Solid Mechanics
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Lecture - 14
Similarity in Properties of Stress and Strain Tensors

Hello everyone! Welcome to Lecture 14! In this lecture, we will look at the similarities between stress and strain tensors and their properties.

1 Similarity between Stress and Strain tensors (start time: 00:22)

Let us list the formulae of stress and strain components derived earlier to observe the similarities between the two. On the stress side, we have normal (σ_{nn}) and shear (τ_{mn}) components of traction while on the strain side, we have longitudinal (ϵ_{nn}) and shear (γ_{mn}) strains as shown below

Stress	Strain
$\sigma_{nn} = \big(\underline{\underline{\sigma}}\,\underline{n}\big) \cdot \underline{n}$	$\epsilon_{nn} = \big(\underline{\underline{\epsilon}}\,\underline{n}\big) \cdot \underline{n}$
$\tau_{mn} = \big(\underline{\underline{\sigma}}\,\underline{n}\big) \cdot \underline{m}$	$\gamma_{mn} = 2\big(\underline{\underline{\epsilon}}\,\underline{n}\big) \cdot \underline{m}$
($\underline{n}$: plane normal, $\underline{m}$: direction along which we measure shear)	($\underline{n}$: direction of line element, $\underline{m}$: direction of a line element perpendicular to $\underline{n}$)

We can notice the similarity in above formulas. We also defined the strain tensor $\underline{\underline{\epsilon}}$ to be

$$\underline{\underline{\epsilon}} = \frac{1}{2}\Big(\nabla\,\underline{u} + \nabla\,\underline{u}^T\Big) \tag{1}$$

Thus, the strain tensor is also symmetric just like the stress tensor and stress and strain formulae are also similar. This means that we can apply all the properties derived for stress tensor to strain tensor. We discuss them now

1.1 Principal directions and principal components (start time: 04:42)

We had discussed about principal stress planes and principal stress components earlier. Likewise, we can also define principal strain directions (not planes) and principal strain components. We know that at a point, principal stress planes are the planes on which the normal component of traction is maximized/minimized. The value of the normal component of traction on these planes are principal stress components. Similarly, at a point in the body, out of the numerous line elements, the directions of those line elements that experience maximum/minimum longitudinal strain are called principal strain directions. The values of longitudinal strain in these directions are called principal strain components. To find them, we can follow the same approach that we followed for finding principal stress planes and principal stress components. We find the eigenvectors and eigenvalues of the strain tensor to obtain principal strain directions and principal strain components.

1.2 Diagonality of matrix form in principal coordinate system (start time: 07:48)

We know that the stress matrix in the coordinate system of principal stress directions becomes diagonal. Similarly, when we represent the strain matrix in the coordinate system of the principal strain directions, it will become a diagonal matrix. As the off-diagonal components will be zero, this means that if we take the two line elements directed along the principal strain directions, there will not be any change of angle between them. To visualize this, we can consider a cuboid whose face normals are along the principal strain directions (given by the eigenvectors of the strain tensor) as shown on the left in Figure 1. Here $\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3$ represent principal strain directions. The cuboid gets deformed as shown on the right in Figure 1. As the angle between line elements along principal directions does not change, the cuboid only changes its size but the deformed shape is still a cuboid.

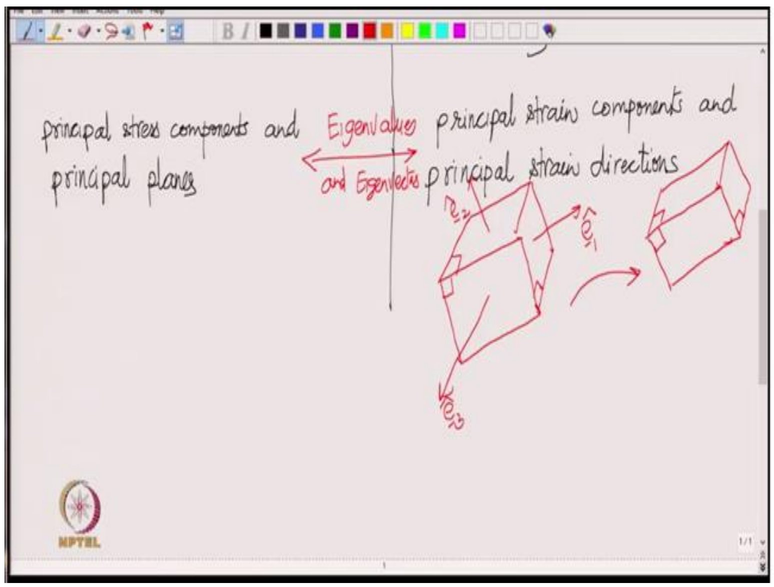

Figure 1: A cuboid whose face normals are along principal strain directions in the reference configuration deforms such that it still retains its cuboidal shape

1.3 Maximum shear (start time: 10:45)

We can also maximize shear strain at a point just like we maximized the shear component of traction. We had found in previous lectures that the planes on which the shear component of traction maximized/minimized lie at an angle of 45° from the principal planes. Similarly, the pair of perpendicular line elements that undergoes maximum change in angle (or maximum shear strain) will be directed at 45° from principal strain directions. This is shown in Figure 2.

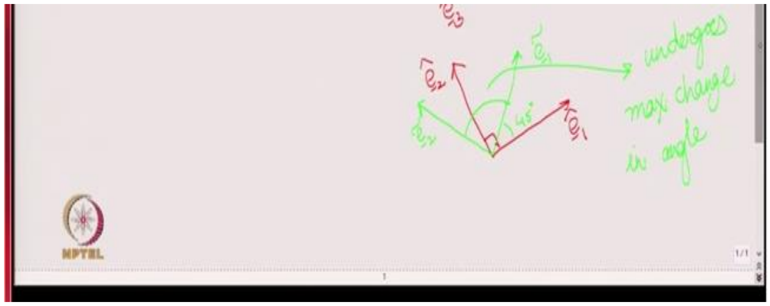

Figure 2: Line element directions corresponding to maximum shear strain ($\underset{\sim}{\tilde{e}}_1$,$\underset{\sim}{\tilde{e}}_2$) shown with respect to the principal strain directions ($\underset{\sim}{\hat{e}}_1$,$\underset{\sim}{\hat{e}}_2$)

1.4 Mohr's circle (start time: 13:13)

We can also think of Mohr's circle for strain. Mohr's circle for stress gave the value of normal traction (σ) and shear traction (τ) on an arbitrary plane. Similarly, if we know the value of longitudinal strain along two perpendicular directions say $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ and also know shear strain between $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$, then one can use Mohr's circle for strain to obtain longitudinal and shear strain for two perpendicular line elements which are oriented at angle ϑ relative to $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ pairs. The strain plane will have its axes as longitudinal strain () and shear strain (γ). However, as the formula for shear strain has an extra factor of 2 when compared with the formula for shear stress, we need to keep the vertical axis in strain plane as $\frac{\gamma}{2}$. This will permit us to draw Mohr's circle for strain in exactly the same manner as we draw Mohr's circle for stress. We had seen while discussing Mohr's circle for stress that it can be drawn only when at least one coordinate axis is along the principal stress direction. Likewise, for Mohr's circle for strain also, we can draw the circle only when at least one coordinate axis is along a principal strain direction. For now let's consider that the third coordinate direction is along a principal strain direction. Thus, the strain matrix in such a coordinate system will look as follows:

$$\left[\underset{\sim}{\epsilon}\right] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \frac{\gamma_{xy}}{2} & 0 \\ \frac{\gamma_{xy}}{2} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \tag{2}$$

For drawing the Mohr's circle, we need to find out the center and the radius of the circle. We first draw the point corresponding to line elements along *x* and *y* directions. Thus, we mark the point $(\epsilon_{xx}, \frac{\gamma_{xy}}{2})$ as shown in Figure 3. The center of the circle will be at the mid point of ϵ_{xx} and ϵ_{yy} on the axis. Now, we have the center of the circle and a point on the circle $(\epsilon_{xx}, \frac{\gamma_{xy}}{2})$. Thus, we can get the radius by joining them as shown in Figure 3. We can then draw the circle itself with the center and radius known. We can extract a lot of information from this circle just like we had seen in the stress case. For example, the principal strain components will be obtained from the points where the circle cuts the axis (shown in red crossed circles in Figure 3). Thus, we have

$$\text{Principal strain components: } \frac{\epsilon_{xx} + \epsilon_{yy}}{2} \pm R, \epsilon_{zz} \tag{3}$$

Here, R can be found by applying Pythagoras theorem to the right angled triangle shown.

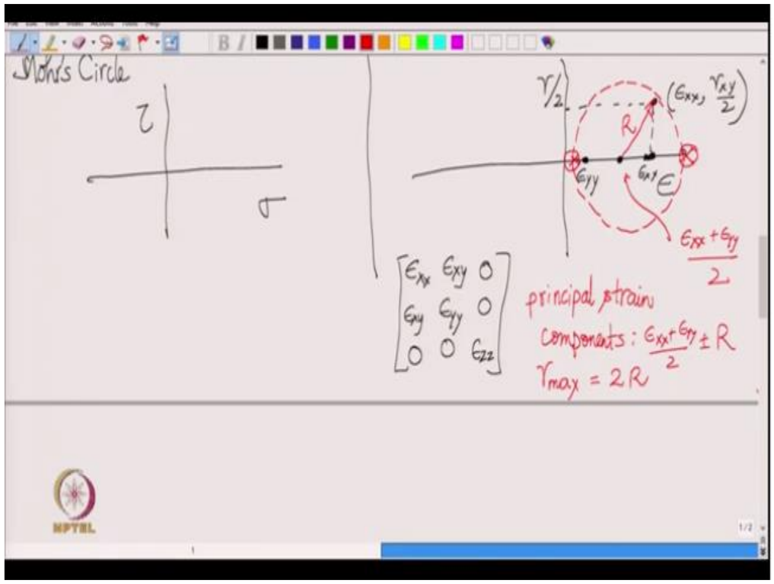

Figure 3: Mohr's circle for strain

Also, maximum shear strain (γ_{max}) can be obtained from the Mohr's circle. $\frac{\gamma_{max}}{2}$ is equal to the radius of the Mohr's circle. Thus

$$\gamma_{max} = 2R \tag{4}$$

Finally, we can find longitudinal and shear strains for any arbitrary line elements. For example, to get the longitudinal strain for a line element which makes an angle ϑ with the $\underline{e}_1$ direction in the clockwise direction, we need to go anticlockwise by 2ϑ on the Mohr's circle from ($\epsilon_{xx}, \frac{\gamma_{xy}}{2}$). For getting the shear strain, we should remember that we need to multiply the value obtained from Mohr's circle graph by 2.

1.5 Invariants (start time: 20:50)

Just like we have invariants of the stress tensor as I_1, I_2 and I_3, we also have invariants of the strain tensor denoted by J_1, J_2 and J_3. They are exactly analogous to each other. J_1 represents the trace of the strain tensor (just like I_1 represents the trace of the stress tensor). J_3 represents the determinant of the strain tensor (just like I_3 represents the determinant of the stress tensor).

1.6 Decomposition of the tensors (start time: 22:08)

We had learnt about the decomposition of the stress tensor into hydrostatic and deviatoric parts. We can also decompose the strain tensor into two parts in a similar way as shown below:

$$\underline{\underline{\epsilon}} = \underbrace{\frac{1}{3}J_1\underline{\underline{I}}}_{\text{volumetric strain tensor}} + \underbrace{\left(\underline{\underline{\epsilon}} - \frac{1}{3}J_1\underline{\underline{I}}\right)}_{\text{strain deviator}} \tag{5}$$

The first part is proportional to identity. It is analogous to the hydrostatic part of stress and is called the volumetric strain tensor. This part is responsible for change in volume and does not affect the shape. The second part is analogous to stress deviator and is called strain deviator. This part is responsible for distorting the body. The trace of the second part is zero by construction.

2 An alternate physical meaning of shear strain (start time: 23:58)

We know already that shear strain denotes change in angle between two perpendicular line elements. There is another physical interpretation of shear strain. Let us consider the following displacement function:

$$u_1 = \alpha X_2, \qquad u_2 = 0, \qquad u_3 = 0 \tag{6}$$

and understand its underlying deformation. After deformation, the position vector of a typical point in the body changes as follows:

$$x_1 = X_1 + \alpha X_2, \qquad x_2 = X_2, \qquad x_3 = X_3. \tag{7}$$

Think of a rectangular slice in the reference configuration of the body in $\underline{e}_1 - \underline{e}_2$ plane (see Figure 4). This slice deforms to a parallelogram according to (6) as shown in the figure inducing change in angle between its two perpendicular edges. Therefore, the displacement prescribed by equation (6) is also called shear displacement. To measure the amount of shear, we can directly notice from Figure 4 that

$$\beta = \tan^{-1}\left(\frac{\alpha X_2}{X_2}\right) = \tan^{-1}(\alpha) \approx \alpha \quad \text{(for small } \alpha\text{)} \tag{8}$$

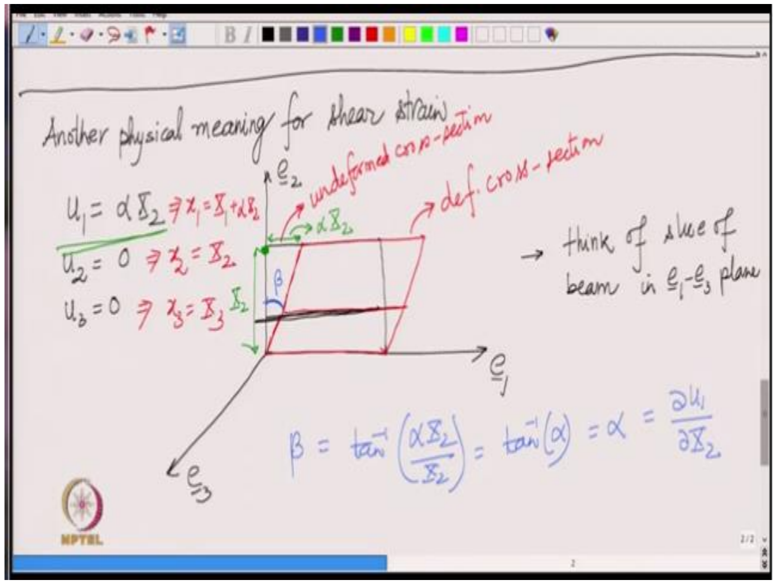

Figure 4: Undeformed and deformed slices of a body undergoing deformation as prescribed by equation (6)

Alternatively, using formula for shear, we can also see

$$\gamma_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = \alpha + 0 = \alpha. \tag{9}$$

Let us now imagine a plane section of the body in ($\underline{e}_1$ − $\underline{e}_3$) plane. These planes have constant X_2 coordinate. According to (6), all points in this plane displace by the same amount in $\underline{e}_1$ direction. Hence, such planes do not deform but simply rigidly translate along $\underline{e}_1$ by αX_2. One can, in fact, think of infinite such planes all parallel to each other having different X_2 coordinate but a given plane having the same X_2 coordinate for all its points (also see Figure 5). The plane having $X_2 = 0$ (lowermost plane in Figure 5) will have the same reference and deformed position. However, higher the X_2 coordinate of a plane, higher is the drift/translation in $\underline{e}_1$ direction. This can be visualized as sliding of a pack of cards in the plane of the card itself. This is an alternate physical interpretation of shearing: sliding of parallel planes in a direction perpendicular to the plane normal. Here, sliding is along $\underline{e}_1$ direction which is perpendicular to $\underline{e}_2$ (the plane normal). We can also let the planes slide in an arbitrary direction perpendicular to $\underline{e}_2$ other than $\underline{e}_1$ and that will also be shear. The angle β in Figure 4 and 5 is the measure of intensity of sliding of parallel planes which also equals the shear strain value.

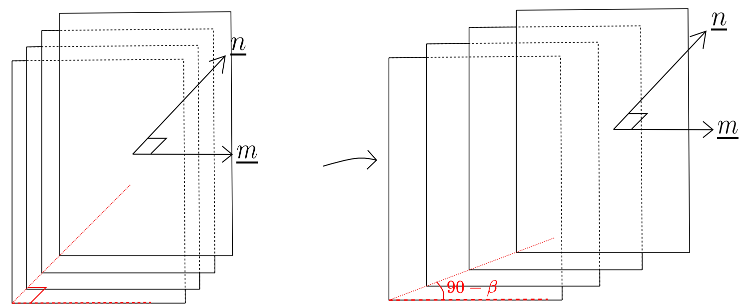

Figure 5: Shearing strain visualised as sliding of parallel planes having normal $\underline{n}$ in the direction $\underline{m}$: $\underline{n} = \underline{e}_2$ and $\underline{m} = \underline{e}_1$ for the displacement function in (6)

Thus we have another physical interpretation of shear strain:

$$\begin{aligned} \text{Shear strain} &= \text{sliding of sets of parallel planes (with normal } \underline{n}\text{) along a direction } \underline{m} \perp \underline{n} \\ &= 2\left(\underline{\underline{\epsilon}}\,\underline{n}\right) \cdot \underline{m} \end{aligned} \tag{10}$$

3 Strain Compatibility Conditions (start time: 37:16)

We know that the strain matrix in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system is

$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} \frac{\partial u_X}{\partial X} & \frac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\right) & \frac{1}{2}\left(\frac{\partial u_X}{\partial Z} + \frac{\partial u_Z}{\partial X}\right) \\ \frac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\right) & \frac{\partial u_Y}{\partial Y} & \frac{1}{2}\left(\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y}\right) \\ \frac{1}{2}\left(\frac{\partial u_X}{\partial Z} + \frac{\partial u_Z}{\partial X}\right) & \frac{1}{2}\left(\frac{\partial u_Y}{\partial Z} + \frac{\partial u_Z}{\partial Y}\right) & \frac{\partial u_Z}{\partial Z} \end{bmatrix}. \tag{11}$$

Because of symmetry, it has six different components which are functions of (X, Y, Z) in general. Suppose, instead of obtaining strain matrix from the derivative of displacement functions, we directly write it by choosing six arbitrary functions for its components, i.e.,

$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz}. \end{bmatrix} \tag{12}$$

Will such a strain matrix correspond to any displacement function? The answer is NO! Basically, using strain-displacement relation in (11), if we integrate the six arbitrary functions to obtain the three displacement components, we may not obtain a consistent displacement function. For example, think of integrating $\epsilon_{XX}(X, Y, Z)$ in X to obtain u_X and then integrating $\epsilon_{YY}(X, Y, Z)$ in Y to obtain u_Y, the resulting function need not satisfy the prescribed function for ϵ_{XY}. Physically, it may happen that the displacement so obtained is such that it leads to two parts of a body overlapping with each other or getting separated as in Figure 6. Thus, there has to be some constraint on the six strain functions which are collectively called *strain compatibility condition*. Thus, a general symmetric matrix does not necessarily represent a strain matrix until it satisfies those *strain compatibility conditions*.

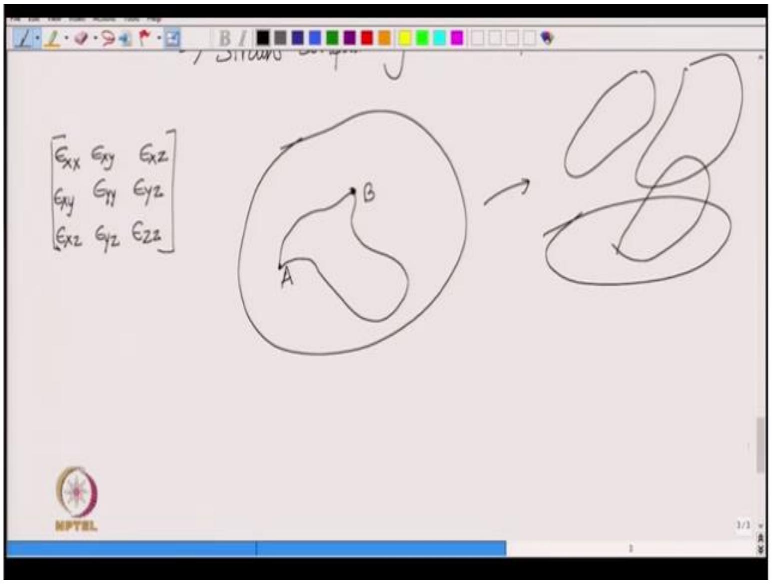

Figure 6: Illustration of a displacement function obtained from an arbitrary symmetric matrix leading to discontinuities and overlaps in the body

3.1 Another interpretation (start time: 46:13)

Another way to look at compatibility condition is that when we integrate a given strain matrix along an arbitrary path in the reference configuration of a body, the displacement so obtained should just depend on the end points of the path and not on the actual path of integration. For example, suppose we start at a point A shown in Figure 7 and want to find the displacement of point B. We can go through different paths and integrate the strain components along the path to find the displacement of B.

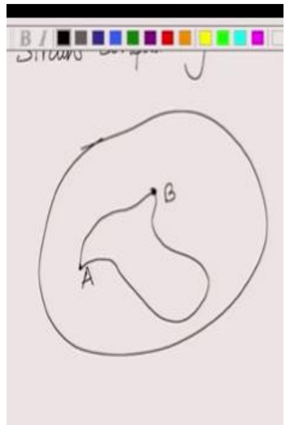

Figure 7: Two possible paths to go from point A to point B in the reference configuration of a body

Regardless of the path followed, the displacement should come out to be the same because the final point is the same and it must have a unique displacement. In other words, the integration has to be path independent and the strain compatibility condition ensures that. There is a rigorous mathematical proof to obtain strain compatibility conditions but that is out of scope of this class. There are six compatibility conditions which can be divided into two sets. The first set is

$$\frac{\partial^2 \epsilon_{xx}}{\partial Y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial X^2} = \frac{\partial^2 \gamma_{xy}}{\partial X \partial Y} \tag{13}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial Z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial Y^2} = \frac{\partial^2 \gamma_{yz}}{\partial Y \partial Z} \tag{14}$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial Z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial X^2} = \frac{\partial^2 \gamma_{xz}}{\partial X \partial Z} \tag{15}$$

We can also verify them by plugging in strain-displacement relation, e.g., consider the LHS of equation (13):

$$\begin{aligned} \frac{\partial^2 \epsilon_{xx}}{\partial Y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial X^2} &= \frac{\partial^3 u_x}{\partial Y^2 \partial X} + \frac{\partial^3 u_y}{\partial X^2 \partial Y} \\ &= \frac{\partial^2}{\partial X \partial Y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ &= \frac{\partial^2 \gamma_{xy}}{\partial X \partial Y} = RHS \end{aligned} \tag{16}$$

The second set of compatibility conditions is

$$\frac{\partial}{\partial Z}\left(\frac{\partial \gamma_{yz}}{\partial X}+\frac{\partial \gamma_{zx}}{\partial Y}-\frac{\partial \gamma_{xy}}{\partial Z}\right) = 2\frac{\partial^2 \epsilon_{zz}}{\partial X \partial Y} \quad (17)$$

$$\frac{\partial}{\partial X}\left(\frac{\partial \gamma_{xy}}{\partial Z}+\frac{\partial \gamma_{xz}}{\partial Y}-\frac{\partial \gamma_{yz}}{\partial X}\right) = 2\frac{\partial^2 \epsilon_{xx}}{\partial Y \partial Z} \quad (18)$$

$$\frac{\partial}{\partial Y}\left(\frac{\partial \gamma_{xy}}{\partial Z}+\frac{\partial \gamma_{yz}}{\partial X}-\frac{\partial \gamma_{xz}}{\partial Y}\right) = 2\frac{\partial^2 \epsilon_{yy}}{\partial X \partial Z} \quad (19)$$

3.2 Special Case (start time: 52:07)

There is a specific situation where five of the compatibility conditions get satisfied automatically. Consider the following situation:

$$\begin{aligned}
\epsilon_{xx} &= \epsilon_{xx}(X,Y),\\
\epsilon_{yy} &= \epsilon_{yy}(X,Y),\\
\gamma_{xy} &= \gamma_{xy}(X,Y),\\
\epsilon_{zz} &= \gamma_{xz} = \gamma_{yz} = 0
\end{aligned} \quad (20)$$

This special case is also called plane strain condition. For such a case, it is easy to check that five of the compatibility conditions (all except equation (13)) are automatically satisfied. So for this special situation, only the following compatibility condition needs to be checked:

$$\frac{\partial^2 \epsilon_{xx}}{\partial Y^2}+\frac{\partial^2 \epsilon_{yy}}{\partial X^2}=\frac{\partial^2 \gamma_{xy}}{\partial X \partial Y} \quad (21)$$

3.3 An example (start time: 54:36)

Suppose the strain components are given by the following functions:

$$\begin{aligned}
\epsilon_{xx} &= 5+X^2+Y^2+X^4+Y^4\\
\epsilon_{yy} &= 6+3X^2+3Y^2+X^4+Y^4\\
\gamma_{xy} &= 10+4XY(X^2+Y^2+2)\\
\epsilon_{zz} &= \gamma_{yz} = \gamma_{xz} = 0
\end{aligned} \quad (22)$$

This case satisfies the special condition defined in (20). To check the compatibility condition (21), we first obtain the required derivatives of strain components, i.e.,

$$\begin{aligned}
\frac{\partial^2 \epsilon_{xx}}{\partial Y^2} &= 2+12Y^2,\\
\frac{\partial^2 \epsilon_{yy}}{\partial X^2} &= 6+12X^2,\\
\frac{\partial \gamma_{xy}}{\partial X} &= 12X^2Y+4Y^3+8Y \Rightarrow \frac{\partial^2 \gamma_{xy}}{\partial X \partial Y} = 12X^2+12Y^2+8
\end{aligned} \quad (23)$$

Upon plugging them into (21), we see that the compatibility condition is indeed satisfied. Thus, the strain matrix prescribed is a valid one and a physical displacement function can be extracted from it.