Solid Mechanics
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Lecture - 7
Principal planes and Principal stress components

Welcome to Lecture 7! In this lecture, we will learn about the concept of principal stress components and principal planes.

1 Definition (start time: 00:30)

By now we have learnt that at any point in the body, we have different traction on different planes. Accordingly, each of the planes also has its own normal component of traction. Among these planes, the planes on which the normal component of traction becomes maximum or minimum are called principal planes and the values of the normal traction on those planes are called principal stress components. The knowledge of such planes and traction on them is important because one of failure theories says that a body will fail at a point if the principal stress component reaches a threshold limit. Whenever we design a machine, the knowledge of principal stress components can help us to know whether our machine will be within the limits of failure or not.

2 Finding Principal Planes (start time: 1:56)

Let us suppose we are interested in finding principal planes at a point $\underline{x}$ in the body as shown in Figure 1. At this point, the normal component of traction on an arbitrary plane with normal $\underline{n}$ is given by

$$\sigma_{nn} = \underline{t}^n \cdot \underline{n} = (\underline{\underline{\sigma}}\, \underline{n}) \cdot \underline{n} \tag{1}$$

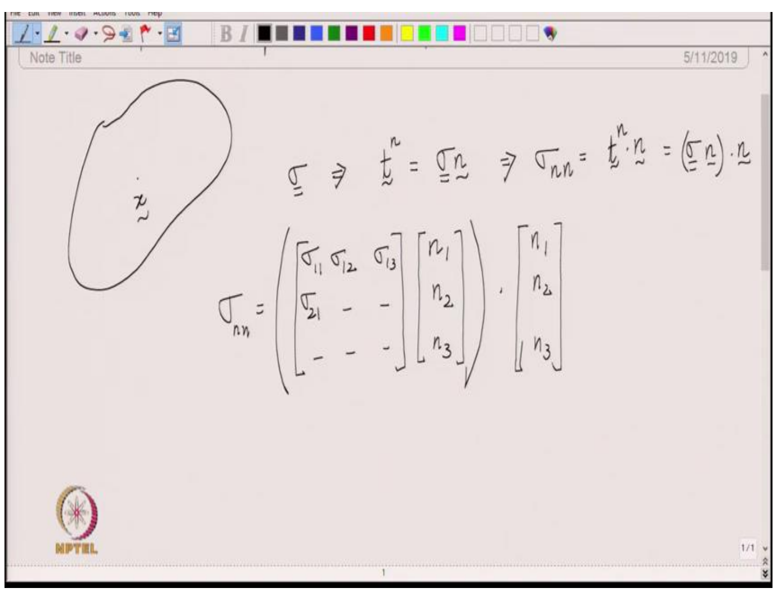

Figure 1: A body with an arbitrary point $\underline{x}$

Our objective is to maximize/minimize it. We know from the first year calculus that once we have a mathematical formula for a quantity (in terms of variables) to be maximized/minimized, we set the derivative of the quantity with respect to all variables to zero and solve the resulting equations to obtain the variables. Let us choose a coordinate system ($\underline{e}_1, \underline{e}_2, \underline{e}_3$) and write formula for σ_{nn} in this coordinate system, i.e.,

$$\begin{aligned} \sigma_{nn} &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \\ &= \sum_i \sum_j \sigma_{ij} n_j n_i \end{aligned} \qquad (2)$$

The normal direction (or the three components (n_1, n_2, n_3)) is an unknown here while the stress matrix is known. However, we know that any direction vector has to be a unit vector. Thus, the three components of $\underline{n}$ must satisfy

$$n_1^2 + n_2^2 + n_3^2 - 1 = 0 \qquad (3)$$

Accordingly, not all three components of the direction are independent, e.g., n_3 can be calculated from the other two by setting

$$n_3 = \sqrt{1 - n_1^2 - n_2^2} \qquad (4)$$

We can substitute the above formula for n_3 in equation (2) and then differentiate the resulting expression for σ_{nn} just with respect to n_1 and n_2. However, the modified expression for σ_{nn} becomes a bit complex differentiating which and further solving the resulting equations is not easy. Another way to maximize/minimize our function (2) is using the Lagrange multipliers which we now discuss.

2.1 Method of Lagrange Multipliers (start time: 06:33)

Whenever a function is to be maximized/minimized in presence of constraints, one uses the method of Lagrange multiplier. Basically, the objective function (the function to be minimized/maximized which is (2) in our case) is augmented by adding/subtracting to it the constraint equation (equation (3) here) multiplied with an unknown Lagrange multiplier. So, the augmented function to be minimized (denoted by f) now becomes a function of the 3 components n_1, n_2, n_3 and the Lagrange multiplier λ as follows:

$$f(n_1, n_2, n_3, \lambda) = \sum_i \sum_j \sigma_{ij} n_j n_i - \lambda \Big(\sum_i n_i n_i - 1 \Big) \qquad (5)$$

The term $\lambda \Big(\sum_i n_i n_i - 1 \Big)$ represents our constraint (equation (3)) and the negative sign in front of it could very well have been a positive sign. This sign does not make any difference to the overall formulation. As the function f has to be minimized/maximized, we take its derivative with respect to each of the unknowns (n_1, n_2, n_3, λ). Let us begin by taking the derivative with respect to n_1, i.e.,

$$\frac{\partial f}{\partial n_1} = \sum_i \sum_j \sigma_{ij} \frac{\partial n_j}{\partial n_1} n_i + \sum_i \sum_j \sigma_{ij} n_j \frac{\partial n_i}{\partial n_1} - \lambda \Big(\sum_i 2 \frac{\partial n_i}{\partial n_1} n_i \Big) \qquad (6)$$

As σ_{ij} is a constant here, it does not get differentiated. As n_1, n_2 and n_3 can be taken to be independent now, we can write

$$\begin{aligned}\frac{\partial n_i}{\partial n_j} = \delta_{ij} &= 1 \quad \text{if } i = j \\ &= 0 \quad \text{if } i \neq j\end{aligned} \tag{7}$$

Now, taking the derivative of f with respect to a general component n_k and using equation (7), we get

$$\frac{\partial f}{\partial n_k} = \sum_i \sum_j \sigma_{ij}\delta_{jk}n_i + \sum_i \sum_j \sigma_{ij}n_j\delta_{ik} - 2\lambda \sum_i \delta_{ik}n_i = 0 \tag{8}$$

Using Kronecker Delta property, we can remove one of the summations from each term and replace the index of summation by the other index in the Kronecker Delta function, i.e.,

$$\sum_i \sigma_{ik}n_i + \sum_j \sigma_{kj}n_j - 2\lambda\ n_k = 0. \tag{9}$$

The first and second terms can be clubbed together because i and j are just dummy variables (variable of summation) which can be replaced with any other variable. Thus

$$\sum_i (\sigma_{ik} + \sigma_{ki})n_i - 2\lambda n_k = 0 \tag{10}$$

Finally, as the stress matrix is symmetric, we get

$$\boxed{\sum_i \sigma_{ki}n_i - \lambda n_k = 0.} \qquad k = 1, 2, 3 \tag{11}$$

Because of writing in a summation form, we have ended up with a simple expression. The summation expression also helped us to take the derivative with respect to a general component n_k. Now, we take the derivative of f with respect to λ, i.e.,

$$\frac{\partial f}{\partial \lambda} = \sum_i n_i n_i - 1 = 0 \tag{12}$$

So, we have 4 equations (3 from equation (11) and 1 from equation (12)) in 4 unknowns (n_1, n_2, n_3, λ). Equation (11) can be written in a matrix form since for each k, the first term on LHS of (11) can be obtained by multiplying the k^{th} row of $[\underline{\sigma}]$ with the column of $\underline{n}$. This leads to

$$[\underline{\sigma}]\ [\underline{n}] = \lambda\ [\underline{n}] \tag{13}$$

We immediately see that this is an 'eigenvalue-eigenvector problem' with

$\underline{n}$: eigenvector of $\underline{\underline{\sigma}}$

λ : eigenvalue of $\underline{\underline{\sigma}}$

We also know from first year mathematics that if $\underline{x}$ is an eigenvector, then a scalar multiple of $\underline{x}$ is also an eigenvector which can be proved as follows:

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \Rightarrow \underline{\underline{A}}(b\underline{x}) = b(\underline{\underline{A}}\,\underline{x}) = b(\lambda \underline{x}) = \lambda(b\underline{x}) \tag{14}$$

Thus both $\underline{x}$ and $b\underline{x}$ are the eigenvectors with the same eigenvalue λ for arbitrary b. Thus, the magniude of our direction vector $\underline{n}$ could be anything as far as equation (13) is concerned but equation (12) restricts its magnitude to be unity. Thus, equations (12) and (13) together give us a unique solution for the direction vector $\underline{n}$. Also when we consider equation (13), the left hand side is nothing but the column representation of traction on plane with normal $\underline{n}$. Thus, we have:

$$\underline{t}^n = \underline{\underline{\sigma}}\,\underline{n} = \lambda \underline{n} \Rightarrow \sigma_{nn} = \underline{t}^n \cdot \underline{n} = \lambda \underline{n} \cdot \underline{n} = \lambda. \tag{15}$$

From this, we immediately infer that the traction on a plane with normal $\underline{n}$ (given by (13)) acts along the direction $\underline{n}$ itself and hence have no shear component. Summarizing, the principal planes of stress at a point have their normals equal to eigenvectors of the stress tensor whereas the principal stress components are given by the eigenvalues of the stress tensor.

3 Properties of Principal Planes at a point (start time: 28:12)

By definition, principal planes are the planes on which the normal component of traction is maximized/minimized. We want to know how many such planes exist at any given point in the body. As stress matrix is a 3 × 3 matrix, it will usually have three eigenvalues and eigenvectors but they need not all be real. However, being symmetric ensures that these eigenvalues and eigenvectors are all real. In fact, for symmetric matrices, the eigenvectors corresponding to different eigenvalues are perpendicular to each other too. To prove this, consider two eigenvectors $\underline{n}_1$ and $\underline{n}_2$ of a symmetric tensor $\underline{\underline{\sigma}}$, with corresponding eigenvalues λ_1 and λ_2 (but distinct). Thus, we have

$$\underline{\underline{\sigma}}\,\underline{n}_1 = \lambda_1\,\underline{n}_1\,, \tag{16}$$
$$\underline{\underline{\sigma}}\,\underline{n}_2 = \lambda_2\,\underline{n}_2. \tag{17}$$

We now dot the first equation with $\underline{n}_2$ and the second one with $\underline{n}_1$. So, we get

$$(\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 = \lambda_1(\underline{n}_1\cdot\underline{n}_2) \tag{18}$$
$$(\underline{\underline{\sigma}}\,\underline{n}_2)\cdot\underline{n}_1 = \lambda_2(\underline{n}_2\cdot\underline{n}_1) \tag{19}$$

Let us consider equation (19) now. From the matrix vector operations discussed in the first lecture, we can take $\underline{\underline{\sigma}}$ to the other side of the dot product by taking its transpose, i.e.,

$$\underline{n}_2\cdot(\underline{\underline{\sigma}}^T\underline{n}_1) = \lambda_2(\underline{n}_2\cdot\underline{n}_1). \tag{20}$$

However, the stress tensor being symmetric ($\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$), we get

$$(\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 = \lambda_2(\underline{n}_2\cdot\underline{n}_1) \tag{21}$$

Now, we subtract equation (21) from equation (18) to get

$$(\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 - (\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 = (\lambda_2-\lambda_1)(\underline{n}_1\cdot\underline{n}_2)$$
$$\Rightarrow (\lambda_1-\lambda_2)(\underline{n}_1\cdot\underline{n}_2) = 0. \tag{22}$$

As λ_1 and λ_2 are distinct, $\underline{n}_1\cdot\underline{n}_2$ has to be zero or the two normals are perpendicular. We have thus proved that principal planes at a point are three in number and are perpendicular to each other. It is also easy to show that if two of the eigenvalues turn out to be the same, then any linear combination of the two eigenvectors is also an eigenvector. For example:

$$\underline{\underline{\sigma}}\,\underline{n}_1 = \lambda\,\underline{n}_1 \quad\Rightarrow\quad \underline{\underline{\sigma}}\,(\alpha\underline{n}_1) = \lambda\,(\alpha\underline{n}_1) \tag{23}$$
$$\underline{\underline{\sigma}}\,\underline{n}_2 = \lambda\,\underline{n}_2 \quad\Rightarrow\quad \underline{\underline{\sigma}}\,(\beta\underline{n}_2) = \lambda\,(\beta\underline{n}_2) \quad . \tag{24}$$

Summing the above two equations:

$$\underline{\underline{\sigma}}\,(\alpha\underline{n}_1+\beta\underline{n}_2) = \lambda\,(\alpha\underline{n}_1+\beta\underline{n}_2) \tag{25}$$

This implies that if two of the eigenvalues repeat, there exists infinite number of eigenvectors all in the plane formed by $\underline{n}_1$ and $\underline{n}_2$ all of which by definition are also the principal planes but all having the same eigenvalue or principal stress component.

4 Representation of stress tensor in the coordinate system of its eigenvectors (start time: 34:26)

If there are three distinct eigenvalues for a stress matrix, the corresponding three eigenvectors will all be perpendicular to each other (we proved it in the previous section). Thus, we can also choose them as the basis for our coordinate system. Let us represent our stress tensor in this coordinate system. We first need to find traction on the planes with normals along the basis vectors of this coordinate system. As the basis vectors are themselves eigenvectors of the stress tensor, traction on those planes will simply be $\lambda\underline{n}$ (no shear components present). Thus, the corresponding stress matrix will be a diagonal matrix as shown below:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \tag{26}$$

Alternatively, given an arbitrary stress matrix in some coordinate system, we can always transform it to become diagonal in the coordinate system whose basis vectors are aligned along the eigenvectors of the stress matrix. In the general case as shown in Figure 2, both normal and shear components of traction are present on the faces of a cuboid element at a point in the body.

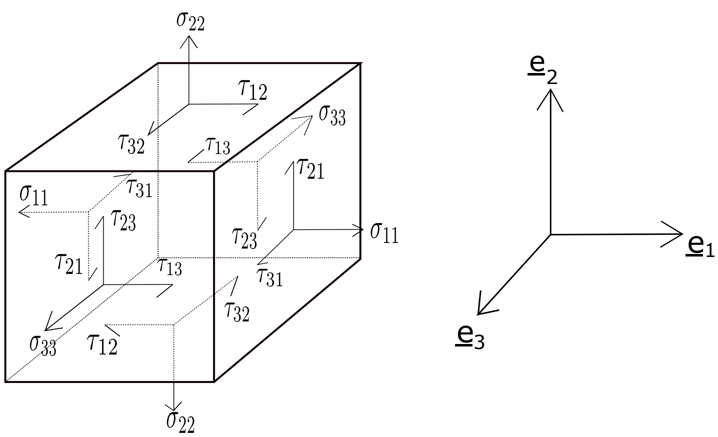

Figure 2: A cuboid element at a point in the body with all the stress components shown on it

But, if we choose the cuboid element in such a way that its faces are along the eigenvectors of the stress matrix as shown in Figure 3, its faces will have no shear component because the faces are also the principal planes. On these planes, we only have normal components (λ_1, λ_2, λ_3) present. So, the faces of this cuboid have no tendency to shear. They can either get pulled apart or pushed inside depending on the sign of λ.

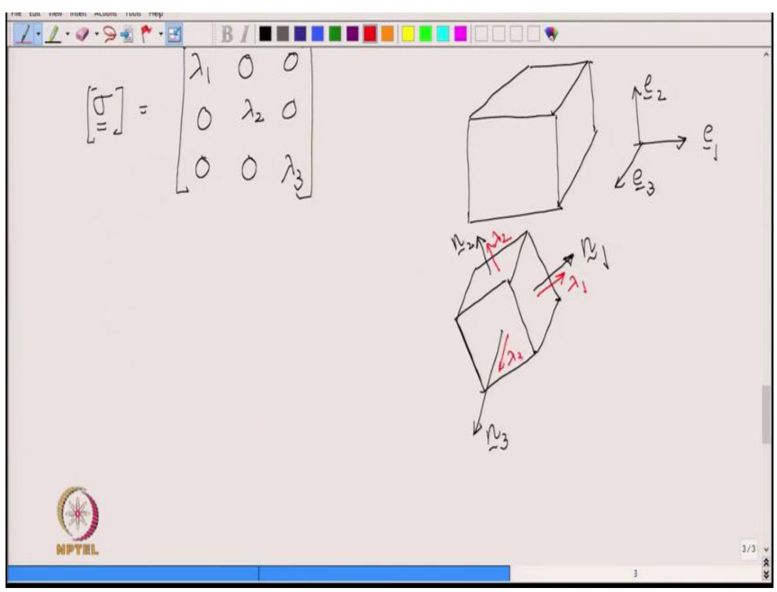

Figure 3: A cuboid element at a point in the body with its face normals along the eigenvectors of the stress matrix at that point: all the faces become shear free