Solid Mechanics

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Lecture - 2
Traction Vector

Welcome to the second lecture! We are going to learn about the concept of traction vector and stress tensor. In the first lecture, we had learnt about vectors and tensors. You should revise that lecture before proceeding here.

1 Introduction (start time: 01:07)

Let us begin with the traction vector. Consider an arbitrary body which is clamped at one part of the boundary and some force acts on another part of the boundary (see Figure 1). The dashed lines are used to denote clamping of a part of the boundary and this part of the boundary is fixed and cannot move. Due to the applied force, the body gets deformed with the clamped part fixed. In this deformed configuration, we say that the body is under some stress (can be thought of as 'not relaxed'). Now, we take a section that divides the deformed body in two parts: Part A and Part B. This section is shown by dashed surface as it is an internal section and we cannot see it. Let's consider Part A. Part B exerts some force on Part A that will be distributed over the cut section. The distributed forces on the section can be in any direction depending on the applied external force.

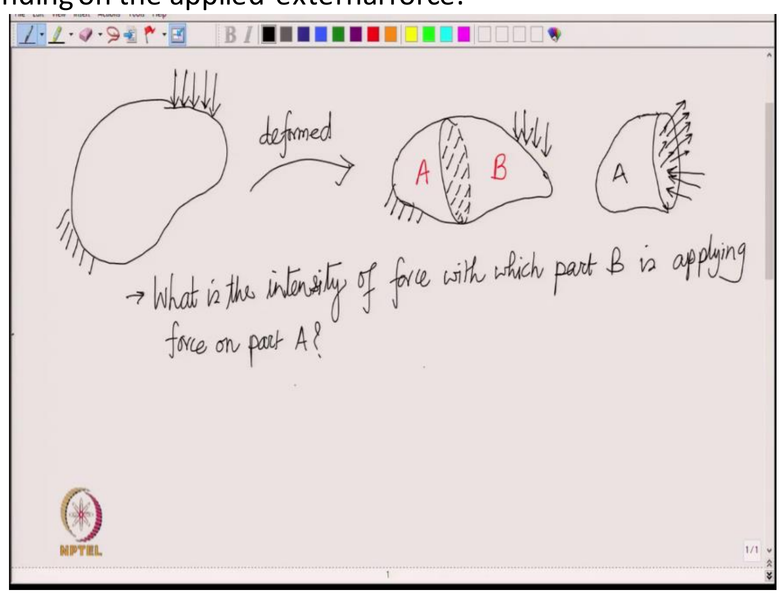

Figure 1: A section divides an arbitrary body into two parts and traction acts on the cut parts.

2 Definition (start time: 05:21)

Traction is defined as the intensity of the force (defined in terms of force per unit area) with which Part B is pulling part A. At different points in the section, the intensity of the force would be different. Let us try to find the intensity at some point $\underline{x}$ ($\underline{x}$ represents the position vector of an arbitrary point on the section). We draw a circle around the point in the plane of the section (see Figure 2). This plane has its normal vector as $\underline{n}$. Let the area enclosed by the circle be denoted by ΔA and total force acting on this circular area be ΔF.

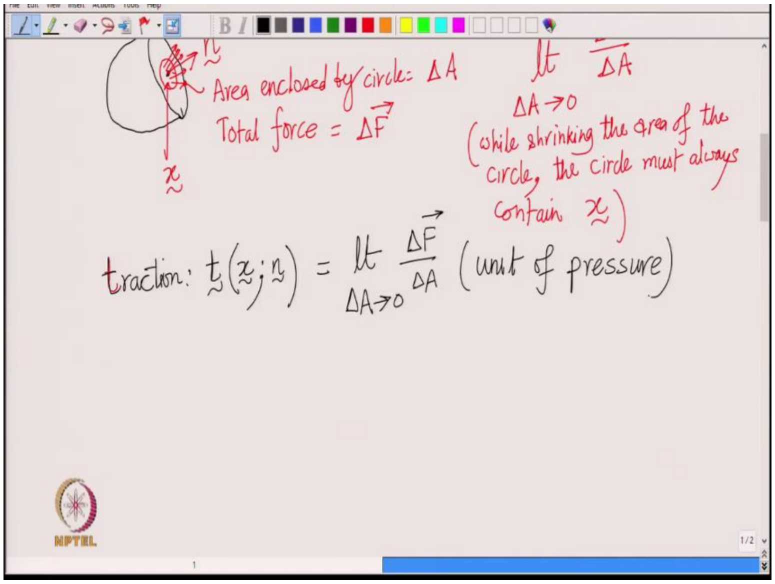

Figure 2: A small area is considered on Part A with its normal and traction shown.

To find intensity at $\underline{x}$, we need to shrink the area of the circle such that the circle must always contain $\underline{x}$. Thus:

$$\text{intensity at } \underline{x} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A} \tag{1}$$

Now, as we shrink the area, the total force acting in that area will also decrease but the limit will attain a value. This value is called traction. It will be represented by $\underline{t}(\underline{x};\underline{n})$ where $\underline{x}$ represents the point at which the traction is being measured and $\underline{n}$ is the normal to the plane that is cut.

$$\underline{t}(\underline{x};\underline{n}) = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A} \tag{2}$$

It has unit of pressure, but it is not same as pressure. For example, traction can act at an angle from the plane normal.

3 Parameters on which traction depends (start time: 11:58)

Consider an arbitrary body under the influence of some force (see Figure 3). Take three points on/inside the body. Now, traction will be different on these three points because by definition, it is a function of the location at which it is measured. At a single point also, say $\underline{x}$, we can define several planes. Traction on each of those planes will be different. To see this more clearly, let's consider an example.

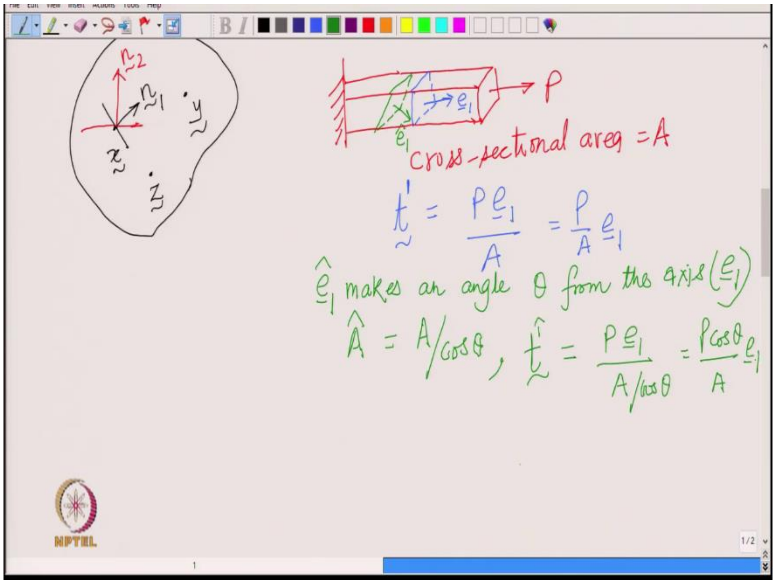

Figure 3: Three points $\underline{x}$, $\underline{y}$ and $\underline{z}$ are considered on an arbitrary body on the left. Two planes are drawn at $\underline{x}$. On the right, a rectangular beam under the action of force P is shown. Two different cross sections are considered on this beam.

Consider a rectangular beam clamped at one end (see Figure 3). A force P is being applied on the other end. The cross-sectional area of the beam is A. First, we take a section whose normal vector is along the axis of the beam. We denote this direction by $\underline{e}_1$. To find the traction on this plane, we find the total force acting on the plane and divide by the area of the plane. The traction on this plane (denoted by $\underline{t}^1$) is given by:

$$\underline{t}^1 = \frac{P\underline{e}_1}{A} = \frac{P}{A}\underline{e}_1 \tag{3}$$

Here, we write the above expression assuming that we have the same traction at every point on this section, but that is not the usual case. Now, let us take a different section. The unit normal to this section ($\underline{\hat{e}}_1$) makes an angle θ with the axis of the beam ($\underline{e}_1$). The area of this section ($\hat{A}$) is given by:

$$\hat{A} = \frac{A}{cos\theta} \tag{4}$$

Total force is still P$\underline{e}_1$. So, the traction on this section (denoted by $\underline{t}^{\hat{1}}$) will be :

$$\underline{t}^{\hat{1}} = \frac{P\underline{e}_1}{A/cos\theta} = \frac{Pcos\theta}{A}\underline{e}_1 \tag{5}$$

Result: Traction not only varies from point to point in the body but even at a given point, traction on different planes is different.

3.1 Number of planes that can exist at a particular point (start time: 19:15)

We go to our body again and consider an arbitrary point $\underline{x}$ (see Figure 4). We can have a plane with normal $\underline{n}_1$ and another with normal $\underline{n}_2$ and so on. We immediately see that we can get infinite number of planes at this point. This means that if we want to know traction even at one point, we need to know traction on all the planes at that point. We will see how we can store the information of all the planes (infinite in number).

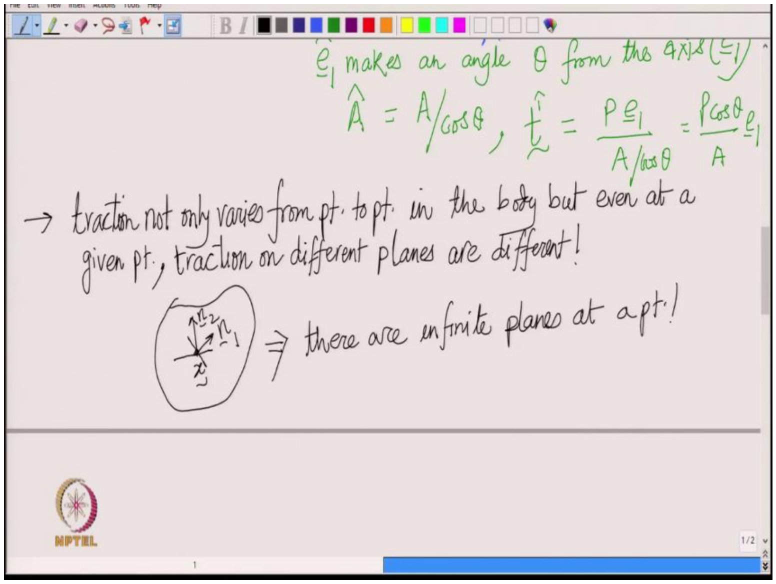

Figure 4: At any point $\underline{x}$ in the body, infinite planes can be drawn out of which two are shown.

4 Importance of traction (start time: 20:30)

By definition, it gives us the intensity of force with which one part of the body is pulling or pushing the other part of the body. If the value of traction is lower than some threshold value, then the body is not going to fracture/fail. But if it is more than the threshold value then it can fail. Now, at a given point also, traction varies from one plane to the other. Thus, probability of failure is higher on the plane on which traction has got a larger value.

Result: Traction can tell us at what point in the body and on what plane at that point would the body fail!

5 Storing information of traction on infinite planes at a point (start time: 21:58)

Theorem : If we know traction on three independent planes we can then find the value of traction on any other plane.

This means that we can just store the value for three planes and then use a formula to get the value of traction on any other plane. Let us work out this theorem. Consider a point $\underline{x}$ on our body (see Figure 5). Suppose we know the value of traction on three different planes at the point $\underline{x}$ and then we want to know what is the value of traction on any other plane. We draw a tetrahedron whose vertex is at $\underline{x}$ and three edges at this point are perpendicular and along the coordinate axes. This tetrahedron has got four faces : OAB (bottom plane), OBC, OAC and ABC. Our tetrahedron is such that along OA we have $\underline{e}_1$, along OB we have $\underline{e}_2$ and along OC we have $\underline{e}_3$.

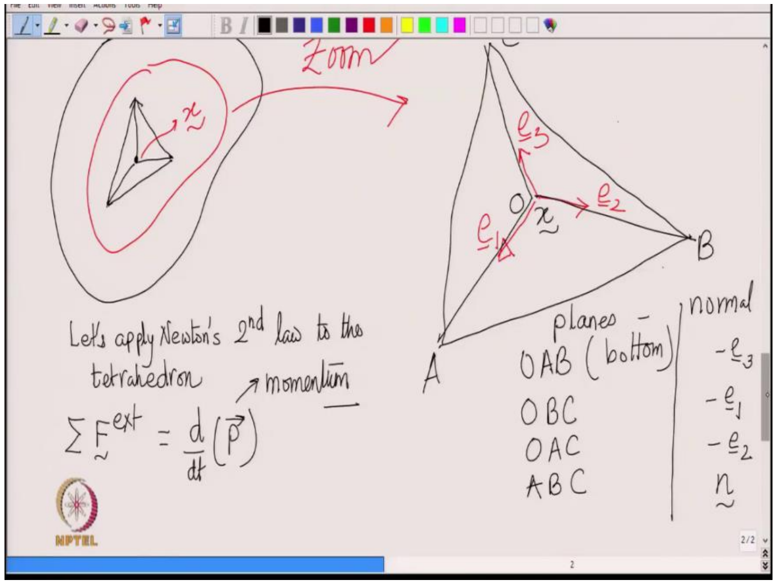

Figure 5: A tetrahedron OABC is drawn at the point of interest $\underline{x}$ in the body.

The plane normals will be:

$\underline{Plane}$	:	$\underline{Normal}\,(outward)$
OAB	:	$-\underline{e}_3$
OBC	:	$-\underline{e}_1$
OAC	:	$-\underline{e}_2$
ABC	:	$\underline{n}(say)$

We have to always look at the outward normal of a plane with respect to the body. Now, if we know the tractions on the three planes $-\underline{e}_1$, $-\underline{e}_2$ and $-\underline{e}_3$ and we want to find the traction on the tilted plane (having normal as $\underline{n}$), first we'll apply Newton's 2^{nd} law of motion to the tetrahedron:

$$\sum \underline{F}^{ext} = \frac{d}{dt}(\overrightarrow{P}) \tag{6}$$

Here, $\underline{P}$ represents the momentum. To visualize the external forces acting on the tetrahedron, we can imagine taking out this tetrahedron from the original body. So, the remaining body will exert forces on the four faces of the tetrahedron (i.e. traction force on all four faces).

The total external force will consist of both contact forces and non-contact forces:

$$\sum \underline{F}^{ext}$$

$$\swarrow \qquad \searrow$$

$$contact\ \ force \qquad non-contact\ \ force$$

$$(traction\ \ force) \qquad (body\ \ force)$$

unit: [force/area]   unit: [force/volume]

A body force acts on every point of the body, e.g. gravitational force is applied by the earth on every point of our tetrahedron. Gravitational body force is given by:

$$\text{Gravitational body force} = (\rho \underline{g} V)/V = \rho \underline{g} \tag{7}$$

Here ρ is the density, $\underline{g}$ is the acceleration due to gravity and V is the volume of the body. To get the total force due to contact force, we need to integrate it over the area on which it is acting. To get the total body force, we need to integrate it over the volume on which it is acting.
Let the area of face OBC be A_1, traction on it be $\underline{t}^{-1}$, area of face OAC be A_2 and traction on it be $\underline{t}^{-2}$, area of face OAB be A_3 and traction on it be $\underline{t}^{-3}$ and area of face ABC be A_n and traction on it be $\underline{t}^{n}$. The exponent in this notation is being named after the plane normal. Let the volume of the tetrahedron be V. Assuming the traction is same on every point of a given face, total force due to a traction will simply be area times the traction.

Total force will then be:

$$\sum \underline{F}^{ext} = \underbrace{\underline{t}^{-1}A_1 + \underline{t}^{-2}A_2 + \underline{t}^{-3}A_3 + \underline{t}^{n}A_n}_{\text{contact force}} + \underbrace{\rho \underline{g} V}_{\text{body force}} = \underbrace{\rho V}_{\text{mass of the tetrahedron}} \times \underline{a}_{CM} \tag{8}$$

We can also write the volume of the tetrahedron in terms of the area A_n. Suppose h is the perpendicular distance of plane ABC from the vertex. So, the volume V is given by:

$$V = \frac{A_n h}{3} \tag{9}$$

We can also relate the areas A_1, A_2, A_3 in terms of A_n. If we project the area A_n along the direction $\underline{e}_1$, that projected area is same as the area of OBC which is A_1. So, A_1 is the projection of A_n along $\underline{e}_1$. Similarly A_2 is the projection of A_n along $\underline{e}_2$ and so on. Thus, A_i is the projection of area A_n along $\underline{e}_i$ (i=1,2,3). Then from geometry, we can then prove that:

$$A_i = A_n (\underline{n} \cdot \underline{e}_i) \tag{10}$$

When we plug equations (9) and (10) in equation (8), we get:

$$A_n(\underline{t}^{-1}(\underline{n} \cdot \underline{e}_1) + \underline{t}^{-2}(\underline{n} \cdot \underline{e}_2) + \underline{t}^{-3}(\underline{n} \cdot \underline{e}_3) + \underline{t}^{n}) + \frac{\rho \underline{g} A_n h}{3} = \frac{\rho A_n h \underline{a}_{CM}}{3}$$

$$\Rightarrow \boxed{\underline{t}^n + \sum_{i=1}^{3} \underline{t}^{-i}(\underline{n}\cdot\underline{e}_i) + \frac{\rho h(\underline{g} - \underline{a}_{CM})}{3} = 0} \tag{11}$$

We want to write down the traction on the plane $\underline{n}$ in terms of the other three tractions at the same point. Now, in our tetrahedron, three planes pass through the point O but the plane with the normal as $\underline{n}$ (the tilted plane) is not passing through the point O. So, we need to shrink the tetrahedron such that in the limiting case, all 4 planes pass through O. We can do this by letting the perpendicular distance h go to zero. When we apply $\lim_{h\to 0}$ to equation (11), the terms proportional to h will vanish. Thus, in the $\lim_{h\to 0}$, the terms corresponding to body force and acceleration drop out.

$$\Rightarrow \underline{t}^n = -\sum_{i=1}^{3} \underline{t}^{-i}(\underline{n}\cdot\underline{e}_i) \tag{12}$$

5.1 Relation between tractions on planes with opposite normals (start time: 47:00)

By definition, $\underline{t}^{-i}$ is the traction on $-\underline{e}_i$ plane whereas $\underline{t}^i$ is the traction on $\underline{e}_i$ plane. So, these two are on the same plane but with normals pointing in the opposite direction. We redraw our body and think of the internal section (see Figure 6). The internal section of the two parts has normals opposite to each other.

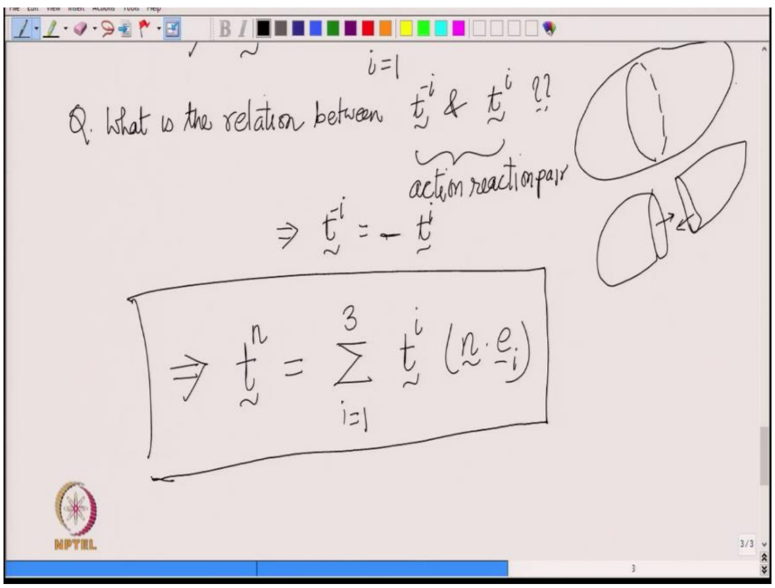

Figure 6: A section is cut in the body and the tractions at this section are shown.

The tractions on these planes will be equal and opposite due to Newton's third law of motion (because they form an action reaction pair):

$$\Rightarrow \underline{t}_{-i} = -\underline{t}_i \tag{13}$$

Using equation (13) in equation (12), our formula finally becomes:

$$\boxed{\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i)} \tag{14}$$

Note : The body force and the acceleration terms dropped out! Thus, the above formula holds even if the body force is present or the body is accelerating!