Solid Mechanics

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Lecture - 13

Local volumetric strain and local infinitesimal rotation

Hello Everyone! Welcome to Lecture 13! In this lecture, we will discuss about two concepts: local volumetric strain and the local rotation tensor.

1 Local Volumetric Strain (start time: 00:31)

We consider a body being deformed as shown in Figure 1. As the body deforms, the volume of every small region (called local volume element) of the body also changes. We can therefore define a quantity called local volumetric strain because the change in volume per unit volume will be different for different parts in the body. Let us think of three line elements $\Delta\underline{X}$, $\Delta\underline{Y}$ and $\Delta\underline{Z}$ forming a parallelopiped at the point of interest $\underline{X}$ in the reference configuration as shown in Figure 1. We should keep in mind that the sides of the parallelopiped are very small so that the parallelopiped lies in a tiny region near $\underline{X}$. As the volume of this region is shrunk to zero, we will be able to define local volumetric strain at the point $\underline{X}$ itself. Upon deformation, the point $\underline{X}$ goes to $\underline{x}$ and the three line elements become $\Delta\underline{x}$, $\Delta\underline{y}$ and $\Delta\underline{z}$ which again generate a parallelopiped.

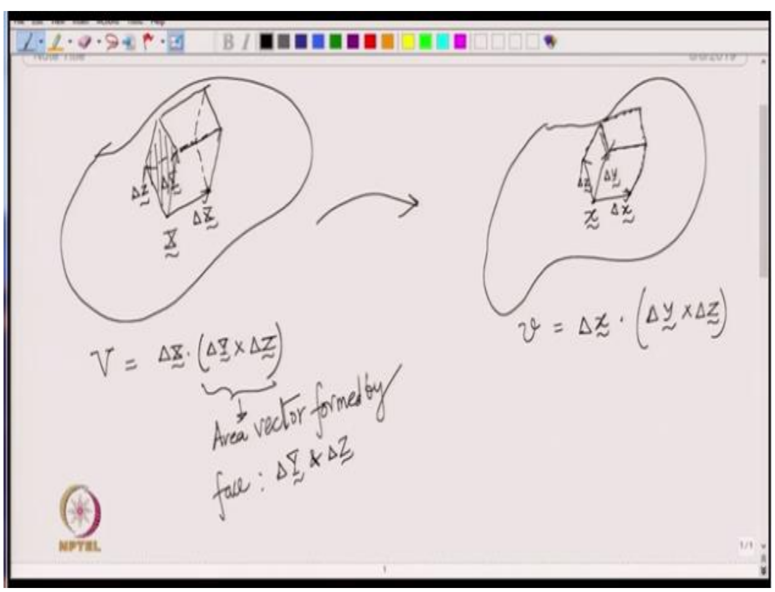

Figure 1: The local volume element in reference and deformed configurations are shown as parallelopipeds.

As the volume of a parallelopiped is given by scalar triple product of vectors forming its sides, the volume of the parallelopiped in the reference configuration (denoted by V) will be

$$V = \Delta\underline{X} \cdot (\Delta\underline{Y} \times \Delta\underline{Z}) \tag{1}$$

Here, the term $(\Delta\underline{Y} \times \Delta\underline{Z})$ gives the area formed by the vectors $\Delta\underline{Y}$ and $\Delta\underline{Z}$. The face of the parallelopiped corresponding to this is shaded in Figure 1. This face is considered to be the base of the parallelopiped. For this base, the height of the parallelopiped will be the component of $\Delta\underline{X}$ along the shaded face' normal. So, the volume is obtained by taking the dot product of the base area vector with $\Delta\underline{X}$. Similarly, the volume of the parallelopiped in the deformed configuration (denoted by v) will be

$$v = \Delta\underline{x} \cdot (\Delta\underline{y} \times \Delta\underline{z}) \tag{2}$$

The volumetric strain (denoted by ϵ_V) is defined as

$$\begin{aligned} \epsilon_V &= \frac{\text{Change in Volume}}{\text{Original Volume}} \\ &= \frac{v - V}{V} \\ &= \frac{v}{V} - 1 \end{aligned} \tag{3}$$

As we want the volumetric strain at the point of interest itself, the original volume has to be shrunk to the same point, i.e.,

$$\epsilon_V = \lim_{V \to 0} \left(\frac{v}{V} - 1 \right) \tag{4}$$

1.1 Formula for volumetric strain (start time: 08:09)

The scalar triple product can also be realized as the determinant of a matrix whose columns are formed by the vectors involved in the scalar triple product. Thus, we can write

$$V = \Delta\underline{X} \cdot (\Delta\underline{Y} \times \Delta\underline{Z}) = det\begin{bmatrix} \Delta\underline{X} & \Delta\underline{Y} & \Delta\underline{Z} \end{bmatrix}, \tag{5}$$

$$v = \Delta\underline{x} \cdot (\Delta\underline{y} \times \Delta\underline{z}) = det\begin{bmatrix} \Delta\underline{x} & \Delta\underline{y} & \Delta\underline{z} \end{bmatrix} \tag{6}$$

Substituting the below formula in (6)

$$\Delta\underline{x} = \underline{\underline{F}}\Delta\underline{X} + O\left(||\Delta\underline{X}||^2\right), \tag{7}$$

we get

$$v = det\left[\underline{\underline{F}}\Delta\underline{X} \quad \underline{\underline{F}}\Delta\underline{Y} \quad \underline{\underline{F}}\Delta\underline{Z}\right] \; + \; O\left(||\Delta\underline{X}||^2 \; ||\Delta\underline{Y}|| \; ||\Delta\underline{Z}||\right)$$
$$+ \; O\left(||\Delta\underline{X}|| \; ||\Delta\underline{Y}||^2 \; ||\Delta\underline{Z}||\right) + O\left(||\Delta\underline{X}|| \; ||\Delta\underline{Y}|| \; ||\Delta\underline{Z}||^2\right) \quad (8)$$

We can simply it further by using the following identity:

$$\left[\underline{\underline{A}}\right]\left[\underline{b}_1 \quad \underline{b}_2 \quad \underline{b}_3\right] \; = \; \left[\underline{\underline{A}}\underline{b}_1 \quad \underline{\underline{A}}\underline{b}_2 \quad \underline{\underline{A}}\underline{b}_3\right] \quad (9)$$

This implies that

$$\begin{aligned} det\left[\underline{\underline{F}}\Delta\underline{X} \quad \underline{\underline{F}}\Delta\underline{Y} \quad \underline{\underline{F}}\Delta\underline{Z}\right] \; &= \; det\left(\left[\underline{\underline{F}}\right]\left[\Delta\underline{X} \quad \Delta\underline{Y} \quad \Delta\underline{Z}\right]\right) \\ &= \; det\left[\underline{\underline{F}}\right] \; det\left[\Delta\underline{X} \quad \Delta\underline{Y} \quad \Delta\underline{Z}\right] = det\left[\underline{\underline{F}}\right] \; V. \end{aligned} \quad (10)$$

Substituting this in (8) and further dividing by reference volume, we get

$$\frac{v}{V} = det\left[\underline{\underline{F}}\right] + O\left(||\Delta\underline{X}||\right) + O\left(||\Delta\underline{Y}||\right) + O\left(||\Delta\underline{Z}||\right) \quad (11)$$

which when substituted in the formula for volumetric strain (4) leads to

$$\epsilon_V = det\left[\underline{\underline{F}}\right] - 1 \quad (12)$$

Let us now write the deformation gradient matrix in terms of components of displacement gradient, i.e.,

$$\begin{aligned} \underline{\underline{F}} \; &= \; \underline{\underline{I}} + \underline{\nabla}\,\underline{u} \\ \Rightarrow \left[\underline{\underline{F}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} \; &= \; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left[\underline{\nabla}\,\underline{u}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} \\ &= \; \begin{bmatrix} 1 + \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & 1 + \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & 1 + \frac{\partial u_3}{\partial X_3} \end{bmatrix} \end{aligned} \quad (13)$$

To obtain its determinant, let us expand it in terms of first row, i.e.,

$$
\begin{aligned}
det\left[\underline{\underline{F}}\right] &= \left(1+\frac{\partial u_1}{\partial X_1}\right)\left[\left(1+\frac{\partial u_2}{\partial X_2}\right)\left(1+\frac{\partial u_3}{\partial X_3}\right)-\frac{\partial u_2}{\partial X_3}\frac{\partial u_3}{\partial X_2}\right] \\
&\quad -\frac{\partial u_1}{\partial X_2}\left[\frac{\partial u_2}{\partial X_1}\left(1+\frac{\partial u_3}{\partial X_3}\right)-\frac{\partial u_2}{\partial X_3}\frac{\partial u_3}{\partial X_2}\right] \\
&\quad +\frac{\partial u_1}{\partial X_3}\left[\frac{\partial u_2}{\partial X_1}\frac{\partial u_3}{\partial X_2}-\frac{\partial u_3}{\partial X_1}\left(1+\frac{\partial u_2}{\partial X_2}\right)\right] \\
&= 1+\frac{\partial u_1}{\partial X_1}+\frac{\partial u_2}{\partial X_2}+\frac{\partial u_3}{\partial X_3}+h.o.t.
\end{aligned} \tag{14}
$$

The higher order terms contain quadratic and higher order combinations of displacement gradient components. As we are working with displacements such that the displacement gradients are very small, the higher order terms can therefore be neglected. Finally, using equation (12), we get the following formula for local volumetric strain:

$$
\boxed{\epsilon_V = \frac{\partial u_1}{\partial X_1}+\frac{\partial u_2}{\partial X_2}+\frac{\partial u_3}{\partial X_3}} \tag{15}
$$

We can notice that the RHS of this equation is equal to the trace of the displacement gradient matrix. Thus, we can also write

$$
\epsilon_V = tr(\underline{\nabla}\,\underline{u}) = tr\left(\frac{1}{2}\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\right) \tag{16}
$$

It should be noted that unlike longitudinal and shear strains, the volumetric strain turns out to be independent of what triplet of line elements is chosen at a point. Thus, the volumetric strain is unique at a point.

2 Strain Tensor (start time: 23:11)

Let us now look at the expressions of all the strains discussed till now collectively:

$$
\begin{aligned}
\text{longitudinal strain}\ :\ \epsilon_{nn}(\underline{X},\underline{n}) &= \frac{1}{2}\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\underline{n}\cdot\underline{n} \\
\text{shear strain}\ :\ \gamma_{mn}(\underline{X},\underline{n},\underline{m}) &= 2\left(\frac{1}{2}\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\underline{n}\cdot\underline{m}\right) \\
\text{volumetric strain}\ :\ \epsilon_V(\underline{X}) &= tr\left(\frac{1}{2}\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\right)
\end{aligned}
$$

We can notice that the expressions for all the strains contain the tensor $\frac{1}{2}(\underline{\nabla}\,\underline{u} + \underline{\nabla}\,\underline{u}^T)$. This quantity is called the (infinitesimal) Strain Tensor (denoted by $\underline{\underline{\epsilon}}$), i.e.

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\nabla}\,\underline{u} + \underline{\nabla}\,\underline{u}^T\right) \tag{17}$$

This is symmetric part of the displacement gradient tensor. In matrix form in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system, it becomes

$$\left[\underline{\underline{\epsilon}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_1} + \dfrac{\partial u_1}{\partial X_2}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_1} + \dfrac{\partial u_1}{\partial X_3}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_1} + \dfrac{\partial u_1}{\partial X_2}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_2} + \dfrac{\partial u_2}{\partial X_3}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_1} + \dfrac{\partial u_1}{\partial X_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_2} + \dfrac{\partial u_2}{\partial X_3}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{18}$$

The individual components of this matrix can basically be obtained using the following formula:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right). \tag{19}$$

3 Local rotation tensor (start time: 27:58)

We again consider two configurations of our body: the reference configuration and the deformed configuration. Our point of interest $\underline{X}$ in the reference configuration gets mapped to the point $\underline{x}$ in the deformed configuration as shown in Figure 2. We have infinite line elements that one can think of at $\underline{X}$ some of which are shown in Figure 2. Any undeformed line element can be transformed to the deformed line element using the relation $\Delta\underline{x} = \underline{\underline{F}}\Delta\underline{X}$ if the undeformed line element is small enough.

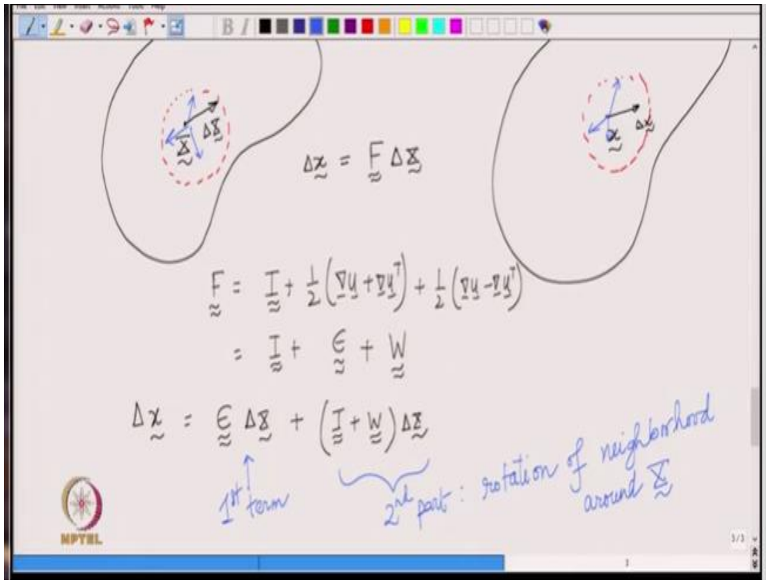

Figure 2: Various line elements in a tiny region at a point $\underline{X}$ in the reference configuration of the body get mapped to line elements in the deformed configuration.

We also know the relation between the deformation gradient tensor and the displacement gradient tensor:

$$\begin{aligned}\underline{\underline{F}} &= \underline{\underline{I}} + \underline{\nabla u} \\ &= \underline{\underline{I}} + \frac{1}{2}\left(\underline{\nabla u} + \underline{\nabla u}^T\right) + \frac{1}{2}\left(\underline{\nabla u} - \underline{\nabla u}^T\right)\end{aligned} \tag{20}$$

The symmetric part of the displacement gradient tensor was defined as the strain tensor ($\underline{\underline{\epsilon}}$). Let us denote the anti-symmetric part of the displacement gradient tensor as $\underline{\underline{W}}$. Thus, we have

$$\underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{\epsilon}} + \underline{\underline{W}} \tag{21}$$

using which we can write

$$\Delta \underline{x} = \underline{\underline{F}} \Delta \underline{X} = \underline{\underline{\epsilon}}\, \Delta \underline{X} + (\underline{\underline{I}} + \underline{\underline{W}})\, \Delta \underline{X} \tag{22}$$

This representation for the deformed line element has a physical meaning associated to it. We again look at Figure 2 where a small region around $\underline{X}$ is shown by the red dotted curve. The line elements in this volume are transformed by multiplying $\underline{\underline{F}}$ to them. According to the representation given in equation (22), the first term $\underline{\underline{\epsilon}}\Delta\underline{X}$ is responsible for straining the small volume around $\underline{X}$, i.e., it generates longitudinal, shear and volumetric strain in this region. If the displacement is such that the strain tensor $\underline{\underline{\epsilon}}$ is $\underline{\underline{0}}$ at a point, then there will be no strain of any kind in the neighbourhood of that point. Stated differently, the line elements in a tiny volume around that point will undergo no change in length or change in angle between them. The tiny volume will also retain its volume. However, due to remaining terms in (22), we will prove later that these line elements will undergo rigid rotation. As $\underline{\underline{W}}$ can vary from point to point, this rigid rotation will be different at different points which is a bit difficult to visualize: the body being deformable (not rigid), tiny volumes around different points of a body need not undergo same rigid rotation. Stated differently, although the body is behaving as a rigid body locally (due to no local strain), as a whole the body deforms due to variation in local rigid rotation. To prove that $\underline{\underline{I}} + \underline{\underline{W}}$ is indeed responsible for local rotation, we first introduce a way to express rotation tensor.

3.1 Rodrigues' Rotation Formula (start time: 38:17)

Suppose, $\underline{a}$ is a unit vector denoting the axis of rotation and ϑ is the angle of rotation about this axis. Then, the rotation tensor is given by

$$\underline{\underline{R}}(\underline{a},\vartheta) = \underline{\underline{I}}\cos\vartheta + \underline{\underline{a}}\sin\vartheta + \underline{a} \otimes \underline{a}\,(1 - \cos\vartheta). \tag{23}$$

Here, $\underline{\underline{a}}$ denotes the skew symmetric tensor corresponding to $\underline{a}$. Stated differently, the vector $\underline{a}$ is the axial vector of the skew symmetric tensor $\underline{\underline{a}}$. This is a general formula for rotation which is valid even when the angle of rotation ϑ is large. Let's consider the case where the angle of rotation (ϑ) is very small. We can then approximate the trigonometric functions using their Taylor's expansion, i.e.,

$$\begin{aligned} \cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + ... &\approx 1 \\ \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ... &\approx \theta \end{aligned} \tag{24}$$

Note that we have kept only the terms that are most significant. Substituting them in equation (23), we get

$$\underline{\underline{R}}(\underline{a},\vartheta) = \underline{\underline{I}} + \vartheta\,\underline{\underline{a}} + \underline{a} \otimes \underline{a}\,(1 - 1) = \underline{\underline{I}} + \vartheta\,\underline{\underline{a}}. \tag{25}$$

Thus, an arbitrary small/infinitesimal rotation can be represented using the above formula which can now be compared with the expression $\underline{\underline{I}} + \underline{\underline{W}}$ in (22). This proves that $\underline{\underline{I}} + \underline{\underline{W}}$ indeed represents local infinitesimal rotation.

3.1.1 An example for rotation (start time: 43:27)

Consider rotation about $\underline{e}_3$ axis by angle ϑ. The rotation matrix for such a rotation was discussed in the first lecture and is given by

$$[\underline{\underline{R}}(\underline{e}_3,\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

If ϑ is taken to be very small, this matrix reduces to

$$[\underline{\underline{R}}] = \begin{bmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

The second term here is a skew symmetric matrix. Its axial vector has only third component non-zero and equal to 1 which is also the column form of $\underline{e}_3$: the axis of rotation in this example. Similarly, the coefficient of skew symmetric matrix is ϑ: the angle of rotation.

3.2 Extracting the axis and angle of local rotation (start time: 46:24)

Let $\underline{w}$ denote the axial vector of $\underline{\underline{W}}$. Then, upon comparing $\underline{\underline{I}} + \underline{\underline{W}}$ with (25), we can conclude the following:

$$\underline{a} = \frac{\underline{w}}{||\underline{w}||}, \quad \theta = ||\underline{w}|| \tag{28}$$

Let us now look at the matrix form of the skew symmetric tensor $\underline{\underline{W}}$ in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system:

$$[\underline{\underline{W}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) & 0 \end{bmatrix} \tag{29}$$

The column form of axial vector of $\underline{\underline{W}}$ will then be

$$[\underline{w}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_3}{\partial X_2} - \frac{\partial u_2}{\partial X_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2}\right) \end{bmatrix} \tag{30}$$

The magnitude of this vector will then be the angle of rotation and the unit vector in its direction will be the axis of local rotation.

3.2.1 An example (start time: 51:13)

Let us consider an example where our coordinate system is ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) and the displacement components are given as follows:

$$u_1 = u_1(X_1,X_2), \qquad u_2 = u_2(X_1,X_2), \qquad u_3 = 0. \tag{31}$$

We have assumed u_1 and u_2 components to be independent of the third coordinate and u_3 is assumed to be zero. So, the strain matrix in this case according to (18) will be

$$[\underline{\epsilon}] = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_1}+\frac{\partial u_1}{\partial X_2}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial X_1}+\frac{\partial u_1}{\partial X_2}\right) & \frac{\partial u_2}{\partial X_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{32}$$

It has basically a 2 × 2 non-zero submatrix. This is also called the plane strain case since all the strains involving $\underline{e}_3$ direction are zero. We can then obtain the axial vector corresponding to local rotation using equation (30) to be

$$[\underline{w}] = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial X_1}-\frac{\partial u_1}{\partial X_2}\right) \end{bmatrix} \tag{33}$$

Let us now see how the above strain and rotation matrices act on the system. As the displacement is confined in $\underline{e}_1 - \underline{e}_2$ plane, we take two line elements in this plane along $\underline{e}_1$ and $\underline{e}_2$ directions with magnitudes ΔX_1 and ΔX_2 respectively (see Figure 3). Let us analyze the effect of strain and rotation on their deformation separately. First, let's consider only $\underline{\underline{\epsilon}}$ acting on the line elements. The line elements do not undergo any rigid rotation for now. Shear strain causes change in angle between the two line elements. The total shear in this case will be

$$\gamma_{12} = 2\epsilon_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \tag{34}$$

which can be thought of as both the line elements rotating by half of this quantity in opposite directions (see the middle of Figure 3). We then superimpose local rigid rotation in the second step. We can deduce the angle of local rigid rotation from equation (33) to be $\frac{1}{2}\left(\frac{\partial u_2}{\partial X_1}-\frac{\partial u_1}{\partial X_2}\right)$. Hence, both the line elements will further rotate by this amount in the same direction.

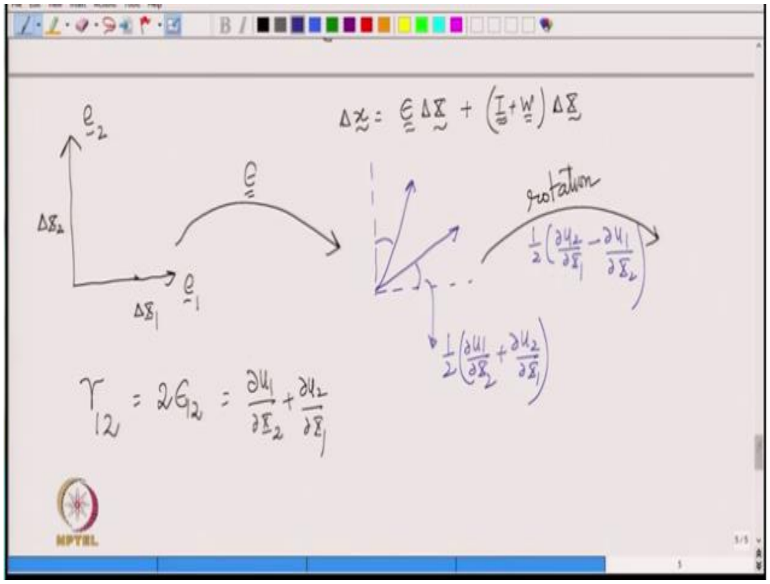

Figure 3: Two line elements in $\underline{e}_1 - \underline{e}_2$ plane in the reference configuration deform by the action of strain and rigid rotation one after the other

We can now add the rotations due to shear and rigid rotation to get the total rotation of the line elements. The total rotation of line element along $\underline{e}_1$ will be

$$\begin{aligned}\text{Total rotation} &= \text{rotation due to shear + local rigid rotation}\\ &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\right)+\frac{1}{2}\left(\frac{\partial u_2}{\partial X_1}-\frac{\partial u_1}{\partial X_2}\right)\\ &= \frac{\partial u_2}{\partial X_1}\end{aligned} \tag{35}$$

For the line element along $\underline{e}_2$, the rotation due to shear is in the opposite direction (clockwise). Therefore

$$\begin{aligned}\text{Total rotation} &= \text{rotation due to shear + local rigid rotation}\\ &= -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\right)+\frac{1}{2}\left(\frac{\partial u_2}{\partial X_1}-\frac{\partial u_1}{\partial X_2}\right)\\ &= -\frac{\partial u_1}{\partial X_2}\end{aligned} \tag{36}$$

The total rotations for the two line elements are exactly the same that we had seen in the last lecture (see section 3.1 in last lecture). This also verifies that we can view the total rotation of a line element as the sum of rotations due to shear and local rigid rotation.