Solid Mechanics
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Lecture - 26
Concept of Shear Center

Hello everyone! Welcome to Lecture 26! In this lecture, we will discuss about a very interesting concept: the concept of shear center.

1 Definition (start time: 00:34)

The shear center is defined as the point in the cross-sectional plane relative to which the net torque due to shear stress distribution (originating because of transverse load only) vanishes. Finding analytical formula for the location of the shear center for a general cross-section isn't possible. However, the same can be derived for thin and open cross-sections such as the one in Figure 1. We show this derivation now.

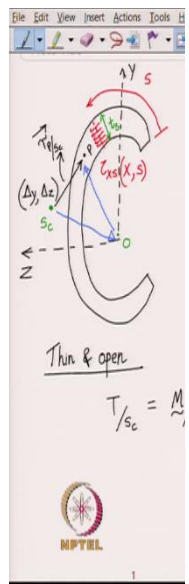


Figure 1: A typical thin and open cross-section.

2 Analysis (start time: 04:46)

We had derived the following formula for shear stress τ_{sx} at arc-length s:

$$\tau_{sx} = -\frac{1}{t_s(I_{yz}^2 - I_{yy}I_{zz})}\left[-V_y(Q_y^s I_{yy} - Q_z^s I_{yz}) + V_z(Q_y^s I_{yz} - Q_z^s I_{zz})\right] \quad (1)$$

which is assumed to be uniformly distributed through the cross-section's thickness. Let us denote by $s_c = (\Delta_y, \Delta_z)$, the location of the shear center as shown in Figure 1. The centroid of the cross-section is located at O. As the net torque T about s_c vanishes as per definition, we can first find the total moment about s_c and then equate its x-component to zero. Let us denote the cross-sectional area by Ω and an arbitrary point in the cross-section by P. The position vector of P with respect to shear center ($\vec{r}_{P/sc}$) can be written as follows:

$$\overrightarrow{r}_{P/s_c} = \overrightarrow{r}_{P/O} + \underbrace{\overrightarrow{r}_{O/s_c}}_{-(\Delta_y \hat{j} + \Delta_z \hat{k})} \tag{2}$$

The total torque T due to shear stress distribution will thus be

$$\begin{aligned} T_{/s_c} &= \underline{M}_{/s_c} \cdot \hat{i} = \iint_\Omega (\overrightarrow{r}_{P/s_c} \times \tau_{xs}\hat{s}) \cdot \hat{i} \; dA \\ &= \iint_\Omega (\overrightarrow{r}_{P/O} \times \tau_{xs}\hat{s}) \cdot \hat{i} \; dA - \iint_\Omega \left[(\Delta_y \hat{j} + \Delta_z \hat{k}) \times \tau_{xs}\hat{s}\right] \cdot \hat{i} \; dA. \end{aligned} \tag{3}$$

In the second integral term, the shear center position ($\Delta_y \hat{j} + \Delta_z \hat{k}$) can be taken out of the integral while the integral of shear stress over the cross-section gives the total shear force $\vec{V}$ in the cross-section. For the first integral, let us write area dA as $dsdt$ and change the area integral into two line integrals: one along the arc-length s and the other along the thickness t. Thus, the above equation can be rewritten as

$$\begin{aligned} T_{/s_c} &= \int_s \int_t (\overrightarrow{r}_{P/O} \times \tau_{xs}\hat{s}) \cdot \hat{i} \; dsdt - \left[(\Delta_y \hat{j} + \Delta_z \hat{k}) \times \iint_\Omega \tau_{xs}\hat{s}\, dA\right] \cdot \hat{i} \\ &= \left[\int_s \left[\int_t \overrightarrow{r}_{P/O}(s,t)dt\right] \times \tau_{xs}(x,s)\hat{s}\, ds\right] \cdot \hat{i} \; - \left[(\Delta_y \hat{j} + \Delta_z \hat{k}) \times (V_y \hat{j} + V_z \hat{k})\right] \cdot \hat{i} \\ &= \left[\int_s \left[\int_t \overrightarrow{r}_{P/O}(s,t)dt\right] \times \tau_{xs}(x,s)\hat{s}\, ds\right] \cdot \hat{i} \; - \left[\Delta_y V_z - \Delta_z V_y\right]. \end{aligned} \tag{4}$$

The expression $\int_t \overrightarrow{r}_{P/O}(s,t)dt$ gives us the first moment which we can replace by the position vector of point P^0 times the thickness of the cross-section at arc-length s, i.e., t_s and P^0 denotes the centroidal point at arc-length s. These centroidal points form the centerline of the cross-section shown as the dashed green line in Figure 1. Thus, we get

$$T_{/s_c} = \int_{s=0}^{L_c} t_s (\overrightarrow{r}_{P^0/O}(s) \times \tau_{xs}\hat{s}) \cdot \hat{i}\, ds - [\Delta_y V_z - \Delta_z V_y] \tag{5}$$

Here, L_c denotes the full arc-length along the cross-section boundary and should not be confused with the length of the beam. Upon working out the above integral, we get

$$T_{/sc} = C_y V_y + C_z V_z - [\Delta_y V_z - \Delta_z V_y] \tag{6}$$

where C_y and C_z are the constants obtained through the integration over the cross-section's arclength. As, the torque about the shear center must be zero, we get the following upon rearranging:

$$(C_y + \Delta_z)V_y + (C_z - \Delta_y)V_z = 0. \tag{7}$$

As this holds for arbitrary V_y and V_z, we finally obtain

$$\Delta_y = C_z, \qquad \Delta_z = -C_y. \tag{8}$$

We can find C_y and C_z by substituting equation (1) for τ_{xs} in equation (5). Let us consider the case when only V_y acts on the cross-section and (y, z) axes are also the principal axes. So, I_{yz} becomes zero and formula (1) for τ_{sx} simplifies to

$$\tau_{sx} = \frac{-V_y Q_y^s}{I_{zz} t_s} . \tag{9}$$

Plugging this expression in equation (5), we get:

$$\begin{aligned} \int_{s=0}^{L_c} t_s(\overrightarrow{r_{P^0}}_{/O}(s) \times \tau_{xs}\hat{s}) \cdot \hat{i}\, ds &= \int_{s=0}^{L_c} t_s \left(\overrightarrow{r_{P^0}}_{/O}(s) \times \frac{-V_y Q_y^s}{I_{zz} t_s} \hat{s} \right) \cdot \hat{i}\, ds \\ &= V_y \underbrace{\left[\frac{-1}{I_{zz}} \int_s \overrightarrow{r_{P^0}}_{/O} \times Q_y^s \hat{s}\ ds \right] \cdot \hat{i}}_{C_y} . \end{aligned} \tag{10}$$

Thus, we get *z*-coordinate of the shear center to be

$$\Delta_z = -C_y = \frac{1}{I_{zz}} \int_s \left(\overrightarrow{r_{P^0}}_{/O} \times Q_y^s \hat{s} \right) \cdot \hat{i}\ ds. \tag{11}$$

Proceeding along similar lines, *y*-coordinate of the shear center turns out to be

$$\Delta_y = C_z = -\frac{1}{I_{yy}} \int_s \left(\overrightarrow{r_{P^0}}_{/O} \times Q_z^s \hat{s} \right) \cdot \hat{i}\ ds. \tag{12}$$

Note that the above formulas hold only if *y* and *z* axes are also the principal axes of the cross-section.

3 Bending-twisting coupling in unsymmetrical cross-sections (start time: 22:35)

We now illustrate that a beam can also twist when subjected to transverse load if the line of action of transverse load does not pass through shear center. We show such a beam in Figure 2 which is clamped at one end and a point load is applied in the transverse direction at the other end at point *A*.

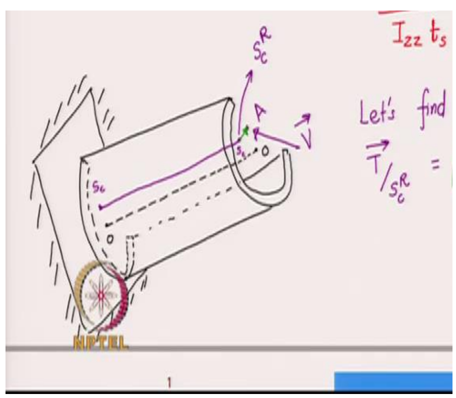


Figure 2: A beam with unsymmetrical cross-section is subjected to a transverse point load

The line joining the shear center of different cross-sections is also shown there. Let us isolate the beam from the clamped end support and include the reaction of shear stress distribution from this support in the left end cross-section in the beam's free body diagram. The net moment on the beam would have to be zero about any general axis in static equilibrium. Consider this axis to be the line of shear centers for which the moment about it would be torque since this axis is perpendicular to the beam's cross-section. The torque due to shear stress distribution in the left cross-section about the shear center of the left end cross-section will be zero by the definition of shear center. However, the torque due to the applied transverse load about the axis of shear center will be non-zero since the load is acting eccentrically to the shear center axis. Thus, a net torque acts on the beam which will also induce twist in the beam in addition to the usual bending due to transverse load. This twist will lead to additional reaction of shear stress distribution from the support on the left end cross-section so that the overall torque on the entire beam finally vanishes. Needless to say if we apply the transverse load such that its line of action passes through the shear center at the right end cross-section, there will be no tendency in the beam to twist and the beam will undergo just bending.

4 Shear center for symmetrical cross-sections (start time: 32:10)

It can be shown using formulae (11) and (12) for the shear center that it always lies on the line of symmetry of the cross-section (if any present). For example, consider several symmetrical cross-sections in Figure 3.

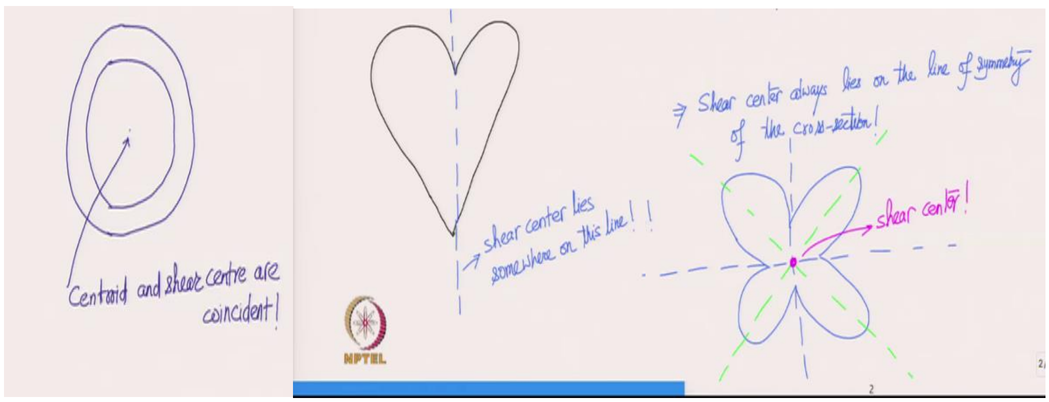


Figure 3: (a) annular cross-section having infinite lines of symmetry (b) heart-shaped cross-section having one line of symmetry (c) Flower shaped cross-section having four lines of symmetry

In Figure (3a), we have an annular cross-section for which all diametrical lines are lines of symmetry. Accordingly, shear center and its centroid coincide. In Figure (3b), we have a heart-shaped cross-section which has just one line of symmetry as shown in the figure. Accordingly, the shear center lies on this line. To fix the other coordinate of shear center, one would have to use the formula derived above. Finally, in Figure (3c), we have a flower-shaped cross-section having four-fold symmetry. As the shear center must lie on all lines of symmetry, it gets uniquely specified here too just by inspection.

4.1 Shear center for an L-shaped cross-section

Consider the thin L-shaped cross-section shown in Figure 4.

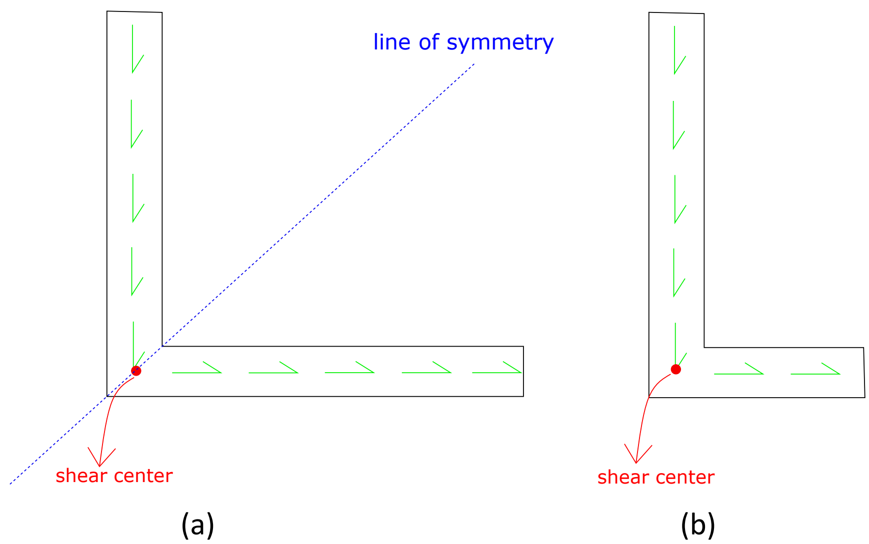
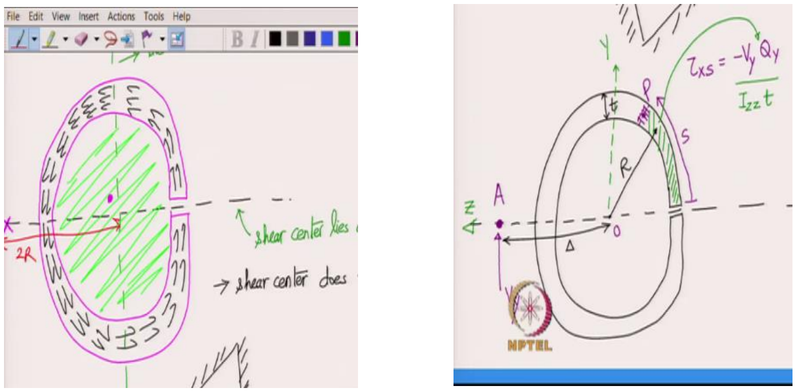


Figure 4: A thin L-shaped cross-section with its shear center and shear stress distribution shown (a) cross-section having one line of symmetry (b) cross-section having no line of symmetry

For the cross-section in Figure (4a), it again has one line of symmetry as shown in the figure. As the cross-section is thin and open, the shear stress flows from one end to the other in the cross-section. By inspection, one can also see that torque due to shear stress distribution vanishes about its corner point: the line of action of shear stress at every point in the cross-section passes through the corner point. Accordingly, the corner point is the shear center which also happens to lie on the line of symmetry. In contrast, for Figure (4b), the two legs of the cross-section are not of same length. Hence, it has no line of symmetry. The corner point is again the shear center since the torque due to shear stress vanishes for the same reason as earlier.

5 Shear center for a cut annulus (start time: 38:14)

Let us think of a thin cross-section which has the shape of an annulus but it is cut as shown in Figure 5a. For such a cross-section, we have only one line of symmetry due to the presence of cut. Thus, the shear center lies on this line of symmetry but its position does not get completely known. As the cross section is thin and open, shear stress flows from one end to the other. If we find torque about any point in the green region (see Figure 5a), the torque due to all the shear stresses contribute in the same direction and hence does not vanish. Thus, the shear center must lie outside this green region. Let us try to find the exact location of the shear center.



(a) (b)

Figure 5: (a) A cross-section of the shape of a cut annulus has one line of symmetry as shown: flow of shear stress is also shown (b) the relevant dimension and variables for the analysis of the cross-section shown

Suppose that the mean radius of the annulus is *R* and the thickness is *t*. As the shear center has to lie on the line of symmetry, $\Delta_y = 0$. Notice that even though there is an infinitesimal cut in the cross-section, any pair of perpendicular lines will still form the cross-section's principal axes, in particular, the (*y* − *z*) axis also form principal axes. We can thus use formula (11) to obtain Δ_z which can be alternatively written in terms of θ coordinate to denote arc-length as follows:

$$\Delta_z = \frac{1}{I_{zz}} \int_0^{2\pi} \left(R\ \hat{e}_r \times Q_y^\theta\ \hat{e}_\theta\right) \cdot \hat{i}\ \ Rd\theta = \frac{R^2}{I_{zz}} \int_0^{2\pi} Q_y^\theta\ d\theta \tag{13}$$

We need to know find Q_y^θ for the region [0,θ] (see Figure 5b) which has also been drawn separately in Figure 6.

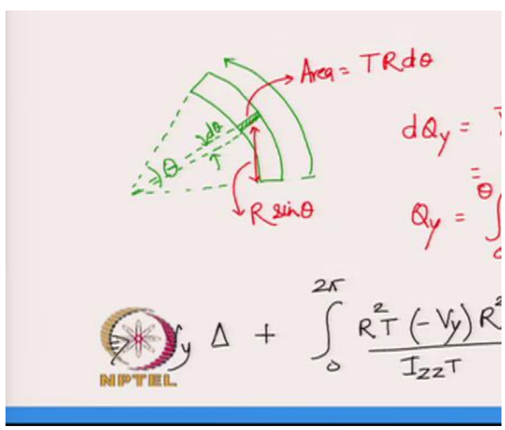


Figure 6: The part of the cross-section from 0 up to θ

Let us identify a tiny strip shown in red in Figure 6 which subtends an angle $d\varphi$ at the center. The *y*-coordinate of the centroid of this tiny strip is $R\sin\varphi$ and its area is $tRd\varphi$. Thus, for this tiny strip, we can write:

$$dQ_y = \bar{Y}\ dA = R^2 t\sin\varphi\ d\varphi \tag{14}$$

We can now find Q_y^θ by integrating over such tiny strips, i.e.

$$Q_y^\theta = \int_0^\theta dQ_y = R^2 t \int_0^\theta \sin\phi\, d\phi = R^2 t(1-\cos\theta) \tag{15}$$

Let us now find I_{zz} for the cross-section about its centroid O. As the cut is really thin, the annulus is approximately complete. So, to calculate I_{xx}, I_{yy} and I_{zz}, we can forget about the thin cut. We thus have

$$I_{zz} = I_{yy} = 1/2\ I_{xx} = 1/2R^2\,(2\pi R\ t) = \pi R^3\ t. \tag{16}$$

Finally substituting equations (15) and (16) into equation (13), we obtain

$$\Delta_z = \frac{R}{\pi}\int_0^{2\pi}(1-cos\theta)\ d\theta = 2R. \tag{17}$$

The shear center thus does not lie on the cross-section. As discussed earlier, if we want to apply shear force on beams having such cross-sections so that the beam does not twist, we must apply shear force so that its line of action passes through shear center. For the present cross-section, we need to create an extension of it (e.g., by a thin rod as shown in red in Figure 7) and apply shear force there so that the beam indeed does not twist.

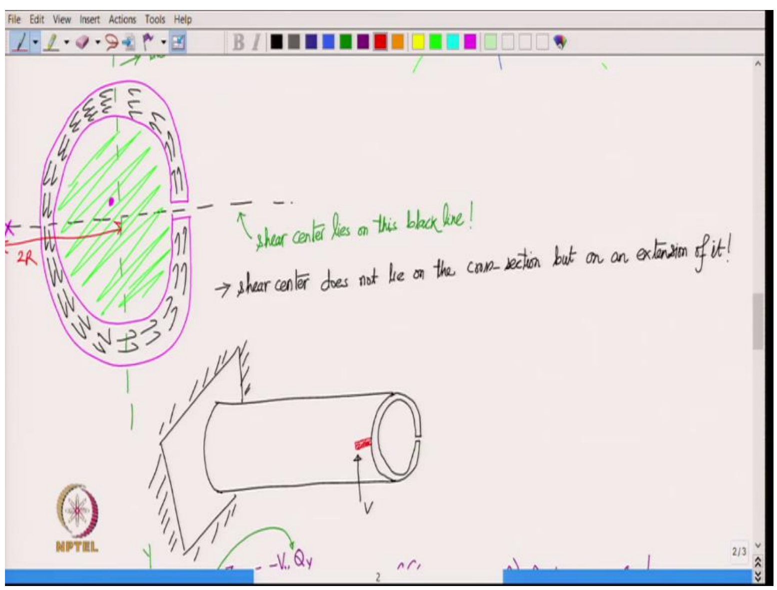


Figure 7: An extension made to the beam's cross-section to avoid twisting of the beam due to application of transverse load