Solid Mechanics
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Lecture - 20
Extension-torsion-inflation in a hollow cylinder

Hello everyone! Welcome to Lecture 20! In this lecture, we will discuss about extension-torsion-inflation in a hollow cylinder.

1 Introduction (start time: 00:28)

1.1 Extension (start time: 00:34)

If we apply axial force on a cylinder by applying normal traction at the end cross-sections as shown in Figure 1, the cylinder gets stretched. Such deformations are called extension.

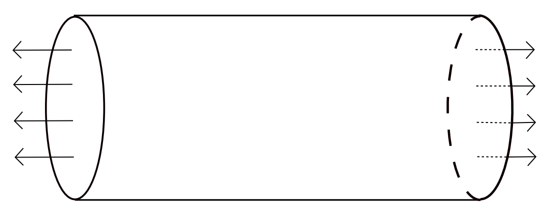

Figure 1: Normal traction applied at the ends of a cylinder to generate stretch in it.

1.2 Torsion (start time: 01:38)

If we hold the two ends of a cylinder and rotate them in opposite directions as shown in Figure 2, different cross sections of the cylinder get rotated by different angles. Such a deformation is called twisting or torsion and requires torque of equal and opposite directions applied at the ends of the cylinder. A torque is a form of moment which is applied along the axis of the beam.

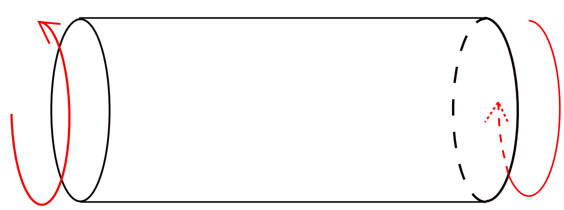

Figure 2: The two ends of a cylinder are rotated in opposite directions to twist the cylinder.

1.3 Inflation (start time: 02:06)

If we apply pressure to a hollow cylinder from within the inner cavity as shown in Figure 3, the radius of the cylinder changes. Similarly, when we fill air in a balloon, the radius of the balloon increases. Such deformations are called inflation.

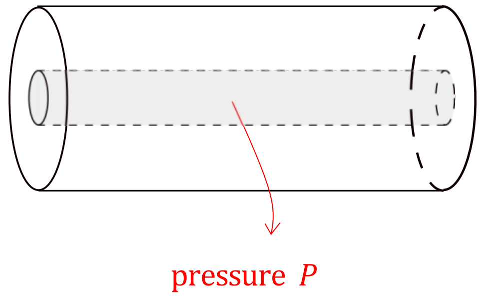

Figure 3: An internal pressure acts on a hollow cylinder.

2 Combined extension-torsion-inflation of a hollow cylinder (start time: 02:44)

2.1 Problem Definition (start time: 02:44)

Let us consider a hollow cylinder subjected to axial force (F), torque (T) as well as internal pressure (P) as shown in Figure 4. Let us also suppose that the end-to-end rotation of the cylinder due to torque is denoted by Ω. Our goal is to find all the displacement and stress components induced within the cylinder. We will solve this problem using cylindrical coordinates. Thus, we need to find u_r, u_θ, u_z, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, $\tau_{r\theta}$, $\tau_{\theta z}$ and τ_{rz}.

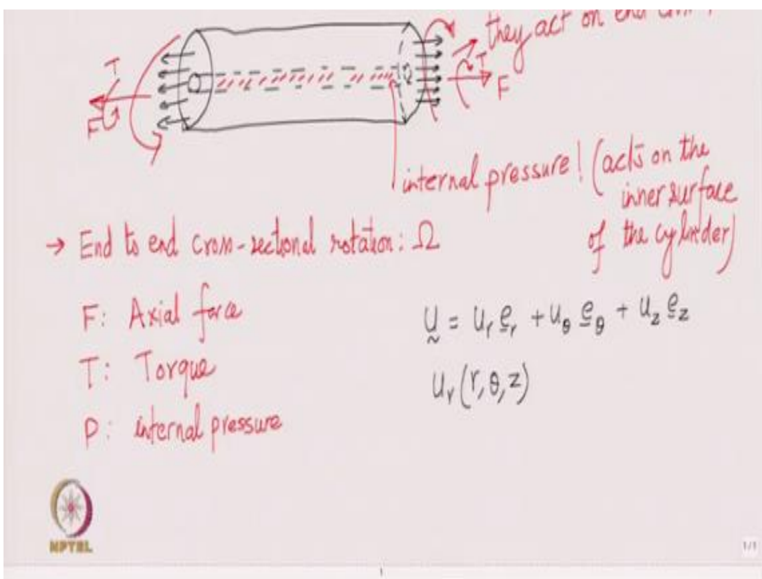

Figure 4: A hollow cylinder which is stretched, twisted and inflated.

2.2 Simplification using physical considerations (start time: 08:17)

Before we start solving the problem mathematically, let us simplify the problem based on physical consideration. The displacement $\underline{u}$, in general, is given by

$$\underline{u} = u_r\underline{e}_r + u_\theta\underline{e}_\theta + u_z\underline{e}_z \tag{1}$$

where each of the displacement components is a function of r, θ and z, i.e.,

$$u_r = u_r(r,\theta,z), \qquad u_\theta = u_\theta(r,\theta,z), \qquad u_z = u_z(r,\theta,z). \tag{2}$$

However, as we are analyzing a special deformation here, some of the dependencies of the displacement components on r, θ and z can be removed as we show now.

2.2.1 Axisymmetry (start time: 09:39)

We can imagine that upon applying axial force, torque and pressure to an axisymmetric cylinder, the deformation induced would also have to be axisymmetric. This automatically implies that none of the displacement components depend on θ, i.e., if we consider any two points in the cylinder with the same r and z coordinates but different θ coordinate, the displacements at the two points are same. For example, if u_r depends on θ, circular cross sections will not remain circular after deformation (see Figure 5). However, such a deformation is possible for arbitrary deformation of the cylinder but for the special case considered here, that is not permissible. Similarly, if u_θ or u_z changes with θ, axisymmetry will be lost. Thus, using axisymmetry, we can write

$$u_r = u_r(r,z), \qquad u_\theta = u_\theta(r,z), \qquad u_z = u_z(r,z). \tag{3}$$

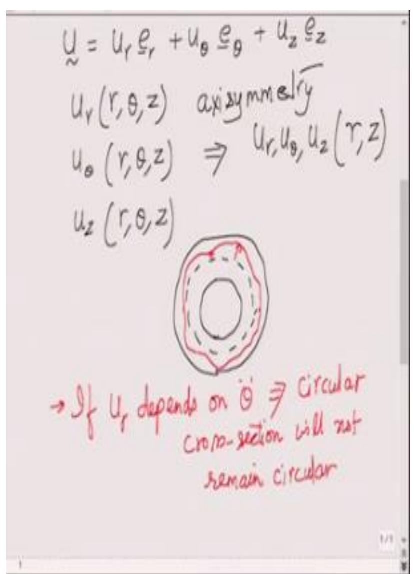

Figure 5: The cross section of a hollow cylinder is shown. A circumferential line element is shown before and after deformation for the case when u_r changes with θ.

2.2.2 Axial Homogeneity (start time: 13:18)

If u_r is a function of z, homogeneity along the axis will be lost. Figure 6 displays such a case where a simple cylinder (constant radius along the axis) with outer radius r_o is is shown. If u_r changes with z, the radius of the cylinder at different points on the axis will be different and the deformed configuration will not be a simple cylinder. This is usually the case when we stretch a cylinder while holding the end cross-sections rigidly. The end cross section radius does not change but the radius of the cross sections between the two ends decreases due to Poisson's effect.

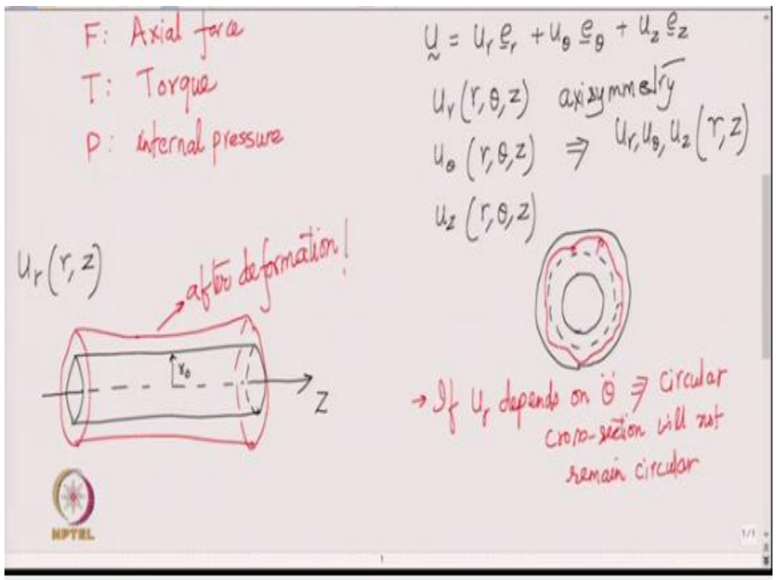

Figure 6: A cylinder before and after deformation is shown for the case where u_r changes with z.

The analysis of such a deformation is more difficult. Let us imagine a simpler physical case where we are allowing the end cross sections to also shrink - we just need to not hold the end cross-sections rigidly and let them relax. In such a case, all the cross sections of the cylinder will undergo same decrease in radius giving us an axially homogeneous deformed configuration. Thus, u_r would be independent of z, i.e,

$$u_r(r,z) \Rightarrow u_r(r). \tag{4}$$

2.2.3 No warping of the cross-section (start time: 17:38)

We can think of the cylinder again and consider its planar cross-section whose normal is along the axis of the cylinder. All the points in the cross-section have the same z coordinate. The displacement component u_z displaces points in the axial direction. If u_z changes with r, two points on such cross sections (having different radial coordinate) would displace in the axial direction by different amounts. This will make the deformed cross section non-planar which is also warping of the ccross-section. We assume no such warping occurs. So, u_z must be independent of r, i.e.,

$$u_z(r,z) \Rightarrow u_z(z) \tag{5}$$

2.2.4 u_θ generated due to torsion (start time: 20:30)

We also have some restrictions on u_θ which we explain now. The extension of the cylinder generates both axial and radial (due to Poisson's effect) components of displacement but no u_θ for isotropic tubes. Similarly, by applying uniform pressure, we can generate radial and axial displacements (due to Poisson's effect) but no u_θ again for isotropic cylinders. An application of torque, on the other hand, causes a typical cross-section to rotate and hence generates only u_θ for isotropic tubes. To quantify this displacement, let us consider a cross section of the cylinder as shown in Figure 7 which rotates by an angle α. An arbitrary point on the cross section having initial coordinate (r,θ,z) displaces to $(r,\theta+\alpha,z)$. The arc of rotation of the point is shown as a solid red curve whose length equals $r\alpha$. This arc would become a straight line in $\underline{e}_\theta$ direction when α is very small. Thus, we get

$$u_\theta = \alpha r \text{ (if } \alpha \text{ is very small)} \tag{6}$$

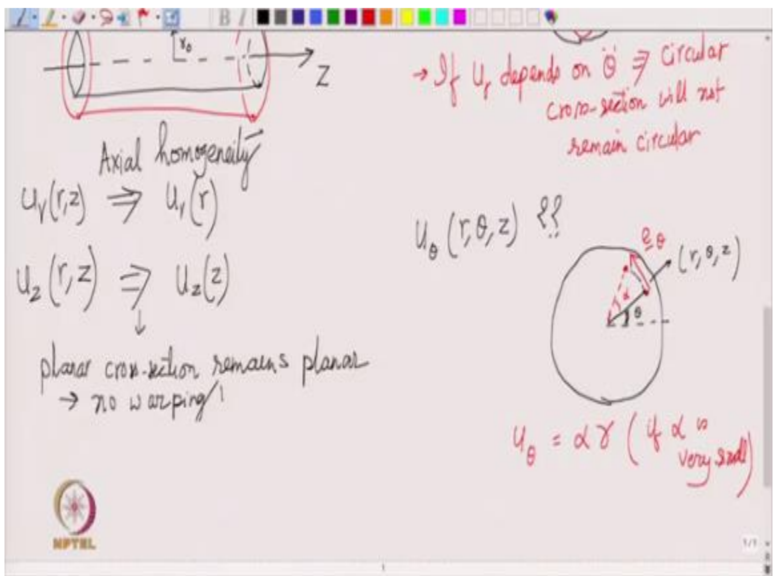

Figure 7: A cross section of a cylinder rotates by angle α: a point on the cross-section displaces due to this rotation

Let us now write α in terms of end-to-end rotation Ω. The right most cross-section rotates by $\frac{\Omega}{2}$ in one direction while the leftmost cross-section rotates by $\frac{\Omega}{2}$ in the other direction as shown in Figure 4. Assuming that this rotation of cross sections is varying linearly along the length L, the rate of change of this angular rotation will be

$$\text{Rate of change of angle of cross section} = \frac{Total\,change\,in\,angle}{Total\,length} = \frac{\Omega}{L}$$

This quantity is also called twist. If we choose the origin at the center of the cylinder, the rotation of the cross section (α) at z = 0 will be zero. Thus, α for a general cross section at axial location z will be

$$\alpha(z) = \text{rate x z} = \frac{\Omega}{L} z \tag{8}$$

Plugging this into equation (6), we get

$$u_\theta = \frac{\Omega}{L} r z. \tag{9}$$

We can observe that u_θ does not depend on θ as required due to axisymmetry. We have finally simplified our displacement functions to

$$u_r = u_r(r), \quad u_\theta = \frac{\Omega}{L} r z, \quad u_z = u_z(z) \tag{10}$$

Note that, except for Ω which is induced due to applied torque, u_θ is fully known while the mathematical forms of u_r and u_θ are unknowns.

2.3 Strain Matrix (start time: 31:54)

To solve for the unknowns, we need to use stress equilibrium equations in cylindrical coordinate system for which we need to express stress components in terms of strain components and finally strain components in terms of displacement components. Let us first find the strain matrix in cylindrical coordinate system. Upon substituting the simplified displacement components (10) in the below formula for strain matrix

$$\underline{\underline{\epsilon}}_{(r,\theta,z)} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left[\dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) + \dfrac{\partial u_\theta}{\partial r}\right] & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left[\dfrac{1}{r}\left(\dfrac{\partial u_r}{\partial \theta} - u_\theta\right) + \dfrac{\partial u_\theta}{\partial r}\right] & \dfrac{1}{r}\left(\dfrac{\partial u_\theta}{\partial \theta} + u_r\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{bmatrix}, \tag{11}$$

we get

$$
\underline{\underline{\epsilon}}_{(r,\theta,z)} = \begin{bmatrix} u_r' & 0 & 0 \\ 0 & \frac{u_r}{r} & \frac{\Omega r}{2L} \\ 0 & \frac{\Omega r}{2L} & u_z' \end{bmatrix} \tag{12}
$$

Note that we have shear strain only in $\theta - z$ plane.

2.4 Stress-Strain relation (start time: 37:22)

Using the below form for stress components in isotropic materials

$$
\sigma_{ij} = \lambda\delta_{ij}\left(\sum_k \epsilon_{kk}\right) + 2\mu\epsilon_{ij} \tag{13}
$$

and substituting strain components from (12), we get

$$
\begin{aligned}
\sigma_{rr} &= \lambda(u_r' + \frac{u_r}{r} + u_z') + 2\mu u_r', \\
\sigma_{\theta\theta} &= \lambda(u_r' + \frac{u_r}{r} + u_z') + 2\mu \frac{u_r}{r}, \\
\sigma_{zz} &= \lambda(u_r' + \frac{u_r}{r} + u_z') + 2\mu u_z', \\
\tau_{r\theta} &= G\gamma_{r\theta} = 2G\epsilon_{r\theta} = 0, \\
\tau_{rz} &= G\gamma_{rz} = 2G\epsilon_{rz} = 0, \\
\tau_{\theta z} &= G\gamma_{\theta z} = 2G\epsilon_{\theta z} = G\frac{\Omega r}{L}.
\end{aligned} \tag{14}
$$

2.5 Solving equilibrium equations (start time: 40:09)

Let us now substitute the above stress components in the below equations of equilibrium:

$$
\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r &= \rho a_r, \\
\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{r\theta}}{r} + b_\theta &= \rho a_\theta, \\
\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\tau_{rz}}{r} + b_z &= \rho a_z.
\end{aligned} \tag{15}
$$

Most of the terms in the above equations become zero which simplifies the equations to

$$
\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r &= \rho a_r, \\
b_\theta &= \rho a_\theta, \\
\frac{\partial \sigma_{zz}}{\partial z} + b_z &= \rho a_z.
\end{aligned} \tag{16}
$$

If we had used Cartesian coordinate system, the terms that became zero due to axisymmetry would have remained non-zero and the formulation would have then become difficult to solve. The cylindrical coordinate system proves handy here because it allows us to use axisymmetry directly.

3 Some examples

Let us now consider some simple axisymmetric deformation examples and find out the corresponding body force components and acceleration components.

3.1 A horizontal cylinder under gravity (start time: 46:48)

If the axis of the cylinder is horizontal and gravity acts vertically downwards as shown in Figure 8, the body force components (b_r and b_θ) will become function of θ as shown below:

$$b_r = \rho g \cos\theta, \qquad b_\theta = \rho g \sin\theta, \qquad b_z = 0. \tag{17}$$

This breaks our axisymmetric assumption.

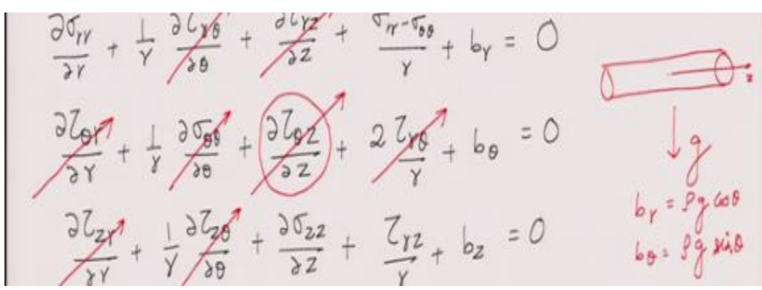

Figure 8: A horizontal cylinder with gravity acting vertically downward

3.2 A vertical cylinder under gravity (start time: 47:52)

When the cylinder is vertical and g acts along the axis of the cylinder as shown in Figure 9, the body force components would be:

$$b_r = 0, \qquad b_\theta = 0, \qquad b_z = -\rho g. \tag{18}$$

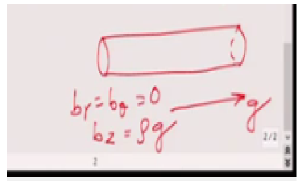

Figure 9: A vertical cylinder under gravity

3.3 A rotating shaft/cylinder without gravity (start time: 48:31)

If the cylinder rotates about its axis with a constant angular speed ω as shown in Figure 10, it will have centripetal acceleration of magnitude $\rho\omega^2 r$ acting in the radially inward direction, i.e.,

$$a_r = -\omega^2 r, \qquad a_\theta = 0, \qquad a_z = 0. \tag{19}$$

There is no body force in this case if we neglect gravity. If we sit on the shaft itself, we need to apply pseudo force to use Newton's second law. Also, in such a frame, the shaft will not be rotating. Thus, a body force in the radially outward direction (called centrifugal force) substitutes the radial acceleration component, i.e,

$$a_r = 0, \qquad a_\theta = 0, \qquad a_z = 0, \qquad b_r = \rho\omega^2 r, \qquad b_\theta = 0, \qquad b_z = 0. \tag{20}$$

In both the view points, the radial equation of equilibrium will become

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0 \tag{21}$$

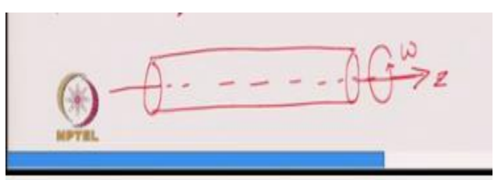

Figure 10: A cylinder rotating with constant angular speed ω about its axis

3.4 Zero gravity and statics problem (start time: 54:26)

For the original problem where only axial force F, torque T and internal pressure P are applied, all body force components will be zero. If it is also a statics problem, all the acceleration components will be zero too. Setting all body force and acceleration components to zero in (16), we see that the equation in θ direction gets trivially satisfied. Thus, the final simplified set of equations for extension-torsion-inflation problem is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{22}$$

We can solve these to obtain the unknown displacement components (u_r, u_z) which we show in the next class.