Introduction to Uncertainty Analysis and Experimentation
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Module - 06
Uncertainty in a Result
Lecture - 18
Single test. Basics of Taylor Series Method

Welcome to the course Introduction to Uncertainty Analysis and Experimentation. In this lecture, we are studying Uncertainty in a Result and we shall learn about the Single test methods; these require the Taylor series method and we will briefly go over the Basics of the Taylor Series Method.

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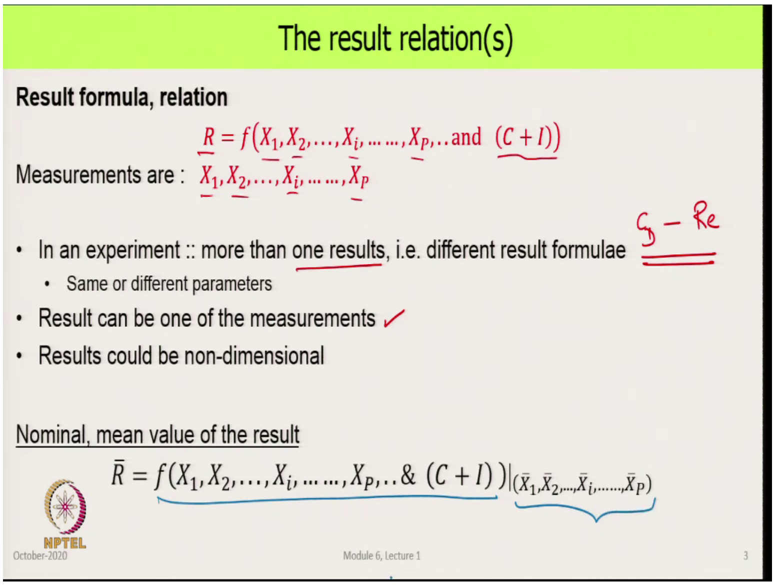

So, what we have in an experiment, a result formula relation one or more depending on the type of the experiment; R is the result which is the function of the various parameters and of course, some constants.

And in the experiments we measure the different parameters X 1, X 2, X i, X p. Now, in this experiment, we could have more than one results; the example as stated in the earlier lecture was drag coefficient and Reynolds number or result could be one of the measurement itself or the results could be non-dimensional as we saw in this case. And the nominal value of the result is a straightforward calculation using the result formula evaluated at the mean values of every parameter.

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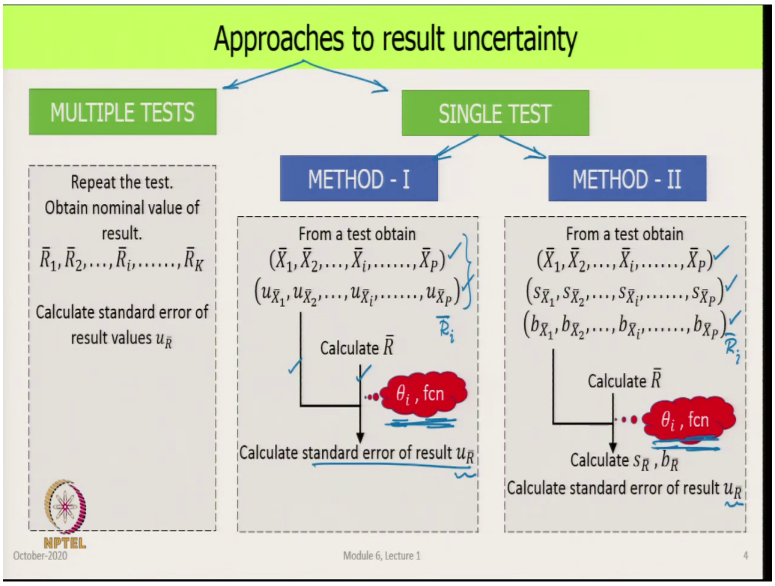

And we said in the previous lecture that, we have two approaches, two result uncertainty; multiple test we looked at it in the previous lecture. Now, we will look at single test and look at these two methods for doing the calculation.

So, before we get into the details, just to mention that what we have written here is that; the unknown here in both these methods is what is being written here a theta i or a function. This we have from the experiment, the mean values of every measurement; this we can calculate from the data set for each measurement, which is the standard uncertainty in every measurement.

Same story here, we can calculate the random standard uncertainty from the measurements and the systematic standard uncertainty is there, all of these comes from the data that we collect from the experiment. We have the result relations with us; but what we do not have is a result of formula or a relation or a mathematical expression by which we can take this and we can take the result formula and calculate U R bar.

So, this is the main idea in going into this lecture that, in the single test, we need a function that connects the measured parameters which are all of these and we could even include in this set the mean value of the result from every experiment R i. Put these numbers into some mathematical relation or a formula and it should tell us, what is the standard error of the result or the standard uncertainty of the result. Now, for doing that, we have two methods which have fundamentally, conceptually very different.

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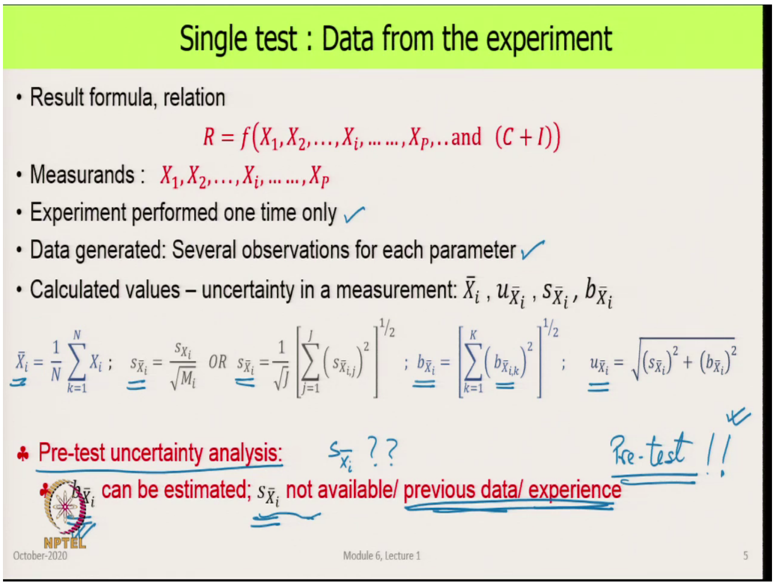

So, before that, we look at what calculations we are able to do and do we have all the formula for that and this is that the experiment we have done once; we got several observations for each parameter and we have this formula, the mean value of the result, the random standard uncertainty in the measurement sorry.

Or we can use this or we can use the elemental method of getting the random standard uncertainty and the random, the systematic standard uncertainty in the result based on the elemental systematic standard uncertainties. And finally, we combine all of these and get this; all these formulas are now known to us.

The thing is that in post-test uncertainty analysis, we have all the data to do all these calculations. In pre-test uncertainty analysis, we do not have the data; so we have a problem, we cannot calculate s X i bar. The random standard uncertainty is in doubt, either we estimate it based on our experience or if we have some idea of previous data, we can adopt it from there; it is after all a pre-test uncertainty analysis stage, where we are using these as nominal values.

But the systematic standard uncertainty, we have enough information and data to calculate that. So, this is a technique, the single test technique is reasonably ok for the pre-test stage, except for this one little item. So, in the whole experimentation process, when we talk of pre-test uncertainty analysis, this is what we would do.

And we take care that, the multiple test option is not available to us; single test is what we have to look at, the random standard uncertainty is not there, this is very much there with us. We can estimate it from previous data or experience and complete our pre-test uncertainty analysis. In post-tests uncertainty analysis, everything is there with us, no issues.

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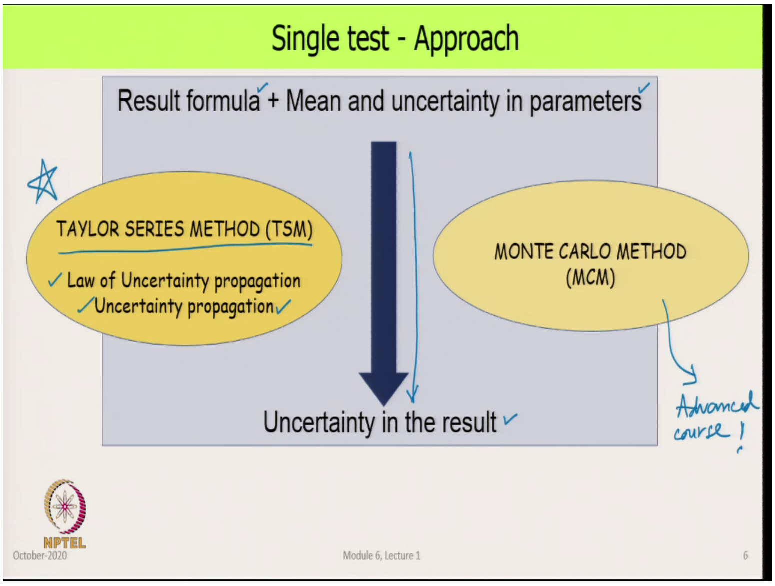

So, let us see, what is the approaches we have. We have the result formula and the mean and uncertainty in every parameter; we are starting with this and our objective is to calculate uncertainty in the result, the mean value of the result is also known. So, we have to come from here to here and we have two broad possibilities; the first is the Taylor series method, TSM.

In this course, we will learn about this; it is also known as uncertainty propagation. And in some texts, you will see this as referred to by the name law of uncertainty propagation. We will just call it TSM; but we will use, well if required uncertainty propagation and not the law of uncertainty propagation.

The second approach that we have is what is called the Monte Carlo Method or the MCM; this is a little more advanced technique. So, we do not do this in this beginning course, this is left for an advanced course in an uncertainty analysis. It is a fairly powerful method and very quickly we can generate data that is of quite value to us. In this course, we will learn only the Taylor series method.

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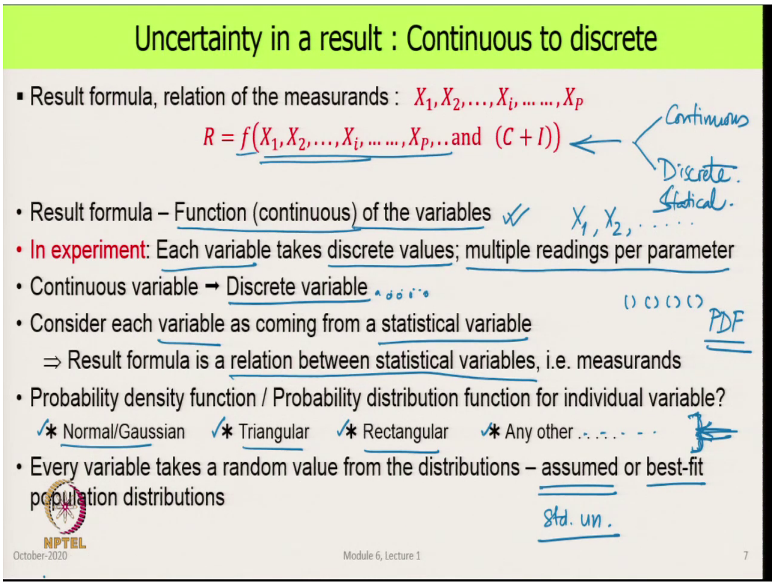

So, here are some of the issues that come up in doing this analysis. When we say we have a result formula f of this, we have basically said that it is a continuous function of every variable; that means its derivatives are define. There are some instances that you have come across, where that may not be the case.

For example, in thermodynamics; when you are looking at properties of a wet state versus super-heated state, there is a discontinuity in the properties at that point; but the rest of the domain those are continues properties. Mostly in most experiments, this is a reasonably good assumption that remains satisfied and we can proceed with this.

So, we have a continuous function; but what we have in an experiment is that, each of these variables all are X 1s, X 2s and all these, which in a result calculation you could have fitted as a continuous line by giving infinitely small changes and making it look like a continuous function.

In the experiment, we take very specific discrete values and we take multiple readings for every parameter. So, for instance if you are seeing the deflection of a spring by putting weights; we are not putting incremental width that every time to get the exact value of the function, but we have putting discrete amounts of weight and seeing what the deflection is.

So, the continuous variable and the continuous function that we had, the variable has now become discrete. So, what we can look at it another way is that, here are all these values of that variable which are coming and one thing I could do is plot a probability distribution function, probability density function. It may not be the exact function, but we could possibly fit something that we are comfortable with and make it into a statistically manageable PDF.

So, what we are saying that, if you look at it backwards again, is that this discrete variable that we had here, which is the value of every parameter; for that parameter, these were variables coming from a statistical variable. So, now, the result formula is become a relation between statistical variables; the X 1s, X 2s, X i's are no longer continuous.

But they have discrete values which are coming from some type of a probability density function or a probability distribution. And the most common probability distribution function we have is they are listed here. The first is the normal or the Gaussian distribution; we could have a triangular distribution, rectangular distribution or many many other variants of any distribution function.

What it tells us or what we are saying in some sense is, what is that population from which these errors are creeping into our experiment; is there a particular probability distributions because of in that, is it that every error has equal probability or is it that certain errors have a high probability than other errors, that is what is the application of these.

So, what we could do is take of this data and if you wanted, we can make a best fit population distribution function on it or we could assume as we do in many of the analysis. We assume it to be coming from one of these distribution functions and then once we have done that it tells you that, the standard deviation can be calculated like this, the standard error can be calculated like this.

And doing that, that assumed distribution helps us in getting what we want which is the standard uncertainty in whatever we are measuring or in the result. So, this is a big change that, we start looking at the result formula; not as a continuous formula of continuous variables, but of discrete values, discrete and we can say statistical variables.

So, it is like saying that, a formula could be that I have some normal function which represents something; in the result formula, I am adding another normal function to it and calculating my result. So, this is what we are looking at.

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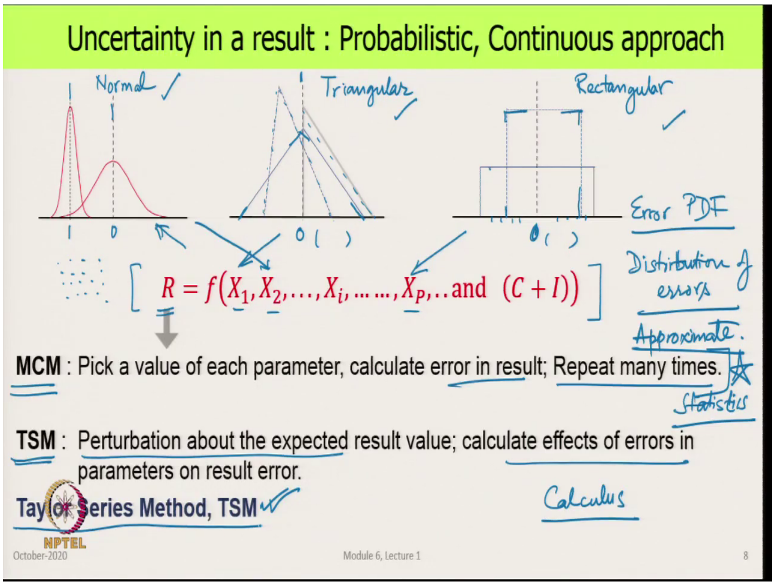

So, here is a graphical representation of what we have just said; we have the result relation over here and here some probability distribution function have been shown. This is the normal or the Gaussian distribution function. So, there is a mean value, across which the distribution is symmetric and it could have different shapes.

If the distribution function has been normalized, this mean would always have a 0 value; if it is not the case, then this would all be absolute values and this would be the mean finite value of that measurand. So, any of these X 1, X 2 or X p could be following say this distribution function.

The second depiction is our triangular distribution function. The simplest one is that, this is the mean value and about that we have this one here, goes up there and comes down over there, this is symmetric. But that not need not be the case, it could be the case where it is axisymmetric like it has been shown here; with this dotted line, it goes up there and from there, it is coming down there.

Or it could be completely one sided that, all the errors are lying on one side with certain decreasing probability. And again as in the case of the normal distribution function, if you have normalized it; this would have a 0 value; otherwise it will have finite values. The third distribution function shown here is the rectangular distribution function.

And this is the mean and the easiest thing would be that, it is symmetric about the mean. What it tells us is that, no matter what the value of this variable; it has equal probability of showing up in the population. Again as before, this could be 0 if it is normalised or it could have a value. And this need not be symmetric as has been shown by this dotted line, where you have this, this curve, this rectangle over here, where the probability of certain values is more than the others compared to the mean value.

So, what we have is that, you could have a situation where X 1 is following triangular distribution function, X 2 is coming from normal distribution function, X p is coming from the rectangular distribution function, that is one case or it could be that all of them are coming from the normal distribution function or that we assume that some are coming from normal, some from triangular or we say all of them are coming from rectangular, any of these possibilities are there.

And it is only through lot of data and very careful experimentation of a very different type, that we can establish the shape of these functions which is essentially distribution of the errors. So, these are the error probability density functions. Very rarely we have the enough information about this; in other cases we get some experience, some data and make our own judgement as to which best fits it and remember this will all be always approximate.

Yet it is not meaningless in that there, just because they are approximate; they are reasonably good and the result at the end has to be interpreted in light of these things. So, the two methods that we have first is the Monte Carlo method, the MCM and what it tells us is that, first pick up the probability density function for each variable; then out of that function, pick out one number at random, that is the mean value, so that is the value and use that value to calculate the result.

Now, do this many many times; this means that, it is not just once, 2, 5 or 10 times that you do, you literally have to do 100, just not 1000s, just not tens of thousands of times. You have to do it as many times that final you say that, look I have got so many data that have come about this variable.

If I plot this, I get the assumed variable from which I was drawing this numbers; then we can say that fine, we have reached that point and that has to be done for every single parameter. So, this is what one does with a computer; you write the program and then it can do all of these things.

This as I mentioned is the topic of advanced courses; we will not study about MCM technique in this course. In the Taylor series method, which is what we will learn, this is what is there. We write the result formula as a continuous equation and say that, if the individual parameters are perturbed by a small amount; how can I calculate the perturbation that comes out in the calculated result?

So, that is what we are doing here. And for this, we need calculus. In this we needed probability and statistics; two approaches, both are very powerful and both are fully usable in any application. So, we will stick in this course with the Taylor Series Method or the TSM.

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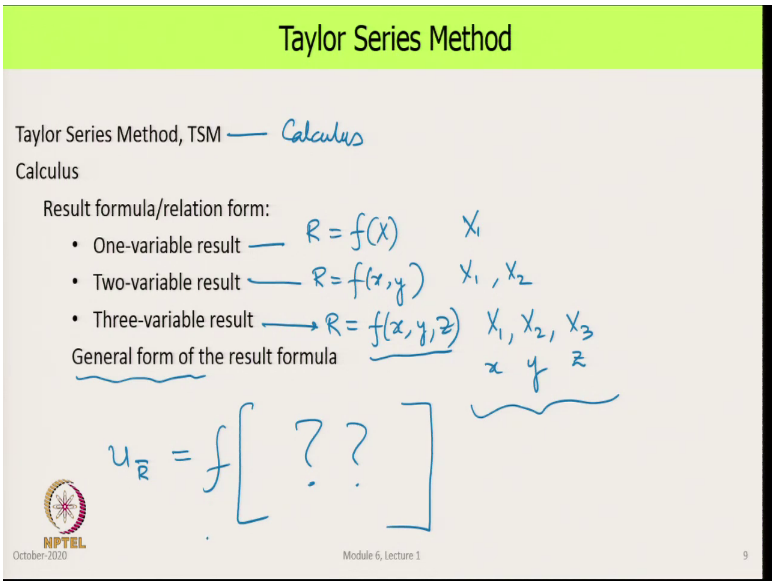

In what we are going to look at now, we will first look at the basic calculus of the Taylor series method. In light of different types of result formula; first we will look at where the result is a function of only one variable; then a two variable result, result is a function of say x and y; a three variable expression, result is a expression of x, y and z.

Now, this could be our X 1, this could be X 1 comma X 2 in the terminology that we are using in this course; this would be X 1, X 2, X 3. So, this would have been x, this is would be y, this would have been z. In calculus books you will find these type of symbols; I have used these symbols in denoting things in the beginning to explain the idea and then later on we will convert that formula into our notations that we have been using in the course.

And after doing this for one, two and three variables, we will generalize this and say, for any number of variables in the result formula; what is the expression that I have which connects standard uncertainty in the result to a function of various things on this side. This is our objective. So, we are in search of this function. Let us start with the basics of the Taylor series method in one variable.

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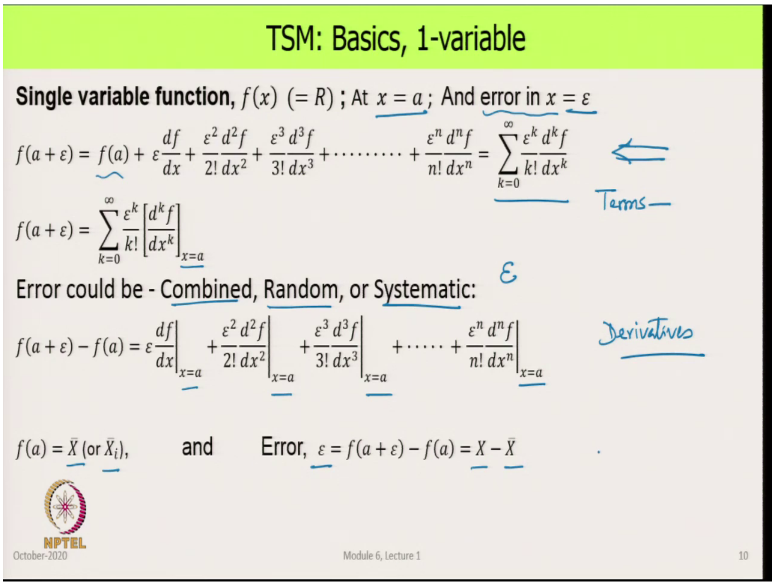

This is something you probably learnt in school, if not in a beginning mathematics course; that is why math's is important. And here I have written the Taylor series expansion as you would see; instead of x, I put it at x equal to a particular value of the variable and the error in x is epsilon, a small perturbation that is there.

So, we say that f a plus epsilon is f a value of the function at a plus epsilon times df by dx plus epsilon square by 2 factorial the second derivative of x and the 3rd derivative and like that nth derivative. And all of this can be written as a summation epsilon to the power k divided by k factorial, the kth derivative of f with respect to x and k goes from 0 to infinity.

So, it tells you that, you can have many terms in this formula. If we are evaluating it at f at x equal to a; then we do a small modification to this formula and say it is that, this derivative is evaluated by putting x equal to a in that formula, strictly that is what it is. These errors epsilon that we have taken, this could be either the combined error or the random error or the systematic error. The name of the error will get replaced by that and so will be the terms on the right side.

Now, what we do is to calculate error in the result, we bring this f a to the left side and say f of a plus epsilon minus f a this is equal to all of this. And so, on the right side, we only have terms which have derivatives, all the evaluated at x equal to a. So, f a in our notation would correspond to the mean value of the variable, error would correspond to X minus X bar. So, this is very basic statistics, all of this is from school and maybe first year mathematics courses.

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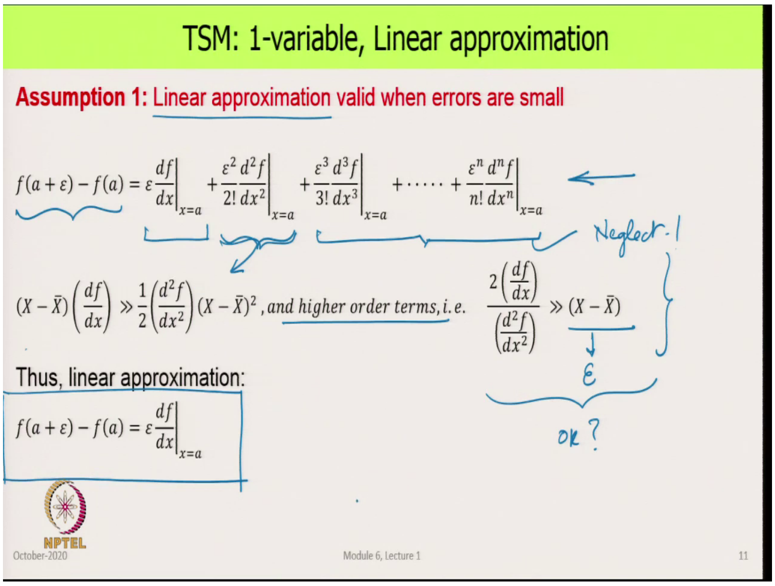

The first assumption that we make in a series of assumptions that we will make is the linear approximation. We come across many engineering applications where we do this approximation and to understand what that is, we looked at that expression that we wrote on the previous slide; that the difference between the function at a plus epsilon and that a is a combination like that. And the linear approximation says that, second and higher order terms are negligible compared to the first order term.

So, we say we will say that these are all very small and for this, we come up with a criteria; that this is very small in comparison to this, so we can write that and write that epsilon as X minus X bar. And when we do that, we get the criteria that 2 df by dx over the second derivative of f is very much greater than X minus X bar, remember X minus X bar is epsilon.

So, if you have to check, this is good enough to check whether the linear approximation is ok or not. In general, most of the applications will be looking at; the third and higher order terms will be definitely much smaller and so, we would be quite ok in saying that these can be neglected.

The second order term can be neglected, if this criteria is satisfied; in which case if that is also neglected, we are left with the simple expression for a linear approximation which is given here that, the value of the function at a plus epsilon minus its value at a is epsilon times df by dx evaluated at x equal to a, ok; something that you learnt in school or in a first year math's course. Now, we come to the issue of uncertainty for this one variable function.

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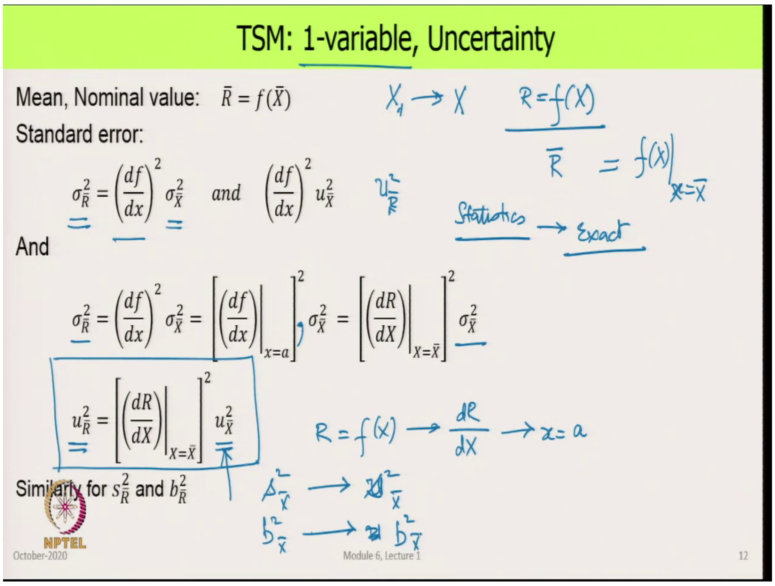

The only parameter we have is X, it could have been called X 1; we have replaced this by X. And our result is R, so R is a function of X; this is our result formula and the mean value of the result is calculated by substituting in this formula X is equal to X bar and this gives us R bar.

And from the previous relation, we can do the statistics on it and I have skipped those steps and given those things in the notes that, rigorously we get from statistics that sigma R bar square is df by dx square plus sigma X bar square. And sigma R bar square is nothing but U R bar square.

The point is that from statistics, we get this expression as an exact expression. And from there we calculate all these parameters; I have just repeated what has been written there in terms of the symbols we have been using. So, sigma X bar square is equal to dR by dX, X equal to X bar whole square divided by sigma X square. So, that is the formula that we have been looking for.

We know the function R as a function of X, from that we can just differentiate and calculate dR by dX then substitute and get it is value at x equal to a; this gives us this term inside the square bracket, the square of that multiplied by the uncertainty in the measurement this is what is U X bar, this gives the square of the standard uncertainty in the result.

So, we got what we wanted, but this is limited to one variable. So, the result at one variable, this is what we have. And we can do the same thing that, if instead of U X bar, we had s X bar square; then this will give us U X bar square. And if we use b X bar square, this is systematic standard uncertainty; this will give us sorry, this will give us s and this will give us b X bar square.

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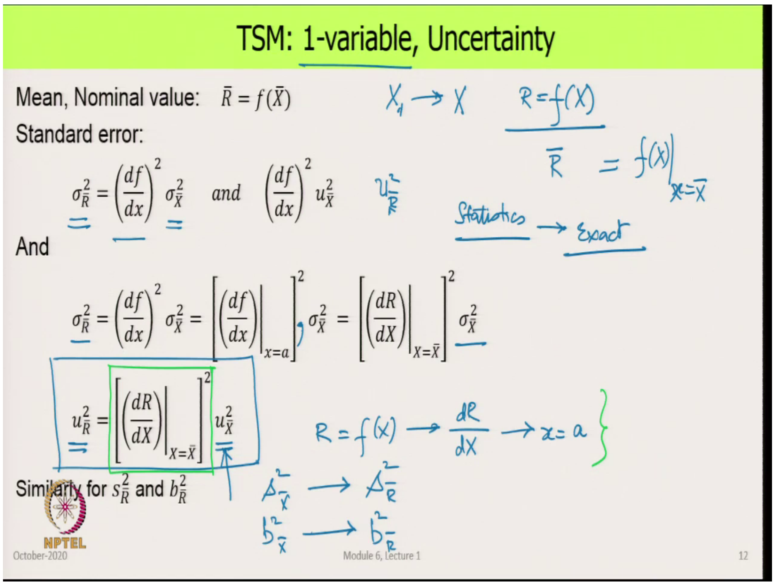

So, this will give us s R bar square, this will give us b R bar square, same formula; which means that in all these cases, this parameter that we have been multiplying with this parameter, this has a fixed value which we got from this calculation. So, that is one of the features that has come out of this discussion.

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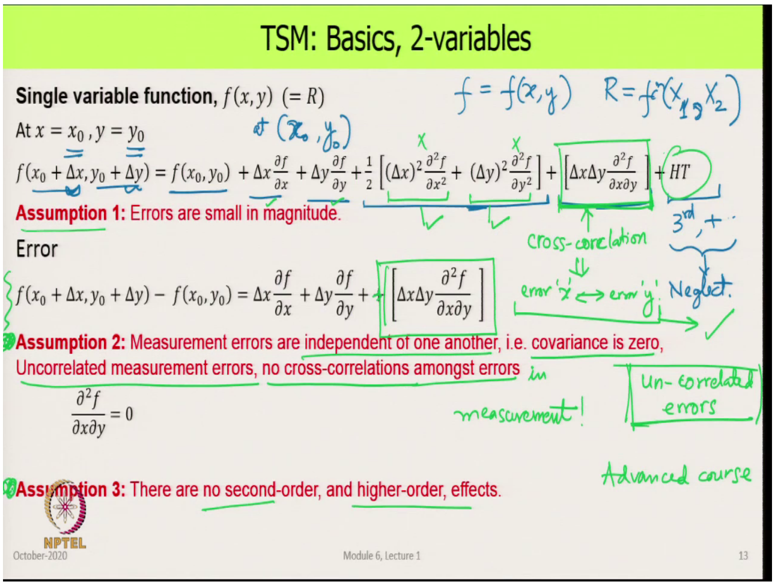

Now, we go to the case where the result expression R is a function of two variables. I started here by putting it as f is a function of x and y or in our terminology this will be R, this is a function of X 1 and X 2, that what is a two variable result. And there would be other constants, we take those for granted.

And the point about which we want the uncertainty, is the point that we designate as x 0, y 0. Now, we use the Taylor series method for two variables unlike becomes slightly more complicated. So, f at this point at this perturbation is delta x delta y, instead of just calling it epsilon; we could have called it epsilon x, epsilon y that is ok or delta x, delta y.

We are expanding this function about the given mean value f x 0, y 0 and we get the first order terms delta x df by dx delta y df by dy. Then come to the second order term and there are two of these. And then there are 3rd and higher order terms, which as I said earlier for all practical purposes are very small, so we will subsequently neglect them.

But these two terms here, this one and this one, we need to spend a little time looking at what they mean. Using the arguments of the one variable expression; we can say that if that criteria were satisfied, we would say that this term is negligible and that this term is negligible in comparison to the first order terms.

In most cases, this is reasonably ok; it is this term which has implications. In that, this loop is telling us about cross correlation; which in terms of uncertainty analysis or experimentation it tells us that, error in one variable say x is in somehow connected to the error in y that means these two errors are not completely independent of one another.

This is an important sub classification of Taylor series method and this cannot always be neglected. So, we have to take care of this. But in this course, which is a basic course; we are looking only at uncorrelated errors. And so, we will neglect this term, keeping in mind that will or it will not always be the case.

And you have to be careful in looking at the application, the experiment that you are doing, what instrument you are using, how you are calibrating them all of that and figure out what is a correlated error. And if there are correlated errors, then we cannot use the linear approximation. This course is specifically limited to uncorrelated errors; cross correlations are left for a advanced course in uncertainty analysis, this is an important topic.

So, by neglecting the higher order terms, this is the expression we got; those terms are gone, we neglected the second order terms which were these, but we have for completeness, kept here this term.

And so, the second assumption that we have just talked about, measurement errors are independent of one another; that means their covariance is zero or they are uncorrelated measurement errors or the same thing as saying that, there are no cross correlations amongst the errors, errors in the measurement.

Under that assumption, so this is assumption number two; d 2 f by dx dy, this is 0. And the third assumption, we have already said that we ignore these terms here that, there are no second order and higher order effects. So, in addition to this first assumption, which was therefore the one variable function; we now have two more expressions coming in.

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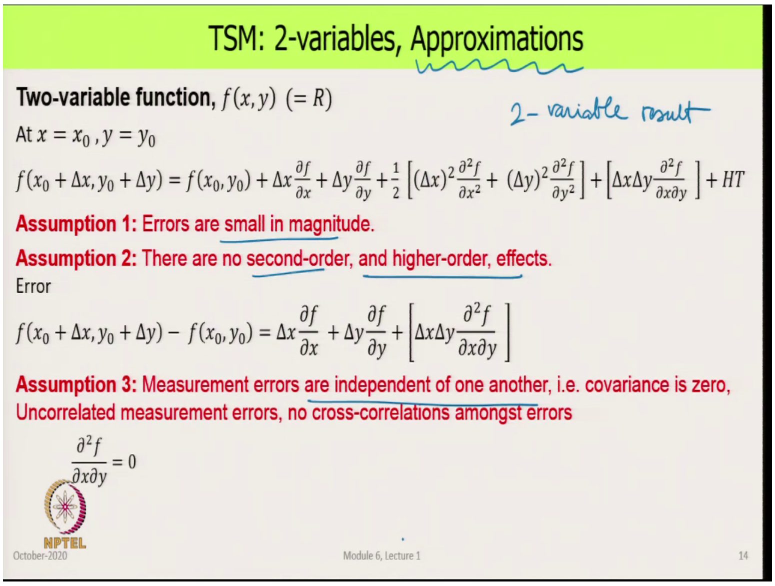

And if these assumptions are satisfied ok; so these are the nothing but the same assumptions rewritten here, errors are smaller in magnitude, no second order and higher order effects, and measurements are independent of one another. So, these are our important approximations in 2 variable result and as we will see if it is three, four or more variables, these will come back.

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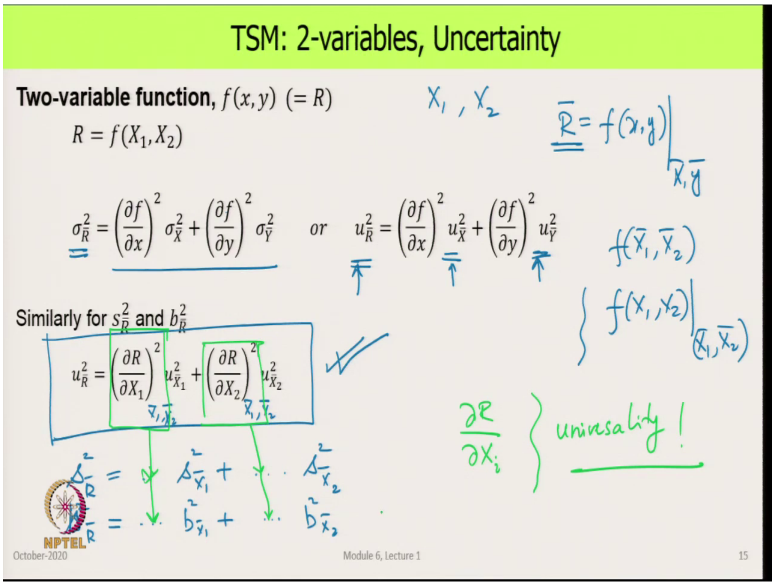

Again I am skipping all the statistics that goes into this; but we can start with the result formula say that, X 1 and X 2 are variables coming from a distribution function, do the analysis and show that R bar is a function of x, y evaluated at X bar, y bar. Or in our case, it is the function evaluated at X 1 bar, X 2 bar or if you want to write it very clearly X 1, X 2 function evaluated at X 1 bar, X 2 bar.

So, the mean value of the result comes from there and the standard deviation or standard error of the result comes from this expression, which is df by dx square plus sigma X square X bar square plus df by dy square sigma Y bar square. Or in terms of uncertainties or standard errors which is what this is; our notation is that this is U instead of sigma.

And with X 1, X 2 notation that we have been following, this is expression that we can write; U R bar square is equal to partial of results with respect to X 1 evaluated at X 1 bar, X 2 bar times U X 1 bar square uncertainty in the X 1 measurement dR by dX 2 square evaluated at X 1 bar, X 2 bar times uncertainty in X 2 bar, so standard uncertainty in X 2 bar square.

And this tells us everything in terms of parameters, numbers, values that we already have. So, this is our answer; instead of U, we could even write this as s bar square, in which case this will be a function of s X 1 bar square plus s X 2 bar square. That will give us the random standard uncertainty of the result. And we can also do is we say, b R bar square; the systematic standard uncertainty in the result, this would be b X 1 bar square plus something b X 2 bar square.

And what you will notice here is that, this thing here, this will come here and then it will come here. This thing here, this value will come here and it will come there. The point is that, if you look at del R by del X 1 or X 2 or later on X i; there seems to be something universal about this.

And that is what we will later on call as a some number and figure out what does it mean; what does it tell us about the setup, what does it tell us about the result, what does it tell us about how errors are going in. So, this sort of relation this is what we said, this is error propagation for two variables; we had an earlier an expression which are just the first part of this, which was error propagation for a single variable result.

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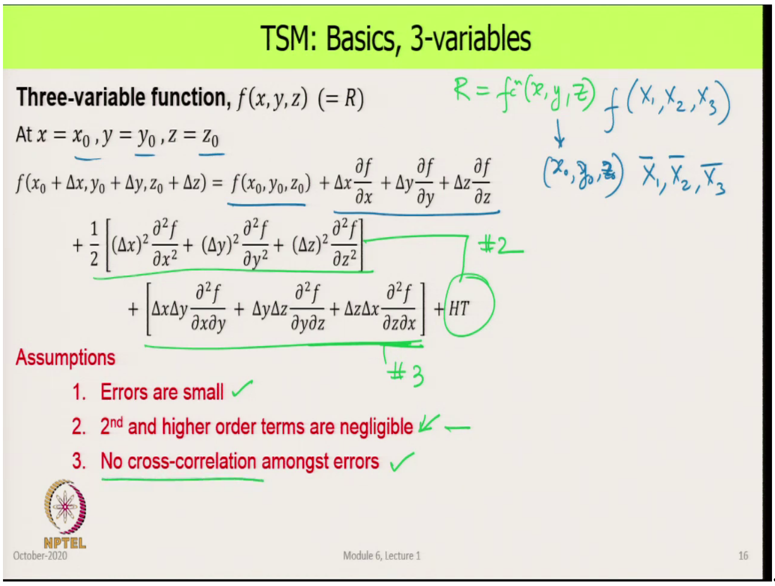

Now, we come to the three variable functions R as a function of x, y, z or in our notation this is a function of X 1, X 2, X 3. And the point about which we are evaluating it is the mean point, which here it is written as x 0 sorry x 0, y 0, z 0 here or in our notation this is that X 1 bar, X 2 bar, X 3.

And now we do Taylor series expansion for three variable expression and this is the value of the function at x 0, y 0, z 0. Then the first order terms and then two types of second order terms; this is one, this is the second one and then the higher order terms. And we use the same assumptions that we just saw for the two variable case, errors are small; 2nd and higher order terms are negligible and there is no cross correlation amongst others.

So, this does not mean that this part is negligible; this we are neglecting, because of assumption number 3. This and this we are neglecting, because of assumption number 2. So, again we are restricting ourselves that there are no cross correlations amongst the errors or the measurements.

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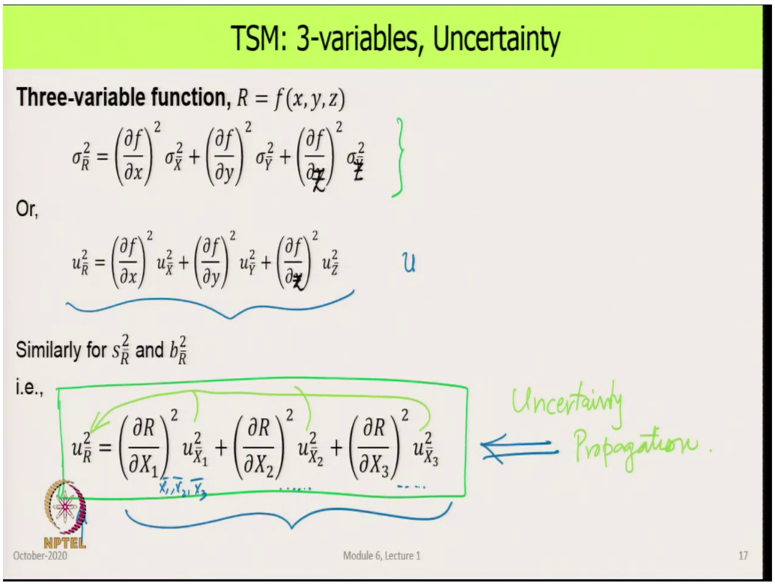

And then you go through the same statistics on it; you will get this expression sigma R bar square is df by dx square sigma X bar square plus df by dy square sigma Y bar square df by d, this should be dz sigma Z bar square or you are instead of sigma, we have been calling the standard error as U.

So, we just substitute by sigma v U and that the function that, tells us what we are looking for. And in terms of the symbols that we have been using our function is right here; U R bar square is del R by del X 1 square U X 1 bar square. Now, this is evaluated at X 1 bar, X 2 bar, X 3 bar; same thing here, same thing here.

So, this is dR by dX 2 square U X 2 bar square dR by dX 3 square U X 3 square and that is what we have been wanting; it tells us a relation between non parameters on the right side to calculate standard error in the result and this is again the propagation of uncertainty. So, what we are saying is, this is an expression when we say propagation of uncertainty; what we are saying is how these errors end up finding a way into the uncertainty in the result that is why this is called uncertainty propagation.

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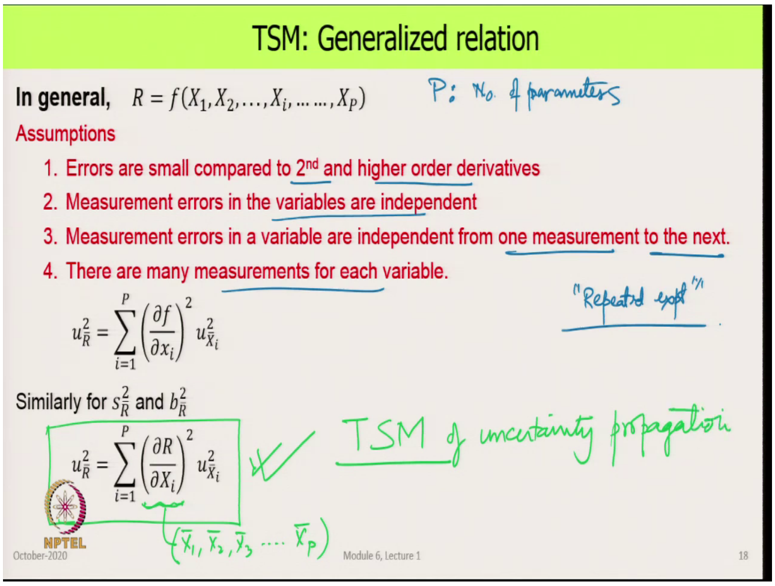

So, now we can generalize it any number of variables that we want; R the function of X 1, X 2, X P, P number of parameters. So, there are that many independent measurements we are doing in the experiment that is P.

And the assumptions under which this analysis is valid. Errors are small compared to 2nd and higher order derivatives that is number 1; number 2, measurement errors in the variables are independent, independent of one another; measurement errors in a variable are independent from one measurement to the next, which is basically saying that, we are using the same setup and not changing say an instrument in between.

So, in some way we can say that, this was the idea that we got in a repeated experiment. And fourth, there are many measurements for each variable; we do not have just one value. And the expression that we have been looking for in terms of the.

So, the expression we have been looking for in terms of our symbols that we have been using; we get U R bar square as a summation from 1 to P dR by dX i square into U X i bar square, noting that this is evaluated at X 1 bar, X 2 bar, X 3 bar X P bar. So, that is something coming in over here. This is our full form of the, you may call it the law of uncertainty propagation or just uncertainty propagation or as what we as always called this the Taylor series method of uncertainty propagation.

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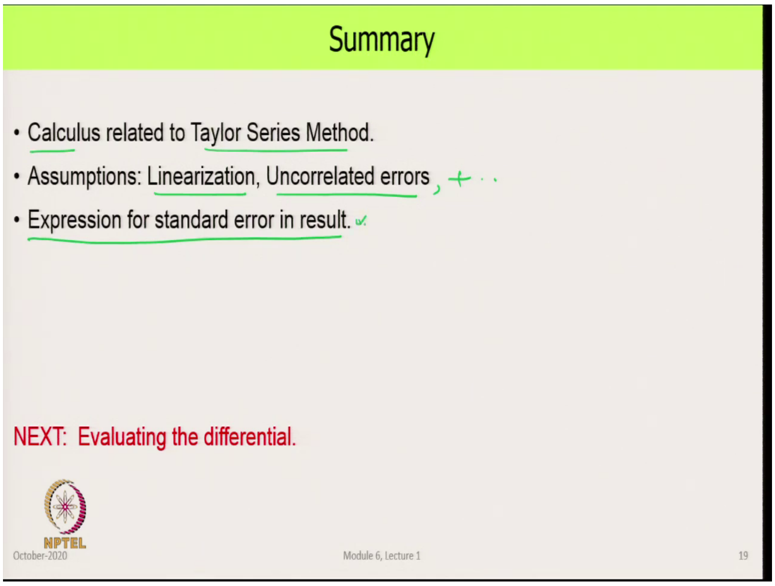

So, we got that result that we have started wanting and we said that in the single test technique, where we treated the functions as continuous; we used calculus and related it to the Taylor series method, then we applied the approximations of linearization, uncorrelated errors and two more.

And finally, we got what you wanted was expression for standard error in the result. So, next point is, how we evaluate those differentials that came in this result from error this result formula? This is what we will take up in the next lecture. With that we conclude this lecture on the single test methodology and we have looked at the basics of how we use Taylor series method for our purpose.

Thank you.