and we will see later on what it is implications are in our calculations.

So, if you have to calculate the value of the uncertainty multiplication factor one way to do it is we calculate the value of theta i. So, for that you differentiate R get dR by dX i differentiate the function respect to that variable, substitute the mean values and this will give you theta i.

We already have the mean value of every parameter in this case you only need X i bar for theta i, the mean value of the result we have already calculated substituted in this expression and we have the answer the numerical value. There is no restriction on the value of theta i, theta i prime this could be any number which is a positive number.

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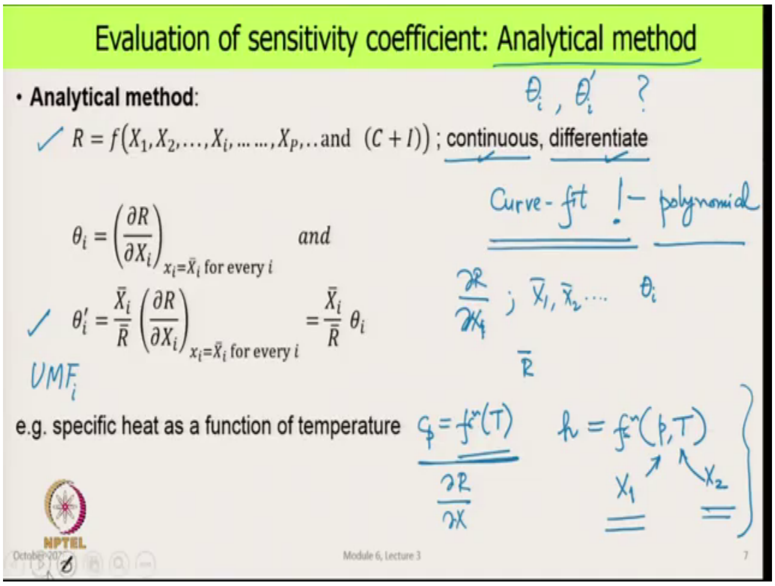

Now, we will look at the techniques by which we can calculate the value of theta i or theta prime. And here we have two methods first we are looking at the Analytical method, where we say that I want to calculate theta i and i want to calculate theta i prime, what is the technique for doing that?

The analytical method is something we just covered in some sense we write the result formula and we assume that this is continuous and differentiable at every point. A word of caution there though that if the result formula came from a curve-fit then we should be very careful in differentiating it, especially when this is a polynomial. In many packages where you make plots it gives you the option of doing a polynomial fit and if you do that polynomial fit.

And think that you got a nice continuous function, which is differentiable we could be in for surprises, there is an being done you can go back to mathematics and see that, that polynomials while they will pass through every point or most of the points between the points they could go haywire, their values could be off from what we think the trend is.

So, you have to be very cautious in differentiating a polynomial curve-fit. And we do that same thing what we mentioned a bit a few minutes back theta i prime is the differential. So, differentiated first we get a function we get the function dR by d X i say dX 1 substitute all the values of X 1 bar X 2 bar all of them and you will get the value of theta i.

And then in addition to this you get R bar and then use the second for line here and get the value of the non-dimensional sensitivity coefficient, so this is X i bar upon R theta i. An example of where we could use this if it is a one function formula a specific heat of a material usually C p this is a function of temperature.

So, this would generally be a polynomial with 3, 4 terms in it and if we want to calculate the sensitivity coefficient of specific heat relative to temperature, we could go through this technique and do that. It could be possible that something say for example specific enthalpy this could be a function say of pressure and temperature in the say superheated vapour or for gas.

In this case, our p becomes X 1, T becomes X 2 and if we have a clear expression over here we can then differentiate it and do the calculations as explained here. So, quite clearly what it looks like well for one parameter fun result, all we have is dR by dX or dX 1 the fairly straightforward simple operation.

The moment there are two parameters you got to evaluate two partial differentials, if there are three becomes three partial differentials there could be a result formula with 4 or 5, 6 functions to have to patiently sit and do 1 month 1 at a time or these days there are enough programs where you just input the function and tell what it will do it does everything for you, in some sense life has become very easy in that sense.

So, this is the analytical method for calculating the sensitivity coefficient as well as the non-dimensional sensitivity coefficient or UMF i.

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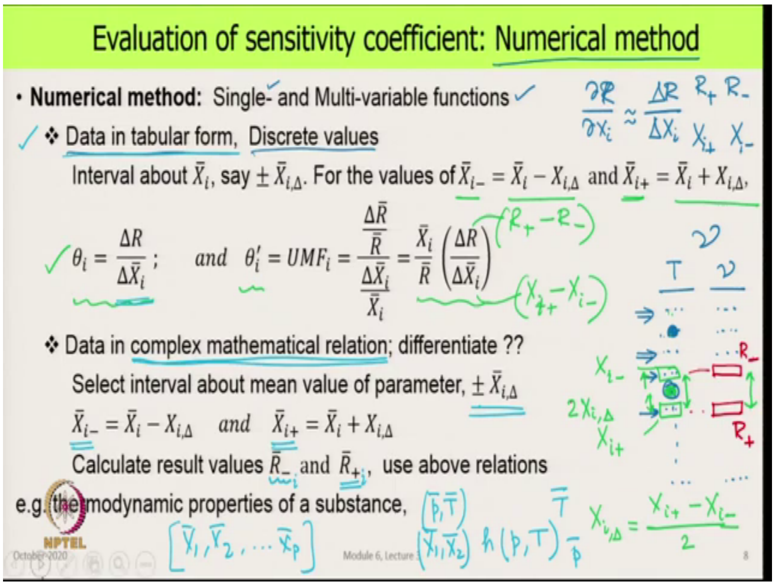

Now, you got to the other technique, where we do not have a continuous function or we have a continuous function which is very complicated so, 60, 70 constants in it. And now we resort to a numerical method for estimating the value of the sensitivity coefficient. So, this is for single or multifunction multivariable functions, in the first case we are only looking at the derivative with respect to one function.

In the second case when we are looking at derivative with respect to one function, the value of all the other functions has to be kept the same. We have 2 possibilities; in this case there are instances where the data that we are using is not as a continuous function or an analytical function, but is given in tabular form. That means, it the data is given at discrete values.

An example of this could be say kinematic viscosity of a fluid, where there is temperature dependence and there is viscosity and in the table we will say for this temperature this is the viscosity, for this temperature this is the viscosity, for this temperature this is the viscosity and like that you would have a long table. And now you want to calculate the sensitivity coefficient at some point some temperature over there.

So, what happens? We have values at this point and we have value at this point. If our point happens to be one of the temperatures listed in the table, then again we have to look at this point and the next point at least these two and then do a numerical technique. Where so be approximate the differential, which was our dR by dX i by delta R by delta X i. And what one does is that R would be an upper value, which comes from the reading which will denote as R plus and the lower value will be R minus.

Similarly, we differentiate this and call it X plus and X i plus and X i minus. So, what we are doing is in this case we say, I will work with this value and this value. So, this is my X i minus this is X i plus and corresponding to this we have this value of the result nu this value here. So, this becomes R plus this is R minus, this is R minus and this value becomes R plus.

And this interval between this we say is 2X i comma delta. So, you can say that the interval half width this is X i plus minus X i minus by 2 and if our point this point is in the middle of this interval or in this case middle of this interval, then X i minus becomes X i minus X i delta X i plus is X i plus delta that interval. So, we are adding half to get this value we are subtracting half to get the lower value.

So, that is how we got these numbers and we are put the bar on top of this, because these are actually numbers in some sense, which is the result that we are working with the result value, the variable value the result value. So, this partial differential, which we are looking at this has now become instead of a difference it is a differential it is now difference ratio. And similarly the non-dimensional sensitivity coefficient or the uncertainty multiplication factor this becomes this value.

So, from reading the table we get the values that we want delta R is R plus minus R minus and in the denominator we have this, which is X plus X i plus minus X i minus. These values are known all these four values are known, from this table we put that value into that calculate delta R calculate delta X i bar mean values of the parameter that we are interested in.

And the result mean value are known, we can put it in here and get this or for the sensitivity coefficient we just take this difference in values divided by this difference in values that is theta i.

So, that gives us all the numbers that we had and we got a value of the sensitivity coefficient from a table which is a discrete set of numbers. If there are more than one functions for example the case of enthalpy being given in a table as a function of pressure and temperature we would have to do the same thing.

While calculating the partial for pressure we have to hold temperature constant, which will be our T bar and while calculating the sensitivity coefficient for temperature. We have to keep pressure constant which is the value at which we are calculating. So, that point at which we want the sensitivity coefficient would be P bar T bar, which is basically X 1 bar and X 2 bar.

So, this is one case data in tabular form. Now, we will look at a second case where we have a function, but it is a very complex mathematical relation. We have 2 options we can have all the patients in the world and time and sit down and actually differentiate it and do exactly what the previous slide showed, perfectly in order.

If it is too complicated and we cannot do that we have yet another option within that, that you go to the web or go get a mathematical program, which automatically differentiates a function. So, you specify the function and say I want differential of this relative to this, it will immediately give you the formula and you say calculate the value at this particular mean values it will do that calculation also.

So, you do not have to go and spend time actually differentiating that function and calculating anything. We have the option of going the numerical way where we say we have the mean point, which is your all your X 1 bar X 2 bar all the, this is our mean value point. About that point one by one we select an interval in one variable keeping the others constant, and then calculate the lower limit calculate an upper value.

For X i bar x bar i minus we get from that function R minus to we are just calculating something in the function we have not done any differentiation. Similarly, you take the value of this put it in the function and get R bar plus and both of these are for the variable i X i and then go back to this expression substitute these values in this formula and we get the answer.

So, that is the second way, second case where a complex mathematical function is too difficult to differentiate we can make a numerical approximation and get the answer. So, that is how we get the value of the sensitivity coefficient and in actually doing uncertainty analysis we want numbers finally. So, this is how we will get this number.

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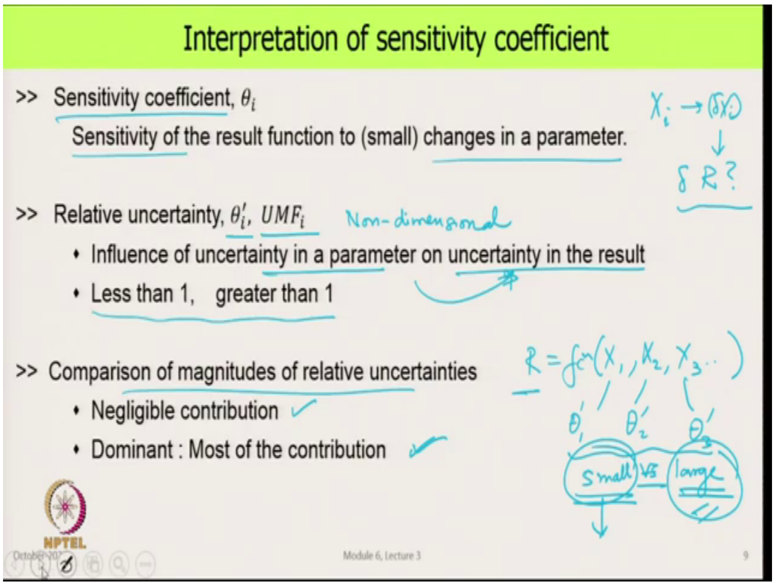

So, now let us spend a few minutes think you know what is the Interpretation of the sensitivity coefficient; we will go into little more detail on this when we look in a later lecture. We look at case studies we look at special cases we look at actual applications where numbers come in that will tell us lot more about what this one number tells us.

Here we will quickly go over some of the major interpretations, first is that as the name suggests it is a function which tells you how sensitive the result function is to small changes in that in a parameter. So, if the parameter changes by a small amount X i changes by a small amount what is the change in the result that it produces there we can call it delta.

And this change as you look at it as a statistical parameter finally tells you that we are basically looking at some statistics. But, this is the very quick definition as the name suggests that how sensitive is a function to a particular result, to a particular parameter.

Then relative uncertainty the symbol is theta i prime or uncertainty multiplication factor UMF i. Now, this remember is non-dimensional, so it tells you the influence of the uncertainty in a parameter on uncertainty in the result and we will see that in a minute when we write the full expression.

That now because this has become a ratio it is a much more direct indicator of influence on the result uncertainty than theta i was, theta i was dimensional and sometimes to be difficult to interpret. But qualitatively that is what it is doing UMF i being a ratio this is generally could be either less than 1 or greater than 1, 1 implying that as much uncertainty was there in a measurement it has not equal sort of a contribution to the uncertainty in the result the same gets carried over.

The third thing that we have written here is that the result formula has many variables, say let us say X 1, X 2, X 3 each one of these we can get theta 1 prime theta 2 prime and theta 3 prime. And these numbers tell us just by looking at them at some numbers are very small some numbers are large, relatively large.

The numbers that is large, which tells us that it is having a greater contribution to result uncertainty, the one which is small is not having that much contribution to the result uncertainty. And if this large one is much much larger than the small one then it also tells us that most of the uncertainty in the result came from the large ones very little came from the small ones.

And if you wanted to simplify the analysis this particular variable whose uncertainty contribution was very small relative to the others could even be neglected. So, that is our first point here and the same time the one which is very large this is dominating the uncertainty of the result and most of the uncertainty in the result comes from that one. So, back to some quick insight into what the sensitive coefficient is, we will come back to it later when we look up examples.

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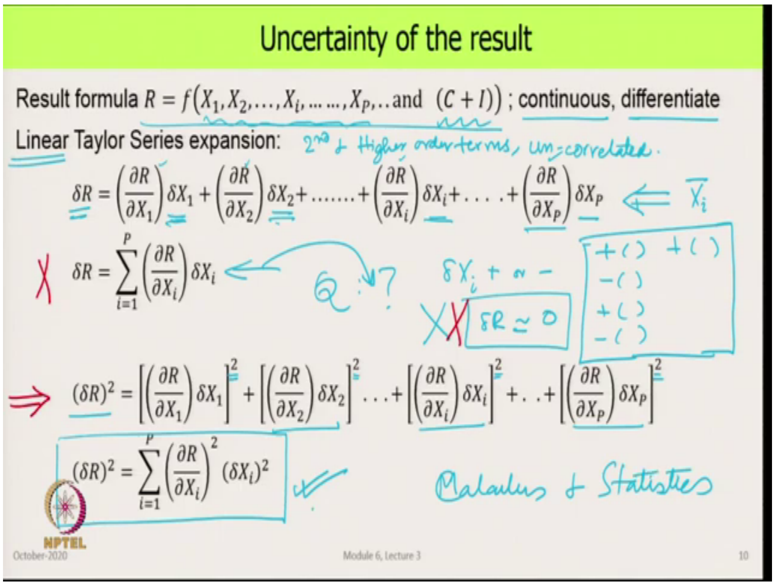

So, now let us see how does this definition of sensitivity and coefficient help us in the uncertainty formula, but before we go there we do one more thing we will come back to our Taylor series expansion formula from a slightly different set of arguments. So, here is how it goes this result is a function of various parameters, which is here and a bunch of constants which are there and we have said that this is a continuous function which is differentiable.

And if we do a linear Taylor series expansion, that means we neglect second and higher order terms and we say that they are uncorrelated measurements. Then the Taylor series expansion says that delta R is dR by dX 1 times the change in dX 1, then same thing for X 2 X i and X p, which can be written as a summation which is shown over here.

Now, the question is why not use this formula itself, after all delta we will think of as an indicator of the uncertainty in the result why not just do this calculation. And here is something that could happen that if we do this calculation delta X i these are numerical values could be as a plus or minus.

So, different parameters some would have plus some error some would have minus some error another would have again plus somebody would have minus there could be another, which is a plus there and so this is the errors that we got in every individual parameter. And if we apply this expression here as it is, it is possible and then we evaluate each one of these at the mean points X i bars.

So, we got all these numbers we put these numbers from these numbers and because some are plus some are minus it could be possible that delta R is very close to 0. So, we have a real strange situation where every measurement is in error, but there is no error in the result. So, that is completely unrealistic.

So, this is something we cannot do and so using this expression to calculate uncertainty in a result is not done. But, what one does he say that since the errors were completely coming in a random manner it is much more realistic to square them and add them up. So, this is what we have here, we took this first term and squared, it we took the second term squared it, like that every term. And that gives you delta R square and you can put all of that as a formula and this is the expression that is there.

And this begins to look very similar to what we got from the Taylor series method analysis of a previous lecture, where we arguments were completely mathematical and statistics based and there was no assumption over there. So, this is the correct formula to use both from these arguments of experimental calculations and also on the basis of calculus and statistics. So, that is the thing to do.

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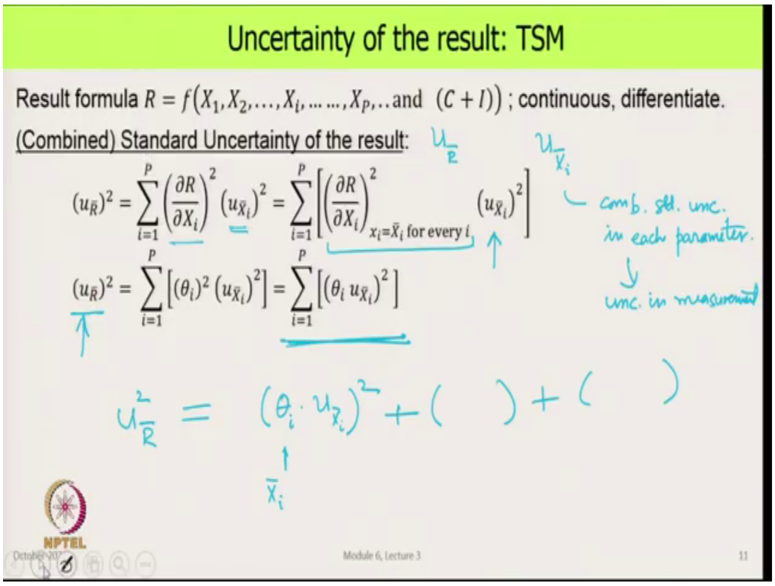

So, now we come to the final expressions what is the Taylor series method expression for uncertainty of a result and using what we had in the previous lecture and what we have just seen. We can say that the combined standard uncertainty of the result, which is u R bar we express this in terms of u X i bar the combined standard uncertainty in each parameter. And the formula is that u R bar square is summation of this square multiplied by this, which for calculation purposes this is purely a functional form. We evaluate the partial differential at the mean value and this number also came from our earlier calculations uncertainty of a measurement.

So, we are building up on what we learned in the previous module and in terms of sensitivity coefficients we can write that this is equal to here. So, what we have done is we say we take we calculate the value of theta i again at the all the X i bars multiplied by u X i bar square the whole thing add up for every term.

And this is u R bar square and this is given here in the formula relation over here. Summation i equal to 1 to p theta i u X i bar this is what our objective was, we wanted to calculate uncertainty of the result once we got the standard uncertainty we can do all the other calculations.

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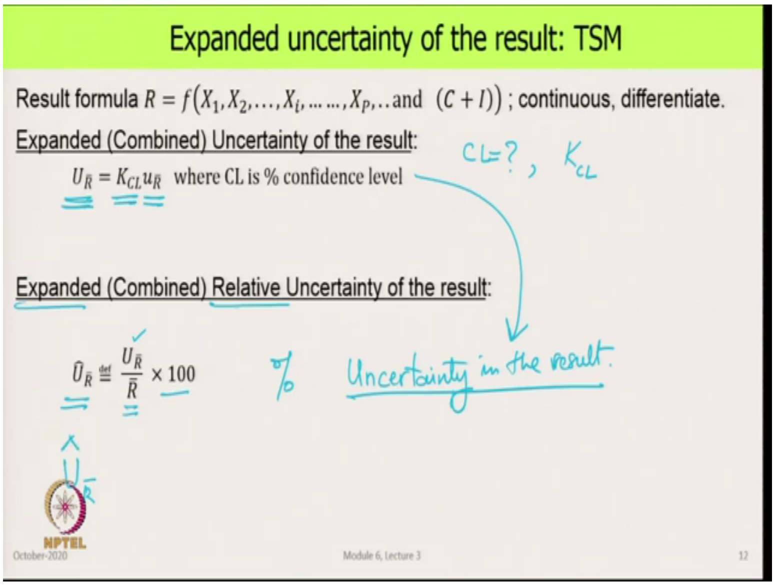

We can use that number and calculate the expanded uncertainty of the result, which is that we select a confidence level what is it that we want, based on that we calculate the multiplication factor K subscript CL. Let me CL is that what confidence level and then multiply K CL by the standard combined standard uncertainty of the result to get the expanded uncertainty of the result.

The word combined is put in brackets because the strict correct name is expanded combined uncertainty of the result, but we drop the word combine and tell us the same thing. We can then convert the expanded uncertainty into an expanded relative uncertainty when the numerator we have the expanded uncertainty divided by the mean or the nominal value of the result multiplied by 100 and this tells you the uncertainty in the result in percentage, which we will refer to everywhere as just uncertainty in the result.

So, this is a relative one this one is the absolute one. So, whenever we come across the term that uncertainty in such and such a parameter is this much, this is what is being reported. If it is got units it is u R bar, if it has no units as a percentage or a ratio it is u hat R bar, so this is u R bar with this thing. So, this gives us the expanded uncertainty of the result.

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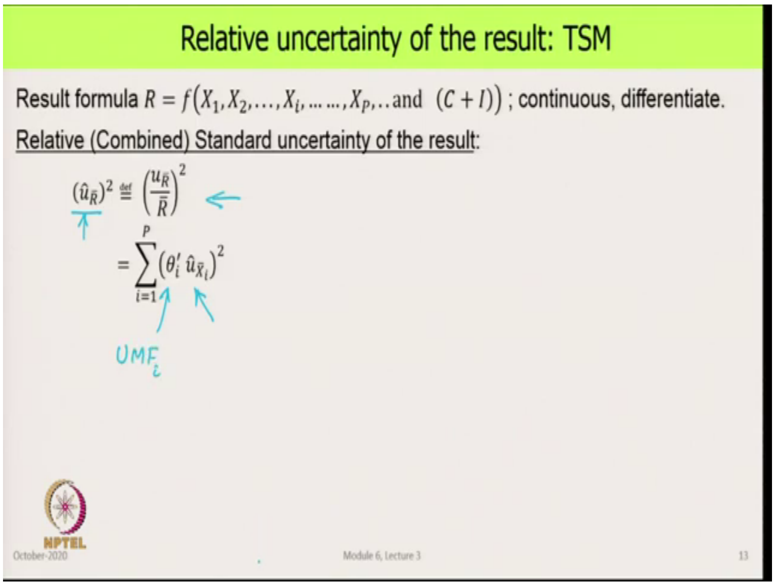

And we can also get another relation for the relative uncertainty in terms of the sensitivity coefficients you can do a little bit of algebra on the previous expressions. Look at this relative standard uncertainty of the result is defined as u R bar upon R bar square that the definition and this we can then see from an earlier result is theta i prime upon u hat X i bar.

So, the relative standard uncertainty of the result we got in terms of relative standard uncertainty of the measurement and the uncertainty multiplication factor for the corresponding variable.

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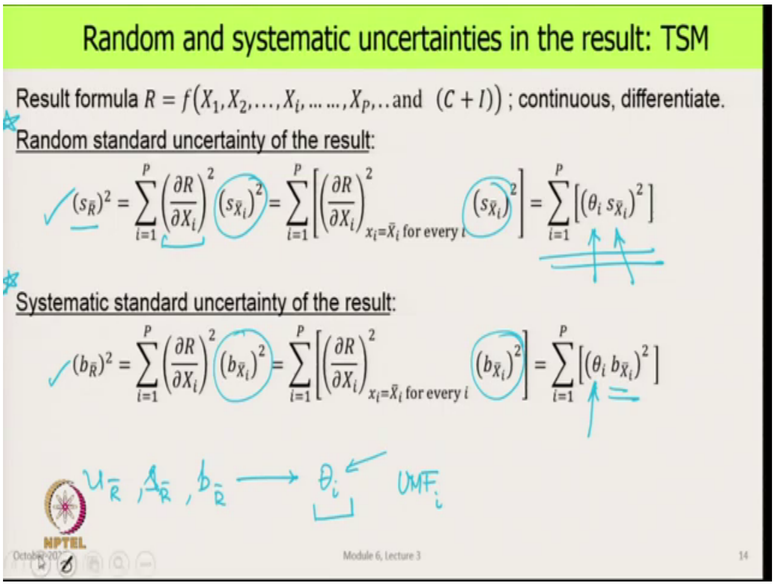

So, this is yet another expression we have and finally we asked how can I calculate the random and systematic uncertainties in the result. And using the Taylor series method we just substitute instead of the combined uncertainty we look at the random uncertainty instead of systemic uncertainty, we look at instead of the combined uncertainty we look at the systemic uncertainty in the second case.

So, s R bar square random standard uncertainty of the result is all of this is same this as change instead of u it has become s, same thing here and same thing over here. So, if we calculate the sensitivity coefficient and we had separately calculated these values we can calculate the random standard uncertainty of the result.

And the same thing we can do with the systemic standard uncertainty b R bar square this is again here we got b X i, b X bar i and here again b the sensitivity coefficient stay the same and we are able to get the systematic standard uncertainty of the result over here and the random standard uncertainty of the result over there.

So, what we saw is that whether it was u R bar, or s R bar, or b R bar in all these cases we are using the same values of the theta i. So, one calculation of this become useful for six more calculations, besides of course it as powerful insight into the propagation of uncertainty. So, that is it another benefit of having defined and then calculated the sensitivity coefficient or the uncertainty multiplication factor UMF i.

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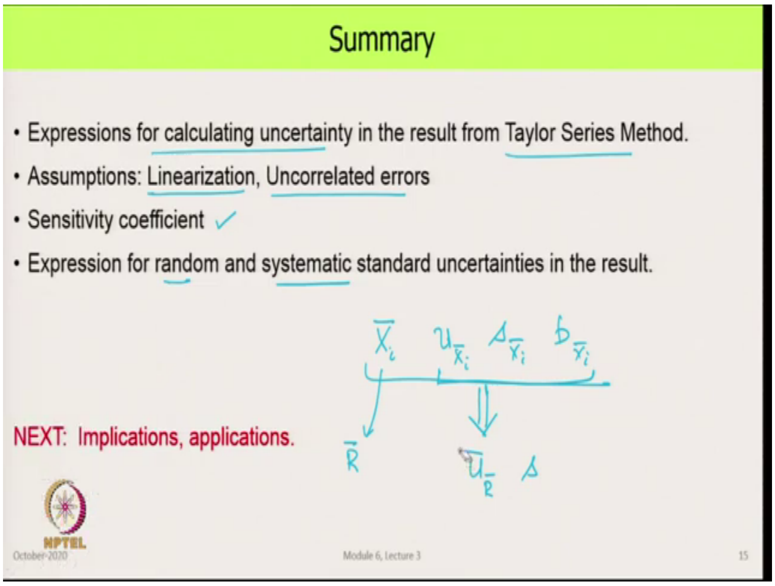

So, with that we got everything that we were wanting to do when we started this module, we got expression for calculating uncertainty in the result from Taylor series method. We also looked at the mathematical and statistical basis of the Taylor series method, the important assumptions that we made was linearization and uncorrelated errors and that errors are small in magnitude.

We came across and define a term the sensitivity coefficient, which we saw mix the mathematics looks more elegant. And finally we got expression not just for the combined standard uncertainty, but you also got expression for random and systematic standard uncertainty in the result.

So, what it tells us that in the first in previous module we got the mean values of every measurement and the u the combined uncertainty standard uncertainty in a measurement s X i bar we got and b X i bar we got. Then we saw what is that function, which we just saw which is that summation equation which converts this and without that expression we directly get this, but from this and all of this to get u bar R sorry.

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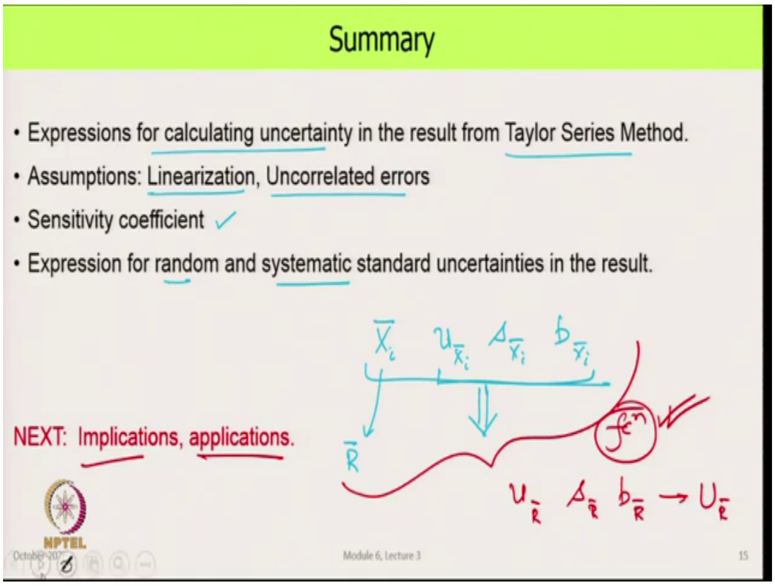

From all of these we saw how we can calculate u R bar combined standard uncertainty in the result so s R bar and b R bar. And then of course from there onwards you can calculate the expanded uncertainty of the result. This is formula is what we were looking for to convert all of this to this in an earlier slide.

We have seen what this formula is and so we have all the mathematical tools required to calculate the uncertainty in a result. In the next lecture, we will look at implications and applications of this and see how this helps us in doing what we have set to do.

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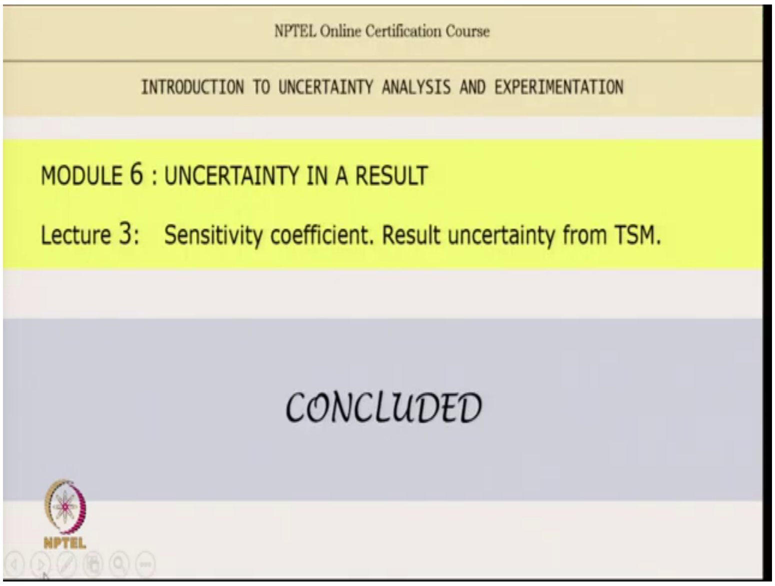

On that note we will conclude this lecture.

Thank you.