Solid Mechanics
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Lecture - 23
Pure bending of rectangular beams

Hello everyone! Welcome to Lecture 23! We will learn about a new concept, i.e., bending of beams.

1 Introduction (start time: 00:27)

You may have come across beams subjected to shear force and bending moment in your first year mechanics course. Figure 1 shows a beam subjected to such loadings. There is a distributed load (force per unit length) acting along the length of the beam, a terminal moment at the right end and also a couple at a point in the beam.

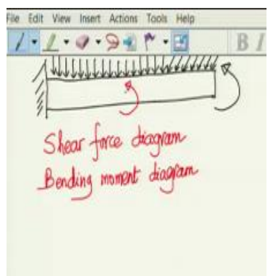

Figure 1: A beam subjected to distributed load, terminal moment and a couple.

You may have learnt how to draw shear force and bending moment diagrams for such cases but assumed the beam to not deform. In this course, we will take another step and learn how beams deform or get curved when they are subjected to such loads. The simplest case is the pure bending of beams. Suppose we have a rectangular beam as shown in Figure 2. The axis of the beam is along *X* axis. Equal and opposite moments of magnitude M_z are applied on its end cross-sections along *Z* direction, i.e., transverse to the beam.

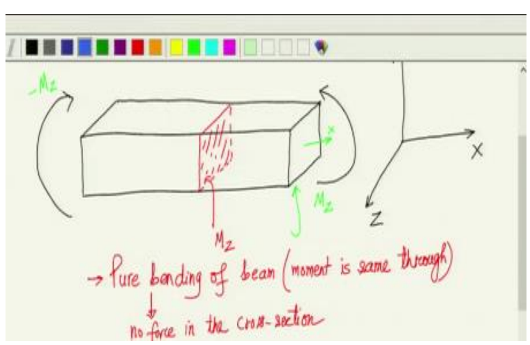

Figure 2: A rectangular beam subjected to terminal moments in a direction transverse to its axis

As the total moment on the system is zero, the beam will be in static equilibrium but the beam also deforms such that it gets curved/bent, e.g., see Figure 3.

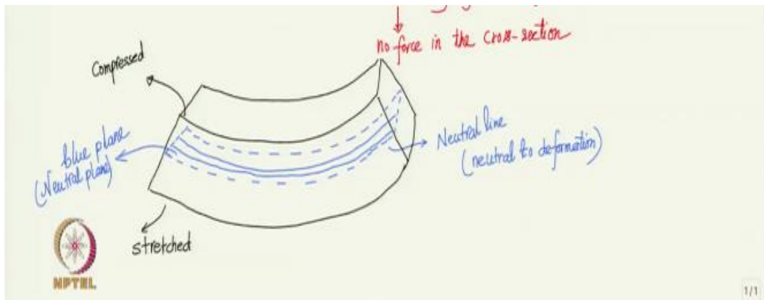

Figure 3: A typical beam under pure bending.

Such a deformation is called bending of beams and the applied moment is called bending moment. Let us cut an arbitrary section in the beam as shown in red in Figure 2. Using moment balance, we can easily conclude that the internal moment on this section is also M_z. Thus, we have the same internal moment acting at every section of the beam. When the same moment acts on all the beam's cross-sections without the presence of any internal force, it is called the case of pure bending. A point to note here is that for a beam to bend, there should be a component of moment perpendicular to the axis of the beam. If the moment acts along the axis, it acts as a torque and leads to torsion of beams, not bending.

2 Analysis (start time: 07:24)

It turns out that in case of pure bending, the beam deforms to become an arc of a circle as shown in Figure 3. Furthermore, planar cross-sections remain planar even after bending.

2.1 Neutral plane and Neutral lines (start time: 07:48)

If we visualize the pure bending deformation, we can note that the lines parallel to the axis of the beam on the top surface get shortened or compressed while those on the bottom surface get stretched. As we go from the bottom surface to the top surface, the deformation of such lines change from getting stretched to getting compressed. So, there must be a surface in between whose all line elements (along the beam's axis) do not undergo any elongation/compression. This surface is called the neutral plane (see the shaded plane in Figure 3) as such lines on this plane remain neutral to deformation. These line elements on the neutral plane are also called neutral lines.

2.2 Longitudinal strain (start time: 11:35)

Let us call the lines parallel to the beam's axis axial/longitudinal fibers and find the amount of elongation/compression of such fibers. We can consider the front view of the beam as shown in Figure 4 to simplify the analysis assuming that the deformation is independent of the coordinate along the width of the beam or z axis. We further choose a local coordinate system such that x-axis is along the horizontal line or tangent to the fibers, *y*-axis is along the vertical line pointing towards the center and *z*-axis comes out of the plane. The neutral line is shown as a solid blue line there.

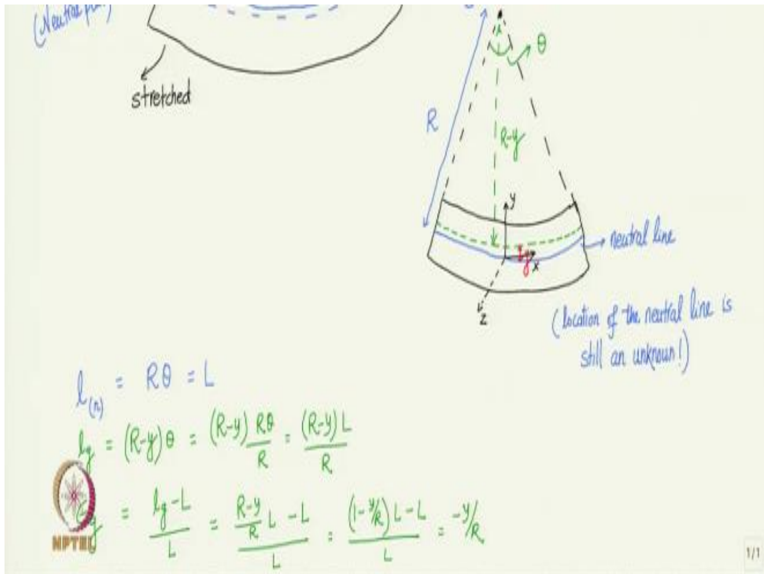

Figure 4: The front view of the deformed beam for the case of pure bending: the center of the circular arcs that the longitudinal fibers deform into is also shown

It should be noted that we have not yet found the location of the neutral line. Let us consider a longitudinal fiber (shown by the dotted green line) at a distance *y* from the neutral line. First of all, we can notice that all these lines, i.e, the dotted green line, the blue line, the topmost and bottommost longitudinal lines form arcs of different concentric circles. If the radius of the circle corresponding to the neutral line is denoted by *R* (also called radius of curvature from now on), the radius of the circle corresponding to the dotted green line will be $R - y$. Let us denote the angle subtended by these arcs at the center as θ. The length of the the neutral line l_n will be

$$l_n = R\theta. \tag{1}$$

which will also equal the beam's length *L* when it is straight, i.e,

$$l_n = R\theta = L \tag{2}$$

since the length of the neutral line does not change. Similarly, the length of the green line will be

$$\begin{aligned} l_g &= (R-y)\theta = (R-y)\frac{R\theta}{R} = (R-y)\frac{L}{R} \quad \text{(using (2))} \\ &= \left(1 - \frac{y}{R}\right)L \end{aligned} \tag{3}$$

As the undeformed length of all longitudinal lines is *L*, the longitudinal strain for the green line will be

$$\epsilon_{xx} = \epsilon_g = \frac{\Delta L}{L} = \frac{l_g - L}{L} = \frac{-y}{R} \quad \text{(using (3)).} \tag{4}$$

We have thus obtained longitudinal strain for a general longitudinal line which is at a distance *y* from the neutral line. Let us again draw the cross-section of the beam as shown in Figure 5.

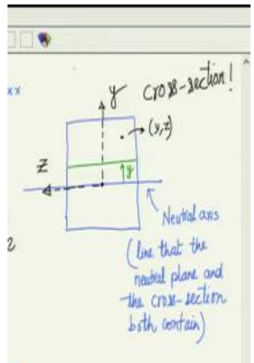

Figure 5: A typical cross-section of a rectangular beam with the neutral axis and in-plane coordinate axes shown

A neutral axis is shown here which is the intersection of the neutral plane with the cross-sectional plane. The green line is shown at a distance *y* from the neutral axis. We then consider a general point (*y,z*) in the cross-section. We found ϵ_{xx} to be $\frac{-y}{R}$, i.e., dependent on *y* but independent of *z*. So, any line with a constant *y* (like the green line) will have the same ϵ_{xx}.

2.3 Longitudinal stress (start time: 21:46)

To get σ_{xx}, we can use the following stress-strain relationship for isotropic materials:

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) \tag{5}$$

Let us look at Figure 2 again. For the case of pure bending, we are only applying moments at the end cross-sections. We are not applying any load on the front, back, top and bottom surfaces. On these surfaces, all components of traction will therefore be zero. Thus, σ_{yy} is zero on the top and bottom surfaces while σ_{zz} is zero on the front and back surfaces. However, we make an approximation that σ_{yy} and σ_{zz} are zero at all points within the beam too. This is a good approximation because as the beam bends, it is allowed to relax freely in *y* and *z* directions. Using this approximation in equation (5), we get

$$\sigma_{xx} = E\epsilon_{xx} = -E\frac{y}{R} \tag{6}$$

which has been plotted in the cross-sectional plane in Figure 6. Notice that σ_{xx} is compressive above the neutral axis and tensile below it.

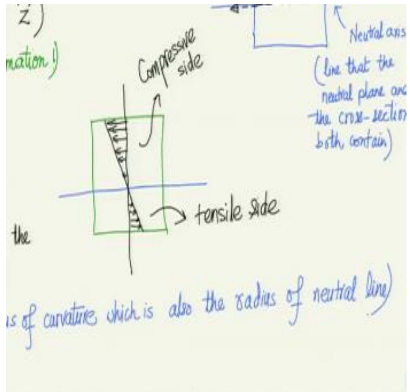

Figure 6: Variation in σ_{xx} along the beam's height.

2.4 Location of the neutral axis (start time: 26:51)

In the pure bending scenario, every cross-section only has bending moment acting on it. No force acts in the cross sections. Hence, the integration of σ_{xx} in the cross-sectional plane must vanish, i.e.,

$$\int_{\Omega_0} \sigma_{xx}\, dA = 0 \quad \Rightarrow -\int_{\Omega_0} \frac{Ey}{R} = 0 \tag{7}$$

Furthermore, *R* can be taken out of the integral since it denotes radius of curvature of the deformed neutral line and hence is a constant for the entire cross-section. This reduces the above integral to

$$\begin{aligned} \frac{-E}{R} \int_{\Omega_0} y dA = 0 \\ \Rightarrow \quad \bar{Y}_c A = 0 \text{ or } \bar{Y}_c = 0 \end{aligned} \tag{8}$$

Here $\bar{Y}_c$ is the centroid of the cross-section. The above equation implies that the neutral axis must coincide with the centroidal axis of the cross-section. This allows us to obtain the location of neutral axis but the radius of curvature *R* of the deformed neutral line is still an unknown.

2.5 Radius of curvature of neutral line (start time: 32:32)

To find *R*, we first obtain expression for the moment (about the center of the cross-section) due to internal traction in the cross-sectional plane as follows:

$$\begin{aligned} \overrightarrow{M}/_O &= \iint_{\Omega_0} (y\hat{j} + z\hat{k}) \times \underline{t}^x dA \\ &= \iint_{\Omega_0} (y\hat{j} + z\hat{k}) \times (\sigma_{xx}\hat{i} + \tau_{yx}\hat{j} + \tau_{zx}\hat{k}) dA. \end{aligned} \tag{9}$$

We have already obtained expression of σ_{xx} but we don't know the variation of τ_{yx} and τ_{zx} in the cross-sectional plane yet. Let us just obtain the component of above moment along z axis which we can equate to the externally applied bending moment, i.e.,

$$\begin{aligned} M_z = \overrightarrow{M}/_O \cdot \hat{k} &= \iint_{\Omega_0} \left[(y\hat{j} + z\hat{k}) \times (\sigma_{xx}\hat{i} + \tau_{yx}\hat{j} + \tau_{zx}\hat{k}) \right] \cdot \hat{k} \ dA \\ &= -\iint_{\Omega_0} y\sigma_{xx} dA = \frac{E}{R} \iint_{\Omega_0} y^2 dA \qquad \text{(using (6))} \end{aligned} \tag{10}$$

Notice that the terms corresponding to shear traction components vanished. The term in the integral above is the second moment of area about z-axis and is denoted by I_{zz}. Thus, we finally have

$$M_z = EI_{zz}\frac{1}{R} \Rightarrow R = \frac{EI_{zz}}{M_z} \tag{11}$$

The quantity $\frac{1}{R}$ is also called bending curvature and is denoted by κ. We can also define a bending modulus if we write M_z in terms of curvature, i.e.,

$$M_z = \underbrace{EI_{zz}}_{\text{Bending Modulus}} \kappa. \tag{12}$$

Notice that the bending modulus is the product of Young's modulus E and geometric property I_{zz}. The curvature that gets induced in the beam on application of bending moment is inversely proportional to this bending modulus. If the bending modulus is high, the curvature induced would be smaller which means a large bending moment is required to curve/bend the beam. Note that large curvature implies more curved beam because the radius of curvature is smaller then whereas small curvature represents an almost straight beam because the radius of curvature is large then. This is also demonstrated in Figure 7.

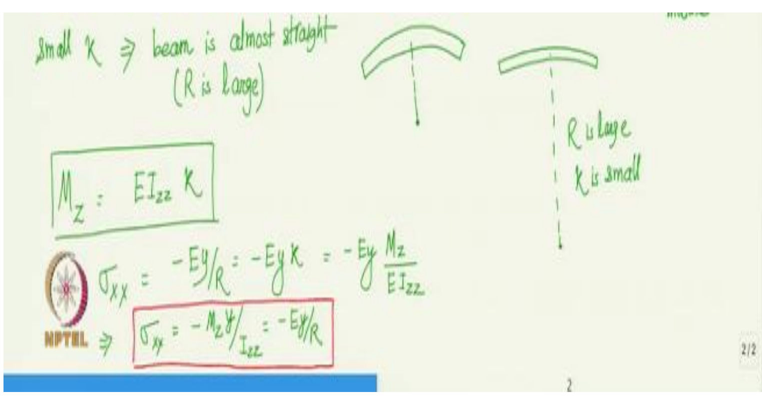

Figure 7: Curved beams with large and small curvatures are shown on the left and right, respectively.

We can now rewrite the expression of σ_{xx} in (6) upon substituting the expression of curvature κ in it, i.e.,

$$\begin{aligned}\sigma_{xx} &= -E\frac{y}{R} = -Ey\frac{M_z}{EI_{zz}} \quad \text{(using (12))} \\ &= \frac{-M_z y}{I_{zz}}. \end{aligned} \tag{13}$$

Thus, we have two formulae for σ_{xx}, i.e.,

$$\boxed{\sigma_{xx} = \frac{-M_z y}{I_{zz}} = -E\frac{y}{R}} \tag{14}$$

2.6 I_{zz} for a rectangular cross section (start time: 47:07)

Figure 8 shows a rectangular cross section of height h and width b. The neutral axis is also shown about which the moment is to be calculated. A small rectangular strip of height dy at a distance of y from the neutral axis is considered.

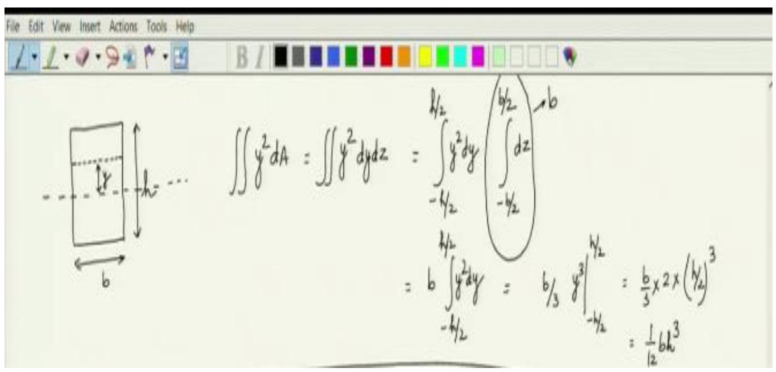

Figure 8: A rectangular cross section with a rectangular strip shown at a distance of y from the neutral axis.

The second area moment about the neutral axis will be

$$\begin{aligned} I_{zz} &= \iint y^2 dA = \iint y^2 dy dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy \int_{-\frac{b}{2}}^{\frac{b}{2}} dz \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 \; b dy = \frac{b}{3} y^3 \Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{1}{12} bh^3. \end{aligned} \tag{15}$$

So, I_{zz} increases with both b and h but is more sensitive to h. A small increase in h brings about a large increase in I_{zz}. We can imagine two rectangular beams made up of the same material and same cross-sectional area (bh) with the first one having smaller height h than the second one. Because of larger h, the second beam will have larger I_{zz} and thus a large bending modulus when compared to the first beam. So, it will be much more difficult to bend the second beam although both the beams have the same cross-sectional area.

2.7 Shape of the deformed cross section (start time: 50:54)

To find the shape of the deformed cross section, we need to find in-plane normal strains ϵ_{yy} and ϵ_{zz}. Using stress-strain relation, we can write

$$\begin{aligned} \epsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) \\ &= \frac{1}{E}(0 - \nu(\sigma_{xx} + 0)) \\ &= \frac{-\nu}{E}\sigma_{xx} = \nu\frac{y}{R} \qquad \text{(using (6))} \end{aligned} \tag{16}$$

Here, we substituted σ_{yy} and σ_{zz} to be zero as mentioned earlier. Similarly, ϵ_{zz} will be

$$\epsilon_{zz} = \nu\frac{y}{R} \tag{17}$$

Let us now consider a line parallel to *y*-axis in the undeformed cross-section as shown by the green line in Figure 9.

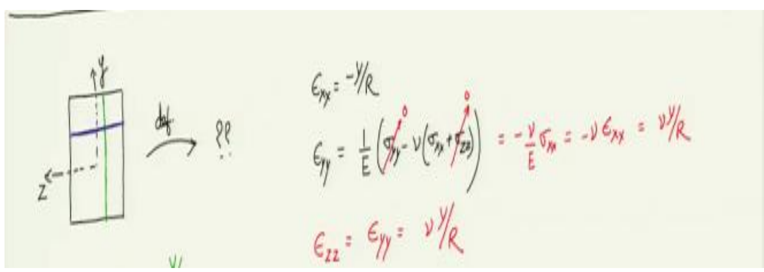

Figure 9: A typical undeformed cross-section of a rectangular beam in pure bending with lines parallel to *y*-axis and *z*-axis shown

As ϵ_{yy} *is* $\nu\frac{y}{R}$, any point above the neutral axis has positive strain while the corresponding point below the neutral axis will have negative strain of equal magnitude. Thus, upper half of the line drawn parallel to *y*-axis will undergo positive longitudinal strain while the lower half will undergo equal but negative longitudinal strain. So, the total length of this line will not change. Now, we consider a horizontal line parallel to *z* axis as shown by the blue line in Figure 9. As ϵ_{zz} *is* $\nu\frac{y}{R}$, every point in the blue line will have same ϵ_{zz}. If these lines are above the neutral axis, they will undergo elongation whereas if they are below the neutral axis, they will undergo compression. Furthermore, this strain being proportional to *y*, the change in length of such lines parallel to *z*-axis will happen linearly. If we draw the deformed cross-section considering the effects of ϵ_{zz} only, it will look like the one shown on the right of Figure 10. It becomes a trapezoid.

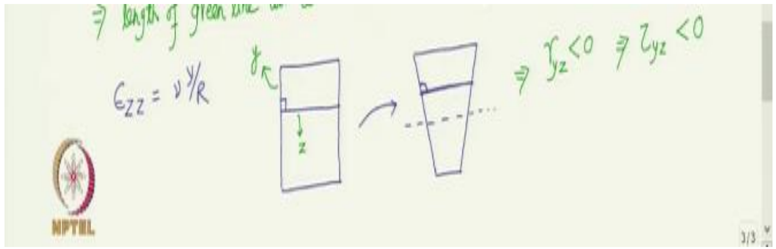

Figure 10: Undeformed and deformed cross-sections considering the effects of ϵ_{zz} only.

If we observe the undeformed and deformed configurations shown in Figure 10 more closely, we see that the lines parallel to y-axis and z-axis which were perpendicular initially now form an angle greater than 90°. This change in angle can happen only if shearing takes place. As the angle has increased, we can infer that γ_{yz} is negative. From stress-strain relation for isotropic materials, we know that if $\gamma_{yz} < 0$, then $\tau_{yz} < 0$. Let us consider the two lateral surfaces of the beam with their normals parallel to z axis. As these surfaces are traction free, all the traction components on the z surface must be zero including τ_{yz}. However, if the beam's cross-section deforms as shown in Figure 10, τ_{yz} on $z-$surface turns out to be non-zero which is a contradiction. Thus, the lines parallel to the z-axis in the cross section must also curve along with undergoing change in length. These lines must curve in such a manner that the angle between the lines parallel to y-axis and z-axis near the $z-$ surface remain perpendicular. This will ensure that shear strain and hence shear stress near the $z-$surface points are zero. The curvature of these lines should be convex upward as shown in Figure 11.

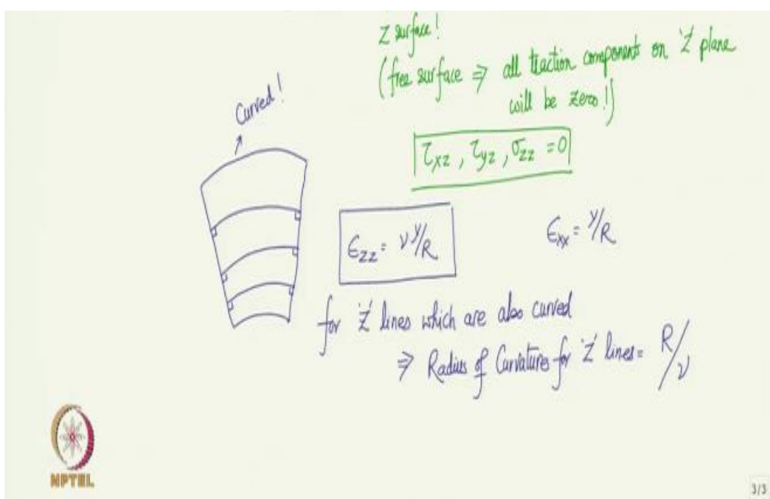

Figure 11: Final deformed configuration of the cross-section in pure bending.

We know that the radius of curvature for the deformed neutral line of the beam is R. If we compare equations (4) and (17), we can say that the radius of curvature of the centroidal line parallel to z axis would be $\frac{R}{\nu}$. With this, we can close the topic of pure bending. In the next lecture, we will discuss about bending of beams when they are subjected to transverse load.