Solid Mechanics
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Lecture - 32
Theories of Failure (Contd.)

Hello everyone! Welcome to lecture 32! This is the last lecture of the course. We will continue with the discussion on theories of failure.

1 Example (start time: 00:28)

Let us discuss a tougher problem than the one worked out in the previous lecture. Suppose we have to design a lever as shown in Figure 1.

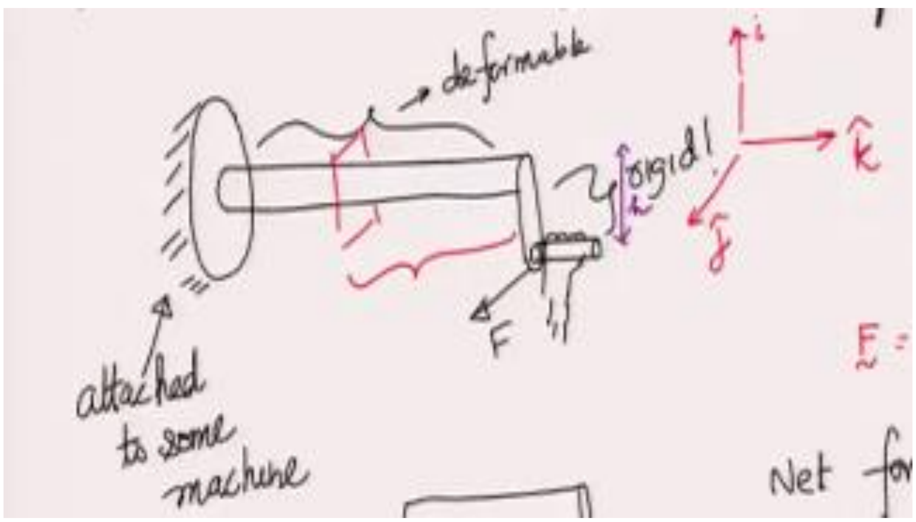

Figure 1: A force *F* is applied to a lever

The lever has a shaft and an arm. There is also a handle to hold the lever where a force *F* is applied usually through hands. The other end of the shaft is usually clamped to a machine component. The shaft is assumed deformable whereas the handle and the arm are assumed to be rigid. We need to design the shaft so that it can bear the applied force. The factor of safety is given to be *N*. The applied force can be written vectorially as

$$\underline{F} = F\,\hat{j} \tag{1}$$

To design the shaft, we need to find the internal contact force and moment in the shaft's crosssection. Let us cut a section in the shaft at a distance *x* from the clamped end and draw the free body diagram of the right part of the shaft as shown in Figure 2.

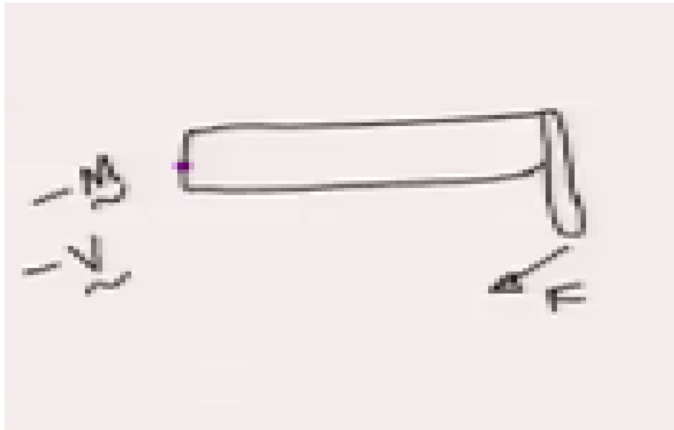

Figure 2: Free body diagram of the right part of the shaft when a section is cut at a distance *x* from the clamped end.

The shear force $-\underline{V}$ and bending moment $-\underline{M}$ acts on its left end since its cross-section normal is along $-z$ axis. Net force balance for the right part of the shaft gives

$$-\underline{V} + F\,\hat{j} = \underline{0} \Rightarrow \underline{V} = F\,\hat{j} \tag{2}$$

Let us now do moment balance about the centroid of the left end which yields

$$\begin{aligned} &-\underline{M} + \left[(L-x)\hat{k} - h\hat{i}\right] \times F\hat{j} = \underline{0} \\ \Rightarrow\quad &\underline{M} = \left[(L-x)\hat{k} - h\hat{i}\right] \times F\hat{j} = \underbrace{-(L-x)F}_{M_x}\,\hat{i}\,\underbrace{-hF}_{T}\,\hat{k}. \end{aligned} \tag{3}$$

The component of moment acting along the axis of the shaft is the torque *T* while the other component is the bending moment M_x. As the shear force acts along *y* axis, it can be denoted as V_y such that $V_y = F$. We now have to find the stress components that generates in the cross-section of the shaft due to these loads. The various non-zero stress components are shown in Figure 3.

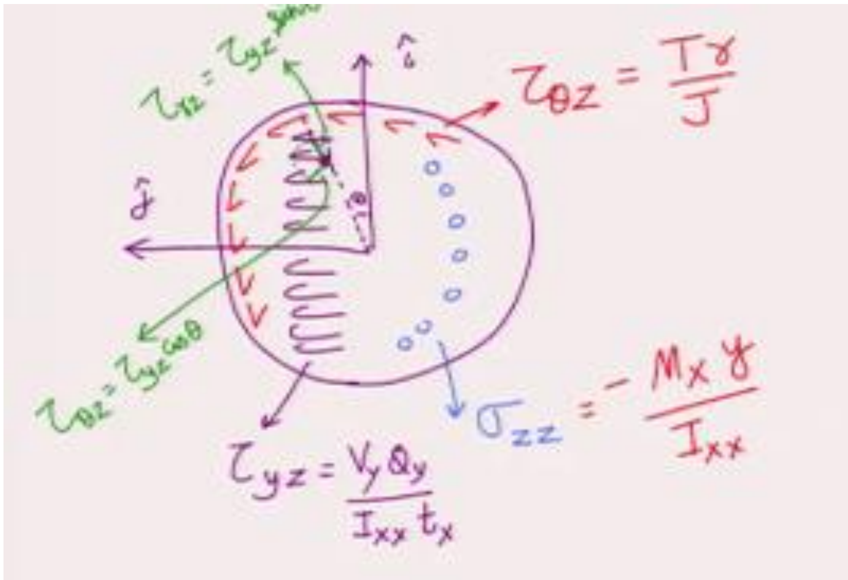

Figure 3: Stress components in the cross-section of the shaft.

Due to shear force V_y, τ_{yz} acts on the cross-sectional plane. The bending moment M_x generates normal stress σ_{zz} while the torque generates shear stress $\tau_{\theta z}$. From the discussion in previous lectures, their values are given by

$$\tau_{\theta z} = \frac{Tr}{J}, \quad \sigma_{zz} = \frac{-M_x y}{I_{xx}}, \quad \tau_{yz} = \frac{V_y Q_y}{I_{xx} t_x}. \tag{4}$$

Here t_x represents thickness of the cross-section in x direction. All other variables have usual meaning. The final stress matrix for a general point in the cross-section can be obtained by superposition of all the above stress components. This stress matrix in the cylindrical coordinate system becomes

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 0 & \frac{V_y Q_y \sin\theta}{I_{xx} t_x} \\ 0 & 0 & \frac{Tr}{J} + \frac{V_y Q_y \cos\theta}{I_{xx} t_x} \\ \frac{V_y Q_y \sin\theta}{I_{xx} t_x} & \frac{Tr}{J} + \frac{V_y Q_y \cos\theta}{I_{xx} t_x} & \frac{-M_x y}{I_{xx}} \end{bmatrix}. \tag{5}$$

Notice that none of the coordinate axes are principal axes. Let us try to use maximum shear stress theory to design the shaft. The torque and shear force are constant along the length of the beam. Thus, the stress components τ_{yz} and $\tau_{\theta z}$ do not vary along the length too. The bending moment varies with x and goes from 0 at the right end to maximum at the clamped end. Thus, the bending stress σ_{zz} is also maximum at the clamped end. So, the cross-section at the clamped end is the critical cross-section and failure will occur first in this cross-section. We thus focus on this cross-section for our analysis which is also shown in Figure 4.

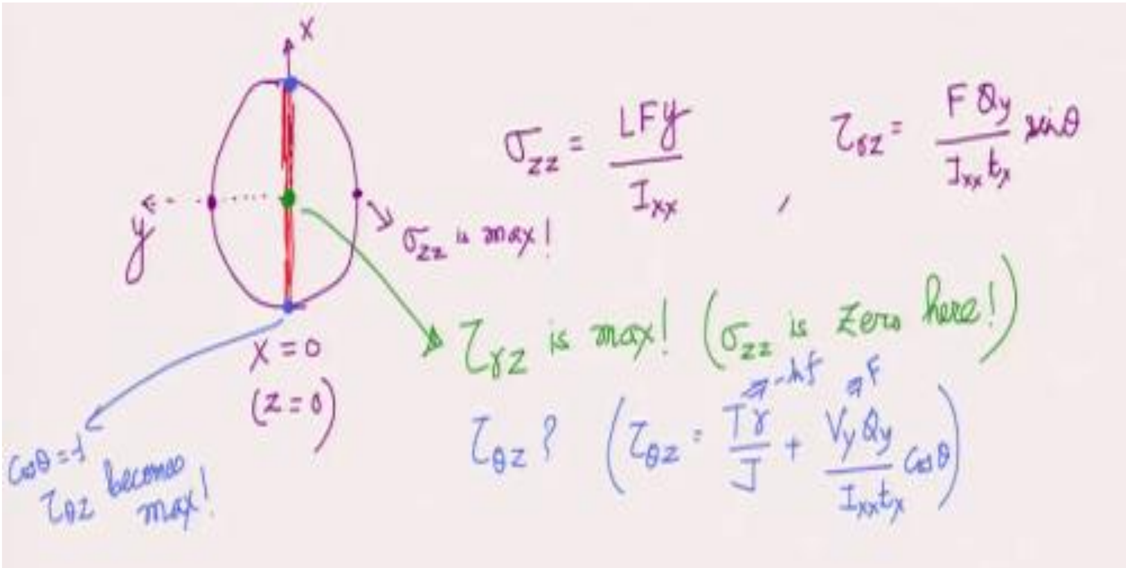

Figure 4: The cross-section at the clamped end with the points experiencing maximum stress components shown

For this cross-section, bending stress σ_{zz} equals

$$\sigma_{zz} = \frac{yLF}{I_{xx}}. \tag{6}$$

It is just a function of *y* and hence it is maximum at the two points shown in Figure 4. The stress component τ_{rz} equals

$$\tau_{rz} = \frac{FQ_y}{I_{xx}t_x}\sin\theta. \tag{7}$$

Here Q_y would be maximum for all points on the line $y = 0$ and $\sin\theta$ is maximum for $\theta = 90^\circ$. Thus, τ_{rz} is maximum only at the center of the cross-section. However, the bending stress is zero at the center of the cross-section. Finally, $\tau_{\theta z}$ is given by

$$\tau_{\theta z} = \frac{Tr}{J} + \frac{V_y Q_y \cos\theta}{I_{xx}t_x}. \tag{8}$$

It is made up of two terms. The first one is maximum when $r = R$, i.e., on the periphery of the cross-section. The second term has t_x which is maximum for all points on the $y = 0$ line. Finally, $\cos\theta$ is 1 at the top and -1 at the bottom. As $T = -hF$, the first term has a negative value and likewise the second term should also be negative in order to maximize the stress component. Thus, we should choose the point where $\cos\theta$ equals -1. Hence, $\tau_{\vartheta z}$ has maximum negative value at the bottom-most point on $y = 0$ line. We have thus found that the three stress components $\tau_{\theta z}$, τ_{rz} and σ_{zz} attain their maximum values at different points. So, this does not turn out to be an easy problem. The point of maximum shear stress in the cross-section cannot be found using pen and paper - one has to solve it on computer. We first have to find the principal stress components (λ_1, λ_2, λ_3) at each point in the cross-section by solving the eigenvalue problem then and get the maximum shear stress as $\frac{\lambda_1-\lambda_3}{2}$. We can then compare this shear stress at all points to get the point experiencing maximum shear stress in the cross-section which would be the critical point from the perspective of failure. We notice that even for a very simple design of a shaft/lever, our analysis led to a stress matrix which cannot be handled using pen and paper. In most practical cases, machines are very complex having multiple parts. Designing them by hand is indeed impossible and we have to resort to computer programming. This completes our discussion on theories of failure.

2 Concept of Thermoelasticity (start time: 24:00)

Let us think of modeling stress-strain relationship when temperature of the material is also changing. If we heat a body, it simply expands with no stress developing in it. This expansion leads to an extra strain called thermal strain (ϵ_T) which does not induce any stress in the body. It turns out that

$$\epsilon_T \propto T \quad \Rightarrow \epsilon_T = \alpha T. \tag{9}$$

Here, T denotes the temperature change and the proportionality constant α is called thermal expansion coefficient. As this strain does not generate any stress in the body, it can be subtracted from the total strain to get the strain term which generates due to stress. This implies

$$\begin{aligned} \epsilon_{xx} - \alpha T &= \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})), \\ \epsilon_{yy} - \alpha T &= \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})), \\ \epsilon_{zz} - \alpha T &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) \end{aligned} \tag{10}$$

which can be further written as

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) + \alpha T \\ \epsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) + \alpha T \\ \epsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) + \alpha T \end{aligned} \tag{11}$$

The temperature change T only affects normal strains and has no effect on shear strains. This is because only the length of line elements change with temperature and not the angle between line elements. Thus, the shear stress-shear strain relation remain unaffected, i.e.,

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}. \tag{12}$$

The equations (11) and (12) together form the three-dimensional thermo-elastic stress-strain relation.

3 Plastic behavior (start time: 28:40)

There are materials whose strain does not vanish upon removal of the applied load. This is in contrast with elastic materials that we have learnt till now - elastic materials come back to their original shape upon removal of the applied load. Consider a steel bar on which a uniformly distributed load σ is applied at both its ends as shown in Figure 5.

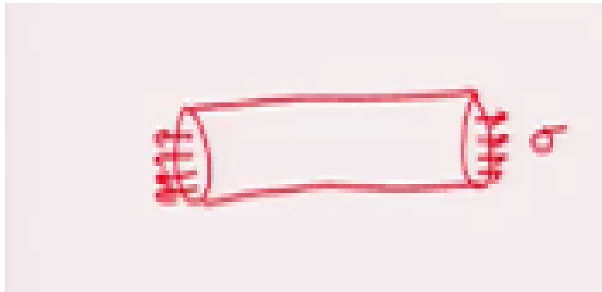

Figure 5: A distributed load σ is applied at the both ends of a circular beam made of steel

We have shown the applied stress σ vs. the longitudinal strain ϵ curve in Figure 6.

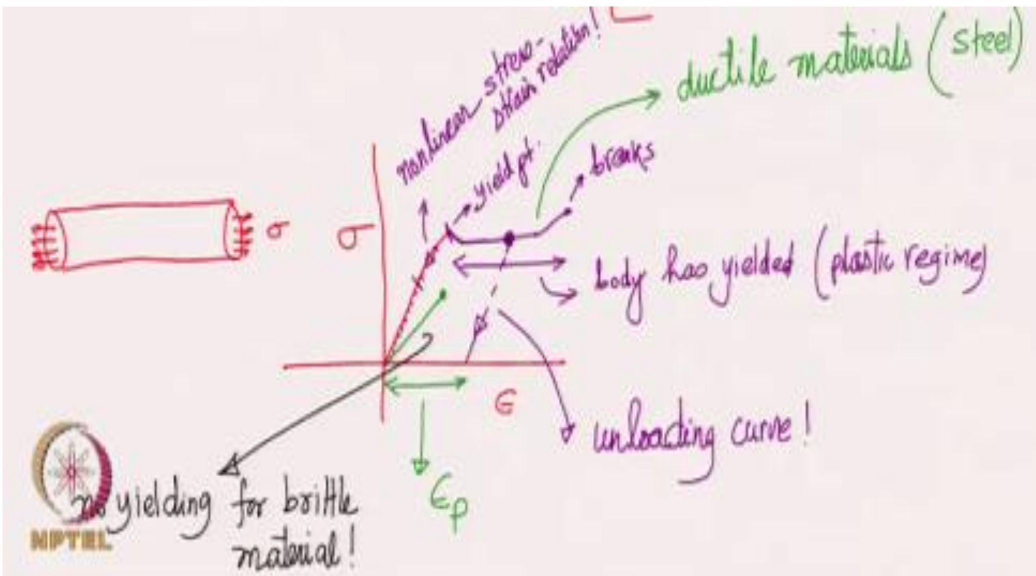

Figure 6: Stress-strain curve for the beam shown in Figure 5.

Initially, this curve is a straight line. This is the regime where linear stress-strain relationship is obeyed. But, the curve does not remain linear throughout. The curved part up to the inflexion point, is the regime of non-linear stress-strain relation because the slope of the curve is changing in this regime. The body is still elastic in this regime which means that if load is removed, the same curve is traced in the backward direction. As we keep increasing the load, there comes a point where the body yields, i.e., the beam elongates a lot even with very small increment in the applied load. The beam eventually breaks. The regime between the yield point and the breaking point is called the plastic regime. If we decrease the applied load in this regime, the original curve is not traced back as shown in the figure. Hence, when the stress becomes zero, the body retains some strain. This strain is called plastic strain (ϵ_p).The yielding behavior is common in other materials too which give them ductile property. We also have brittle materials like glass. They have a different stress-strain relationship (see Figure 6). The stress-strain curve is linear after which the body simply cracks, i.e., there is no yielding in case of brittle materials. There are several other material models which are usually covered in follow-up courses at postgraduate level. Still, the linear stress-strain relation that we covered in this course is useful because all materials follow this behavior when the applied load or the deformation of the body is small enough.