Solid Mechanics
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Lecture - 25
Bending of Unsymmetrical Beams

Hello everyone! Welcome to Lecture 25! We will discuss bending of unsymmetrical beams today.

1 Introduction (start time: 00:26)

Think of bending of symmetrical beams, e.g., a beam having rectangular cross-section with its axis along x-axis and the cross-sectional sides along *y* and *z* axes. If we apply moment on the beam acting along z-axis, the neutral axis of the cross-section was shown to pass through its centroid but the direction of neutral axis was simply assumed to be parallel to the direction of applied moment, i.e., z-axis in this case (also see Figure 1).

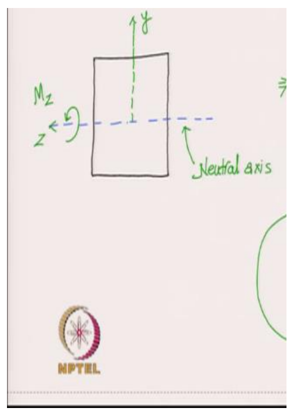

Figure 1: A typical cross-section of a beam having rectangular cross-section

This assumption is not true in general though. For example, Figure 2 shows a typical cross-section of an unsymmetrical beam.

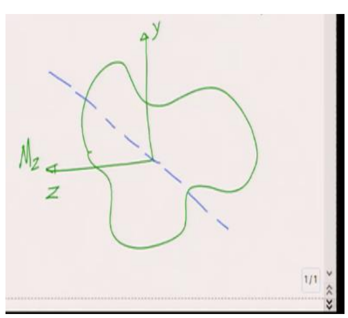

Figure 2: A typical cross-section of an unsymmetrical beam - direction of applied moment and neutral axis need not be parallel.

The neutral axis for this case turns out to be inclined relative to the direction of bending moment. This is non-intuitive since it means the axis of bending (neutral axis) can be in a direction other than the direction of the applied moment. Let us investigate this observation rigorously.

2 Pure bending of unsymmetrical beams (start time: 03:40)

Think of an unsymmetrical beam with its axis along x-axis as shown in Figure 3.

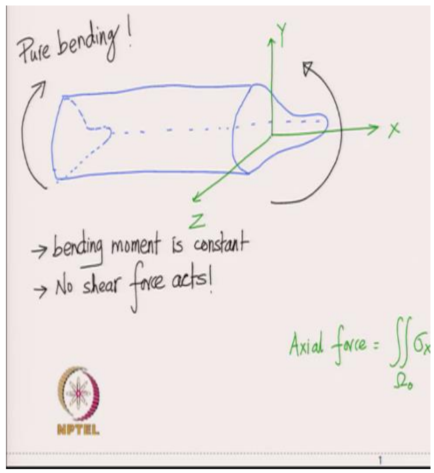

Figure 3: Pure bending of an unsymmetrical beam.

To start with, let us consider the case of pure bending, i.e., a terminal bending moment is applied transverse to the beam's axis. Thus, the bending moment will be constant all along the beam and no shear/axial force will be present in the beam's cross-section. Figure 4 shows a typical cross-section of this beam. As this is the case of an unsymmetrical cross-section, the direction of neutral axis is an unknown too. Let us assume some arbitrary direction (in the plane of the cross-section) for neutral axis which need not pass through the centroid of the cross-section (see Figure 4).

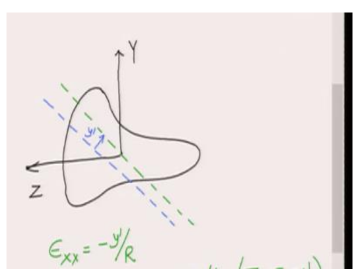

Figure 4: The cross section of the beam shown in Figure 3 with the neutral axis shown in blue.

The beam bends into a circle of radius R about the neutral axis. We then think a line parallel to the neutral axis but at a distance y' from it (shown by the green line in Figure 4). The longitudinal strain ϵ_{xx} for all points lying on this line will be given by

$$\epsilon_{xx} = -\frac{y'}{R} \tag{1}$$

following similar logic as earlier for symmetrical cross-sections. Further assuming σ_{yy} and σ_{zz} to be zero as earlier, σ_{xx} at such points will be given by

$$\sigma_{xx} = E\epsilon_{xx} = -E\frac{y'}{R} \tag{2}$$

The axial force in the cross-section will thus be

$$\iint_{\Omega_0} \sigma_{xx}\, dA = -\frac{E}{R}\iint_{\Omega_0} y' dA. \tag{3}$$

However, as no axial force acts on the cross-section in case of pure bending, this implies

$$-\frac{E}{R}\iint_{\Omega_0} y' dA = 0 \quad \Rightarrow \iint_{\Omega_0} y' dA = 0 \tag{4}$$

Thus, the first moment of the cross-section relative to the neutral axis must be zero which simply means that the neutral axis has to pass through the centroid of the cross-section. So, just like the case of symmetrical cross-sections, neutral axis passes through the centroid even for unsymmetrical cross sections.

We now redraw the cross-section keeping in mind that the neutral axis passes through the centroid as shown in Figure 5 and again consider a point *A* at a distance of *y'* from the neutral axis.

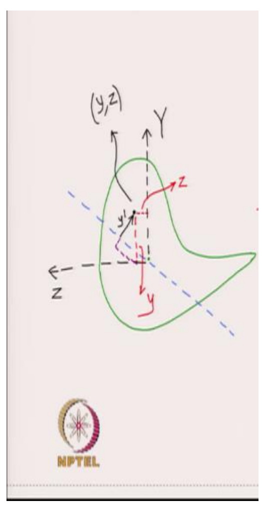

Figure 5: A typical cross-section of the beam with neutral axis passing through the centroid

Let the coordinates of this point be *A*(*y*,*z*) and express *y'* in terms of *y* and *z*. Suppose the neutral axis makes an angle β with the y-axis as shown in Figure 6. Construct two lines, one parallel to the neutral axis from the point *B*(0,*z*) and the other perpendicular to the neutral axis from point *A*(*y*,*z*) as shown in Figure 6.

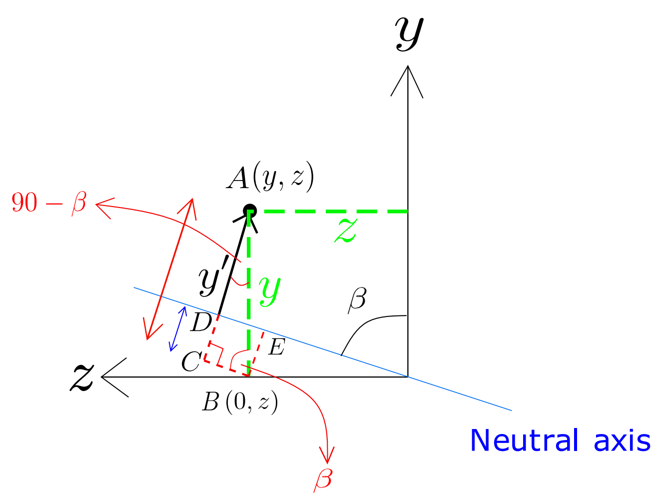

Figure 6: Zoomed view of Figure 5 with relevant dimensions.

The point at which these two lines intersect is denoted by *C*. As the neutral axis makes an angle β with the vertical line, the line *BC*, constructed parallel to the neutral axis, will also be at an angle β from the vertical line. The other angle ($\angle BAC$) in the right triangle *ABC* thus becomes $90^\circ - \beta$. Also, the angle that the neutral axis makes with the z-axis will also be $90^\circ - \beta$. Think of another point *D* at which the neutral axis and *AC* intersect. The length y' will be equal to the difference of the lengths *AC* and *CD* in Figure 6. We now draw a line perpendicular to the neutral axis from the point *B* which intersects the neutral axis at *E*. As *BCDE* becomes a rectangle, *CD* = *BE*. Thus, we get

$$y' = AC - CD = AC - BE = y \sin\beta - z \cos\beta. \tag{5}$$

Substituting this in equations (1) and (2), we get

$$\epsilon_{xx} = \frac{-(y \sin\beta - z\cos\beta)}{R}, \quad \sigma_{xx} = \frac{-E(y\sin\beta - z\cos\beta)}{R} \tag{6}$$

Let us now obtain moment about the cross-section's centroid due to normal stress distribution σ_{xx} which should equal the externally applied moment, i.e.,

$$\begin{aligned}\vec{M}_{/O} &= \iint_{\Omega_0} (y\hat{j} + z\hat{k}) \times \sigma_{xx}\hat{i}\ dA \\ &= \iint_{\Omega_0} (y\hat{j} + z\hat{k}) \times \frac{E}{R}(z\cos\beta - y\sin\beta)\hat{i}\ dA \\ &= \frac{E}{R}\iint_{\Omega_0} \left[(y^2\sin\beta - yz\cos\beta)\hat{k} + (z^2\cos\beta - yz\sin\beta)\hat{j}\right] dA.\end{aligned} \tag{7}$$

Here *E* and *R* could be taken out of the integration. Let us define second area moments of the cross-section as follows:

$$\iint y^2 dA = I_{zz}, \quad \iint yz dA = I_{yz}, \quad \iint z^2 dA = I_{yy}. \tag{8}$$

Using them in (7), we get

$$\overrightarrow{M}_{/O} = \frac{E}{R}\Big[\underbrace{(I_{zz}\sin\beta - I_{yz}\cos\beta)}_{M_z}\hat{k} + \underbrace{(I_{yy}\cos\beta - I_{yz}\sin\beta)}_{M_y}\hat{j}\Big] \qquad (9)$$

So, if we apply external moments M_y and M_z on the beam, we can observe from the above equation that

$$M_y = (I_{yy}\cos\beta - I_{yz}\sin\beta)E\kappa, \qquad M_z = (I_{zz}\sin\beta - I_{yz}\cos\beta)E\kappa. \qquad (10)$$

Here κ denotes the bending Curvature as in previous lectures which equals inverse of radius of curvature R. The angle β, which represents the angle between the neutral axis and y-axis and the bending curvature κ are the two unknowns and we also have two equations in (10) to obtain them. To obtain β, we divide equations (10a) and (10b) which yields

$$\frac{M_y}{M_z} = \frac{I_{yy}\cos\beta - I_{yz}\sin\beta}{I_{zz}\sin\beta - I_{yz}\cos\beta} \qquad (11)$$

This equation can be used to find β for the general case. For a special case when we apply moment only about z-axis, i.e., $M_y = 0$, the numerator of the above equation can be set to zero to yield

$$I_{yy}\cos\beta - I_{yz}\sin\beta = 0 \quad \Rightarrow \tan\beta = \frac{I_{yy}}{I_{yz}} \qquad (12)$$

For cross-sections which are symmetric about y−axis and moment is applied about z−axis, the mixed moment of area I_{yz} vanishes which when substituted in the above equation yields

$$\tan\beta = \infty \Rightarrow \beta = 90^\circ. \qquad (13)$$

This means that the neutral axis makes an angle of 90° with the y-axis. In other words, it coincides with z-axis or the direction of applied bending moment something that we had simply assumed earlier.

Coming back to the general situation for unsymmetrical beams, once we get β from equation (11), we can substitute it in either of the equations (10a) or (10b) to obtain bending curvature κ. For example, in the former case, we would get

$$\kappa = \frac{M_y}{E(I_{yy}\cos\beta - I_{yz}\sin\beta)} \qquad (14)$$

Finally, substituting κ from the above equation in equation (6), we get

$$\begin{aligned}\sigma_{xx} &= \frac{-M_y(y\sin\beta - z\cos\beta)}{(I_{yy}\cos\beta - I_{yz}\sin\beta)} \\ &= \frac{M_z(yI_{yy} - zI_{yz}) + M_y(yI_{yz} - zI_{zz})}{I_{yz}^2 - I_{yy}I_{zz}} \quad \text{(on substituting } \beta \text{ from (11)).}\end{aligned} \qquad (15)$$

2.1 Special Case: When y and z axes are aligned along principal axes (start time: 29:40)

Let us consider a special case where y and z axes are aligned along the principal axes of the crosssection but the bending moment is allowed to act in arbitrary direction (see Figure 7).

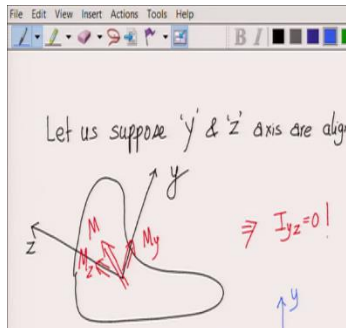

Figure 7: The cross section of an unsymmetrical beam where y and z axes are aligned with the principal axes.

We can resolve this moment along the y and z axes as M_y and M_z, respectively. As y and z axes are principal axes, $I_{yz} = 0$. Substituting this in equation (15) simplifies it greatly, i.e.,

$$\sigma_{xx} = \frac{-M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}} \tag{16}$$

If we apply moment about z axis only, i.e., $M_y = 0$, we get

$$\sigma_{xx} = \frac{-M_z y}{I_{zz}} \tag{17}$$

which is the same result that we had obtained earlier for symmetrical cross sections with only M_z present. If we look at equation (16), we observe that the first term comes with a negative sign whereas the second term comes with a positive sign. To understand this, consider the cross section shown in Figure 8.

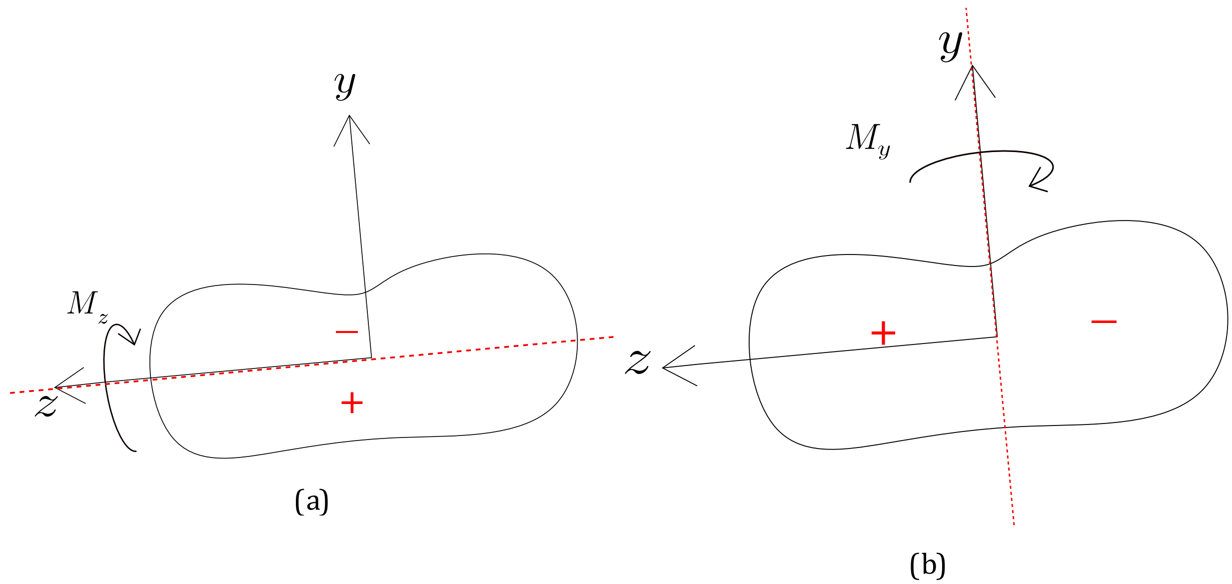

Figure 8: Thin cross-sections of beams with shear stress distribution shown for open ones

When we apply moment about z axis, the bending is such that the top side (+y side) is under compression whereas the bottom (-y side) is under tension. Thus, we have a negative sign in front of the first term in equation (16). But, if we apply a moment about y-axis, the beam bends in such a way that the part of the beam on the left side (+z side) undergoes tension while the other part undergoes compression. Since, the positive side is under tension, there is a positive sign in front of the second term in equation (16). We can also conclude that even though the cross section is unsymmetrical, if we apply moment along principal axis, the bending happens as if it were a symmetrical cross-section! The neutral axis gets aligned with the direction of the applied moment too leading to the formula for bending stress which is the same as the one for symmetrical cross-sections.

3 Non-uniform bending of unsymmetrical cross-sections (start time: 37:05)

In non-uniform bending, a shear force is also present in the cross section due to which the bending moment varies along the length of the beam. The direction of neutral axis will again be governed by equation (11). To obtain shear stress distribution in the cross-section, let us look at the line which is at a distance of y' from the neutral axis. For symmetrical beams, for simplicity, we had assumed uniform shear stress distribution on lines parallel to neutral axis as shown in Figure 9.

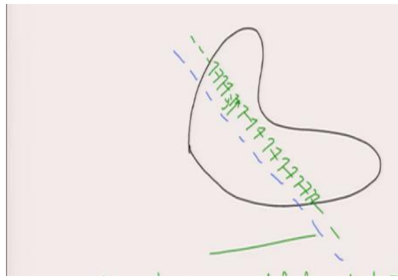

Figure 9: Shear stress along a line parallel to the neutral axis if we make an assumption of uniform shear distribution.

Will such an assumption be valid for unsymmetrical beams also? It turns out that we cannot make such an assumption. This is because in that case, at points on the cross-sectional boundary, the shear component of traction would not be tangential to the cross-section's boundary. Whenever we have a beam having symmetrical or unsymmetrical cross-section and its lateral surface is free of externally applied load, then, along the periphery of the cross-section, shear traction has to be directed along the periphery as shown in Figure 10.

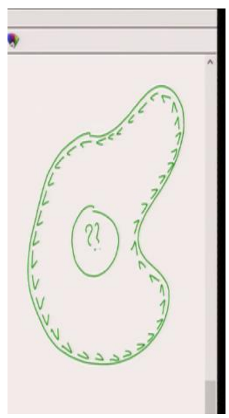

Figure 10: Shear stress distribution along the periphery of the cross-section of a beam when the lateral surface of the beam is traction free

However, we cannot comment about shear stress distribution at points away from the periphery just using the free surface condition. Essentially, we cannot assume uniform shear stress distribution barring us from obtaining any analytical result. However, we do obtain analytical result for special types of cross-sections - they have to be thin and open which we discuss now.

3.1 Shear stress distribution in thin and open cross-sections (start time: 42:11)

Figure 11 shows three different cross sections all of which are thin, i.e., their thickness is very small.

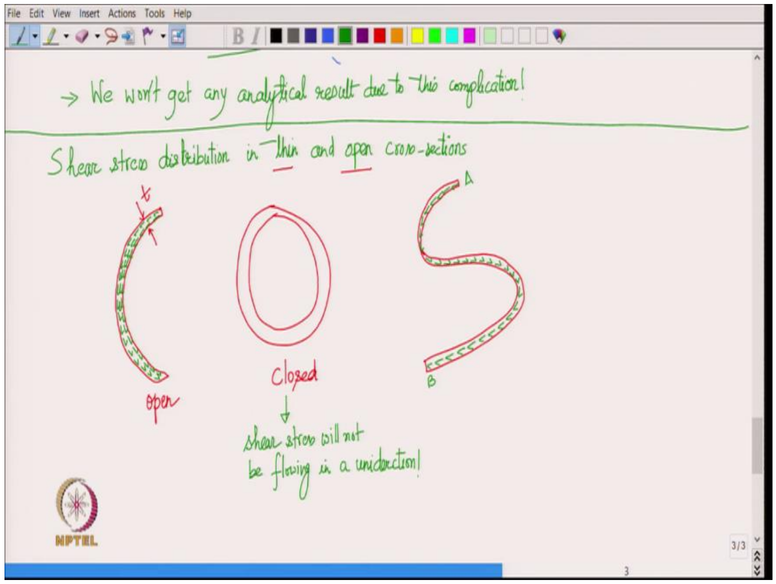

Figure 11: Thin cross-sections of beams with shear stress distribution shown for open ones

The second figure is a closed cross-section while first and third are open cross-sections. We know that the shear stress at points which are close to the periphery of the cross-section has to be directed along the periphery itself. At points away from the periphery, the direction of shear stress is an unknown. For thin cross-sections, however, the thickness is so small that all points can be safely assumed to lie near the periphery. We can thus assume that the shear stress distribution is along the periphery at all points through the thickness too. In fact, one can also assume the magnitude of shear stress to be uniform through the thickness due to such a small thickness. Moreover, for open cross sections, the shear stress must flow from one end of the cross-section to the other end as shown in the open cross-sections in Figure 11 - this flow may be directed oppositely too which gets known only after solving. For closed cross-sections, there are no definite ends though which does not allow the flow to be unidirectional. So, we will consider only thin and open cross-sections as they are easy to analyze.

Let us consider a general thin and open cross-section beam as shown in Figure 12.

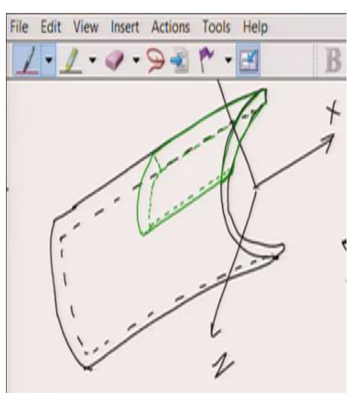

Figure 12: A small element is considered in a beam with thin and open cross-section

As the shear flows from one end to the other and that it is uniform through the thickness, one can paramterize its distribution using an arc-length coordinate s along the cross-section's periphery as shown in Figure 13.

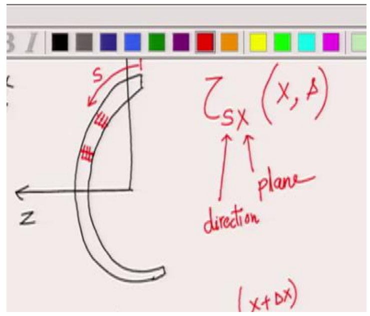

Figure 13: A typical cross-section of the beam shown in Figure 12 together with the arc-length coordinate s.

We can further safely assume the shear to flow in the direction of increasing s. If the distribution comes out to be negative, it would simply mean that the shear actually flows in the other direction. We can also denote the shear stress as τ_{sx}: x denotes the plane of the cross-section in which shear stress is acting while s dentoes the direction of shear stress. Furthermore, τ_{sx} will be a function of s and x only. To find τ_{sx}, we consider a small element at one end of the beam as shown in Figure 12. All the forces acting on this element in the x-direction are shown in Figure 14.

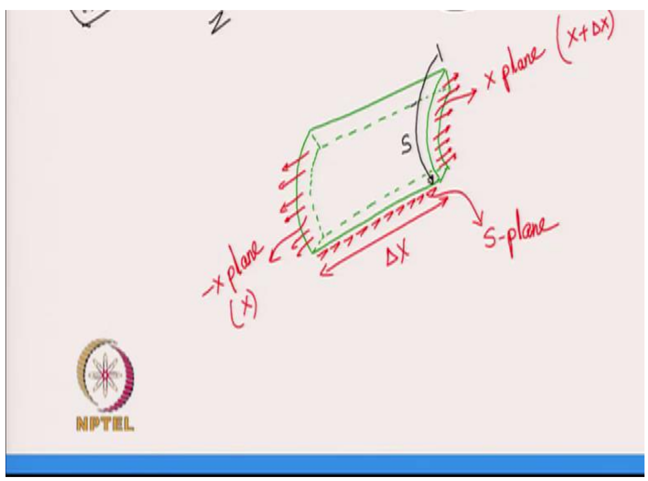

Figure 14: The small element considered in Figure 12 with all forces in the x-direction shown.

The end faces (whose normals are in s direction) will be called the s faces. On $+x$ and $-x$ planes, σ_{xx} acts. The $-s$ plane at $s = 0$ is traction-free while τ_{xs} acts on $+s$ plane at arc-length s in $+x$-direction. The total force on this element in x-direction must be zero for equilibrium, i.e.,

$$\iint_{\text{x-plane}} [\sigma_{xx}(x+\Delta x, y, z) - \sigma_{xx}(x, y, z)]dA + \iint_{\text{s-plane}} \tau_{xs}(\xi, s)dA = 0 \tag{18}$$

Here, ξ denotes the local coordinate in x-direction. It varies from x to $x+\Delta x$ for points in the small element. As the above equation is valid for all elements regardless of the size Δx, we can shrink the size Δx to zero. Let us first divide both the sides by Δx and then take limit $\Delta x \to 0$, i.e.,

$$\lim_{\Delta x \to 0} \iint_{\text{x-plane}} \frac{[\sigma_{xx}(x+\Delta x, y, z) - \sigma_{xx}(x, y, z)]dA}{\Delta x} + \lim_{\Delta x \to 0} \iint_{\text{s-plane}} \frac{\tau_{xs}(\xi, s)dA}{\Delta x} = 0. \tag{19}$$

In the second integral, as the integrand τ_{xs} does not vary through the thickness, the integration through the thickness yields $\tau_{xs}\, t_s$ where t_s denotes the thickness of the cross-section at s. The second area integral thus gets converted into just a line integral along x, i.e.,

$$\begin{aligned}
&\iint_{\text{x-plane}} \sigma'_{xx}(x, y, z)dA + t_s \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} \tau_{xs}(\xi, s)d\xi}{\Delta x} = 0 \\
\implies \quad &\iint_{\text{x-plane}} \sigma'_{xx}(x, y, z)dA + t_s\ \tau_{xs}(x, s) = 0 \\
\text{or,} \quad &\tau_{xs}(x, s) = -\frac{1}{t_s} \iint_{\text{x-plane}} \sigma'_{xx}(x, y, z)dA.
\end{aligned} \tag{20}$$

We can now plug in the expression of σ_{xx} from equation (15) which yields

$$\tau_{xs} = -\frac{1}{t_s} \iint_{\text{x-plane}} \frac{M'_z(yI_{yy} - zI_{yz}) + M'_y(yI_{yz} - zI_{zz})}{I_{yz}^2 - I_{yy}I_{zz}} dA. \tag{21}$$

We can then replace the derivatives of moments with shear forces using the following relation which we had derived in the last lecture:

$$\frac{dM_z}{dx} = -V_y, \quad \frac{dM_y}{dx} = V_z \tag{22}$$

Upon plugging them into (21), we get

$$\tau_{xs} = -\frac{1}{t_s} \iint_{\text{x-plane}} \frac{-V_y(yI_{yy} - zI_{yz}) + V_z(yI_{yz} - zI_{zz})}{I_{yz}^2 - I_{yy}I_{zz}} dA. \tag{23}$$

As shear forces and the second area moments are constants for a cross-section, the denominator of the integrand can be taken out of the integration. We just need to work out the integrals $\iint y dA$ and $\iint z dA$ in the numerator. These integrals are over the x-plane of the small element only and not over the entire cross section. Figure 15 shows the area of this integration going from the arc-length coordinate 0 to s as shaded.

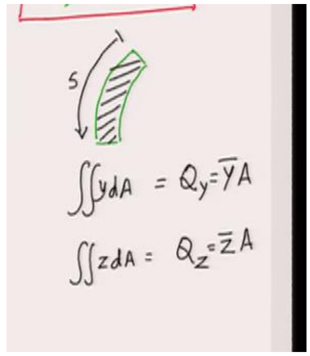

Figure 15: x-plane of the small element considered in Figure 14 denoted as the shaded area.

The integrations $\iint ydA$ and $\iint zdA$ over this shaded part of the x-plane would give us the y and z coordinates of the centroid of the shaded area multiplied with the area of shaded region which we denote by

$$Q_y^s = \iint ydA = \bar{Y}^s A^s, \quad Q_z^s = \iint zdA = \bar{Z}^s A^s \tag{24}$$

Here, $\bar{Y}^s$ and $\bar{Z}^s$ are the centroidal coordinates just the shaded region (from s = 0 to s = s) and not the whole cross-section. Similarly, A^s is the area of just the shaded region. Upon plugging them into equation (23) and further noting that $\tau_{xs} = \tau_{sx}$, we get

$$\tau_{sx} = -\frac{1}{t_s(I_{yz}^2 - I_{yy}I_{zz})}\left[-V_y(Q_y^s I_{yy} - Q_z^s I_{yz}) + V_z(Q_y^s I_{yz} - Q_z^s I_{zz})\right] \tag{25}$$

This is the general formula for shear stress distribution in thin and open cross-sections. We can again consider the special case where y and z axes are aligned along the principal axes in which case I_{yz} vanishes. The above formula then simplifies to

$$\tau_{sx} = \frac{1}{I_{yy}I_{zz}t_s}\left[-V_y Q_y^s I_{yy} - V_z Q_z^s I_{zz}\right] = -\frac{V_y Q_y^s}{I_{zz}t_s} - \frac{V_z Q_z^s}{I_{yy}t_s} \tag{26}$$

If we further assume $V_z = 0$, our formula reduces to

$$\tau_{sx} = -\frac{V_y Q_y^s}{I_{zz}t_s} \tag{27}$$

This is the same expression that we had derived earlier for symmetrical cross-sections except that there is a negative sign here. This difference arises because the direction of increasing s that we have chosen coincides more with $-y$ direction than $+y$ direction.