Solid Mechanics
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Lecture - 1
Working with Vectors and Tensors

In this lecture, we will get familiar with vectors, tensors and various mathematical operations involving them.

1 A vector and its representation (start time: 00:53)

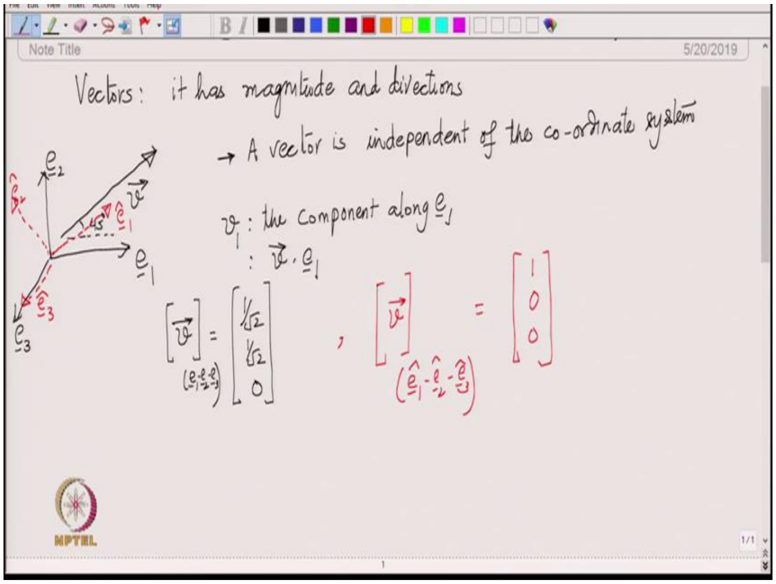
A vector has both magnitude and direction. It is represented by an arrow as shown in Figure 1. The length of the arrow represents the vector's magnitude while the arrow's orientation represents the vector's direction. We have also shown a Cartesian coordinate system here whose basis vectors are ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$). The component of a vector $\vec{v}$ along the basis vector $\underline{e}_i$ is given by

$$v_i = \vec{v} \cdot \underline{e}_i. \tag{1}$$

Geometrically, this denotes the projection of the vector on to the basis vector $\underline{e}_i$. The three components of a vector can be written together in a column and denoted by the symbol $\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$, i.e.,

$$\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{2}$$

The subscript in $\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$ signifies the coordinate system relative to which the vector components have been obtained. At this point, one should note that $\left[\vec{v}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$ and $\vec{v}$ are not the same: the former is the representation of the latter in the specified coordinate system. More importantly, a vector is independent of the coordinate system but its representation changes from one coordinate system to the other. To elaborate this point, think of a unit vector $\vec{v}$ lying in space and being viewed from two different coordinate systems having basis vectors ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) and ($\hat{\underline{e}}_1$,$\hat{\underline{e}}_2$,$\hat{\underline{e}}_3$) respectively (see Figure 1). The red coordinate system can be obtained by rotating the black coordinate system by 45° relative to $\underline{e}_3$ axis. The vector $\vec{v}$ itself lies in $\underline{e}_1$ - $\underline{e}_2$ plane and makes an angle of 45° from $\underline{e}_1$ axis. This also implies that $\vec{v}$ is directed along $\hat{\underline{e}}_1$, i.e., the first basis vector of the red coordinate system.



Vectors: it has magnitude and directions

→ A vector is independent of the co-ordinate system

v_1 : the component along $\underline{e}_1$

: $\vec{v}\cdot\underline{e}_1$

$$[\vec{v}]_{(\underline{e}_1\underline{e}_2\underline{e}_3)} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} , \quad [\vec{v}]_{(\hat{\underline{e}}_1\text{-}\hat{\underline{e}}_2\text{-}\hat{\underline{e}}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Figure 1: A vector $\vec{v}$ observed from two different coordinate systems

The representation of this vector in the two coordinate systems will thus be:

$$[\vec{v}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad [\vec{v}]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Thus, the representation (column form) of a vector varies from one coordinate system to the other but the vector itself is still directed in the same way in the space and hence is independent of the coordinate system. It is also useful to note the below form for a vector:

$$\begin{aligned} \vec{v} &= \sum_{i=1}^{3} (\vec{v}\cdot\underline{e}_i)\ \underline{e}_i = \sum_{i=1}^{3} v_i\ \underline{e}_i \\ &= \sum_{i=1}^{3} (\vec{v}\cdot\hat{\underline{e}}_i)\ \hat{\underline{e}}_i = \sum_{i=1}^{3} \hat{v}_i\ \hat{\underline{e}}_i \quad . \end{aligned} \tag{4}$$

From now on, we will denote a vector $\vec{v}$ by $\underline{v}$, i.e., instead of an overhead arrow, we will use an underbar.

2 Mathematical operations with vectors (start time: 07:45)

In this section, we will talk about various ways in which two vectors can be operated together.

2.1 Dot Product (start time: 08:05)

The dot product between two vectors yields a scalar quantity and hence it's also called scalar product. Basically, the dot product of two vectors is the summation of the product of the corresponding components of the two vectors. The dot product is defined as follows:

$$\begin{aligned} \underline{a} \cdot \underline{b} &= \sum_{i=1}^{3} a_i b_i = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}_{1\times 3} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1} = \left[\underline{a}\right]^T \left[\underline{b}\right] \\ &= ||\underline{a}|| \; ||\underline{b}|| \; cos(\theta) \end{aligned} \tag{5}$$

2.2 Cross Product (start time: 09:47)

The cross product of two vectors yields a vector due to which it is also called vector product. In a coordinate system, say ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$), the cross product can be written as follows:

$$\left[\underline{a} \times \underline{b}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \left[\underline{a}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} \times \left[\underline{b}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} (a_2b_3 - a_3b_2) \\ (a_3b_1 - a_1b_3) \\ (a_1b_2 - a_2b_1) \end{bmatrix} = \underset{\underset{skew\ symmetric\ matrix}{\searrow}}{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{7}$$

Thus, the cross product of two vectors can also be realized as the product of a skew symmetric matrix (a matrix whose diagonal elements are 0 and off diagonal elements are negative of each other) times the column of the second vector. The components of the skew-symmetric matrix are formed by the components of the first vector *a*. In order to easily remember how to form the skew-symmetric matrix from the components of *a*, one can remember the following trick: to get a component in the i^{th} row and j^{th} column, the component of *a* that will be used will be the third index (other than i and j). For example, for 1st row and 2nd column of the matrix, third component a_3 will be used. One then just has to remember where to place the negative signs. We also say

$$\left[\underline{a}\right] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = axial \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{8}$$

Basically, whenever we have a skew symmetric matrix, we can form a column from the three independent entries of that matrix and call the resultant column the axial/axial vector of that skew symmetric matrix. Geometrically, it is defined as follows:

$$\underline{a} \times \underline{b} = ||\underline{a}|| \; ||\underline{b}|| \; sin(\theta) \; \underline{c} \tag{6}$$

Here, $\underline{c}$ is a unit vector perpendicular to the plane formed by $\underline{a}$ and $\underline{b}$. From equations (5) and (6), we can observe that both dot product and cross product of two vectors are independent of the coordinate system. This is because the magnitude of vectors and the angle between the vectors do not change when we change the coordinate system. Thus, coordinate representations of dot product and cross product may be different but the physical result will remain the same.

2.3 Tensor Product (start time: 15:28)

This is a different kind of product which we may not have heard of yet. Through this product, we will also introduce a general notion of tensor. The tensor product of two vectors yields what is called a second order tensor. It is denoted as

$$\underline{a} \otimes \underline{b} = \underline{\underline{C}} \tag{9}$$

Here, $\underline{\underline{C}}$ (with double underline or double tilde) denotes a second order tensor. The tensor product can be represented as follows in a coordinate system:

$$[\underline{\underline{C}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = [\underline{a} \otimes \underline{b}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = [\underline{a}]_{3\times 1}\,[\underline{b}]^T_{3\times 1} \tag{10}$$

Notice that the tensor product implies that the second vector is transposed. This is in contrast with the dot product where the first vector is transposed. The above definition implies that the representation of the second order tensor $\underline{\underline{C}}$ is a matrix whose individual components are given by

$$C_{ij} = a_i b_j. \tag{11}$$

For vectors, we have a single index subscript whereas for the tensor $\underline{\underline{C}}$, we have two subscripts. Thus, vectors are called first order tensors. $\underline{\underline{C}}$ is a second order tensor and that is the reason we denote it by double underline/double tilde.

3 Second order tensors and their representation (start time: 18:21)

The tensor product could also be written as follows:

$$\begin{aligned} \underline{\underline{C}} = \underline{a} \otimes \underline{b} &= \left(\sum_i a_i \underline{e}_i\right) \otimes \left(\sum_j b_j \underline{e}_j\right) \\ &= \sum_i \sum_j a_i b_j \; \underline{e}_i \otimes \underline{e}_j \end{aligned} \tag{12}$$

Upon contrasting the above form with the expansion of a vector in equation (4), we make a note that, just like a general vector is expressed as a linear combination of three basis vectors, a tensor can be expressed as a linear combination of nine basis tensors. Each of the basis here ($\underline{e}_i \otimes \underline{e}_j$) are themselves tensors. Thus, a general second order tensor can be written as

$$\underline{\underline{C}} = \sum_i \sum_j C_{ij} \; \underline{e}_i \otimes \underline{e}_j \tag{13}$$

The nine coefficients C_{ij} are in general independent of each other. The coefficient C_{ij} can be thought of as the component of the tensor $\underline{\underline{C}}$ along the basis tensor $\underline{e}_i \otimes \underline{e}_j$. Using (10), we can also say the following for basis tensors, e.g.,

$$[\underline{e}_1 \otimes \underline{e}_2]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

Thus, just like basis vectors $\underline{e}_i$ have unique column form, each of the basis tensors has a unique matrix form. Using (13) and (14), it is easy to see that the coefficient C_{ij} in (13) also forms the i^{th} row and j^{th} column of the matrix representation of $\underline{\underline{C}}$ in $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ coordinate system, i.e., $\left[\underline{\underline{C}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)}$. Just like vectors, the matrix form of a tensor changes from one coordinate system to the other but the tensor itself does not change, e.g., if

$$\underline{\underline{C}} = \sum_i \sum_j C_{ij}\ \underline{e}_i \otimes \underline{e}_j = \sum_i \sum_j \hat{C}_{ij}\ \underline{\hat{e}}_i \otimes \underline{\hat{e}}_j$$

$$\downarrow \text{(matrix form)} \downarrow$$

$$[C_{ij}] \qquad\qquad [\hat{C}_{ij}] \tag{15}$$

Thus, we get different matrices in different coordinate system but the tensor itself is still the same. As a final remark, tensors can be of any order which are all independent of the coordinate system but their representations change from one coordinate system to the other. Scalars, e.g., are zeroth order tensors and all vectors are first order tensors. We can also have third and fourth (even higher) order tensors.

4 Mathematical operations involving tensors (start time: 26:52)

We will now discuss how a tensor operates with another tensor or a vector.

4.1 Multiplication of a second order tensor with a vector (start time: 26:52)

$$\begin{aligned} \underline{a} = \underline{\underline{C}}\,\underline{b} &= \left(\sum_i \sum_j C_{ij}\ \underline{e}_i \otimes \underline{e}_j\right)\left(\sum_k b_k\ \underline{e}_k\right) \\ &= \sum_i \sum_j \sum_k C_{ij}\ b_k\ (\underline{e}_i \otimes \underline{e}_j)\ \underline{e}_k. \end{aligned} \tag{16}$$

When a second order tensor is multiplied with a first order tensor, then the second vector from the tensor gets dotted with the first order tensor. This is how the multiplication is defined:

$$\underline{\underline{C}}\,\underline{b} = \sum_i \sum_j \sum_k C_{ij}\ b_k\ \underline{e}_i\ (\underline{e}_j \cdot \underline{e}_k) \tag{17}$$

$$= \sum_i \sum_j \sum_k C_{ij}\ b_k\ \underline{e}_i\ \delta_{jk}. \tag{18}$$

Here δ_{ij} is the Kronecker delta function and is defined as

$$\delta_{ij} = 1 \textit{ if } i = j$$

$$= 0 \textit{ if } i \neq j$$

Now consider the summation over k in (18), due to the Kronecker delta function present there, only the terms having j=k will contribute to the summation and the others will be zero. Thus, we can get rid of the summation over k and replace k by j at all places, i.e.

$$\Rightarrow \underline{a} = \underline{\underline{C}}\,\underline{b} = \sum_i \underbrace{\left(\sum_j C_{ij} b_j\right)}_{a_i} \underline{e}_i \tag{19}$$

As any general vector can be written as $\underline{a} = \Sigma\, a_i \underline{e}_i$, the above term in parentheses comes out to be a_i. Thus, when we multiply a second order tensor with a vector, we get a vector whose components are given by

$$a_i = \sum_j C_{ij} b_j = \underbrace{\begin{bmatrix} C_{i1} & C_{i2} & C_{i3} \end{bmatrix}}_{i^{th}\ row\ of\ [\underline{\underline{C}}]} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{20}$$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [\underline{\underline{C}}]\ [\underline{b}] \tag{21}$$

Thus, we simply multiply the matrix representation of $\underline{\underline{C}}$ with the column form of $\underline{b}$ to get the column form of the resulting vector $\underline{a}$. Let us recall the cross product definition in (7) where we had written it as a skew symmetric matrix times a vector. On further noting the multiplication we just saw in (21), we immediately conclude that the cross product of two vectors can also be thought of as a second order tensor times the second vector where the second order tensor corresponds to the first vector, i.e.,

$$\underline{c} = \underline{a} \times \underline{b} = \underline{\underline{a}}\,\underline{b}. \tag{22}$$

Here, $\underline{\underline{a}}$ is the skew symmetric tensor formed from the first vector $\underline{a}$.

4.2 Extracting the coefficients in matrix representation of a tensor (start time: 35:57)

To get the coefficient, say C_{kl} of the matrix form of a tensor $\underline{\underline{C}}$ in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system, we'll verify that

$$C_{kl} = (\underline{\underline{C}}\,\underline{e}_l) \cdot \underline{e}_k \tag{23}$$

The right hand side of the above expression is in tensor form. Let us write it in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system for k=1, l=2:

$$(\underline{\underline{C}}\,\underline{e}_2) \cdot \underline{e}_1 = \left(\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_{12} \\ C_{22} \\ C_{32} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_{12}$$

This verifies our assertion (23). Stating differently, by using equation (23), we are also able to extract the component of an arbitrary tensor $\underline{\underline{C}}$ relative to the basis tensor $\underline{e}_k \otimes \underline{e}_l$.

4.3 Mutiplying two second order tensors (start time: 39:08)

There are various ways in which two second order tensors can be operated together. We will consider the usual multiplication of two tensors which yields another second order tensor:

$$\begin{aligned}
\underline{\underline{C}} &= \underline{\underline{a}}\,\underline{\underline{b}} \\
&= \left(\sum_i \sum_j a_{ij}\, \underline{e}_i \otimes \underline{e}_j\right)\left(\sum_k \sum_l b_{kl}\, \underline{e}_k \otimes \underline{e}_l\right) \\
&= \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl}\, (\underline{e}_i \otimes \underbrace{\underline{e}_j)\ (\underline{e}_k}_{\text{inner ones are dotted by definition}} \otimes \underline{e}_l) \\
\Rightarrow \underline{\underline{C}} &= \sum_i \sum_j \sum_k \sum_l a_{ij} b_{kl}\, (\underline{e}_i \otimes \underline{e}_l)\ \delta_{jk}
\end{aligned} \tag{24}$$

Using Kronecker delta property, we remove the summation over j and replace j with k everywhere:

$$\begin{aligned}
\Rightarrow \underline{\underline{C}} &= \sum_i \sum_k \sum_l a_{ik} b_{kl}\, \underline{e}_i \otimes \underline{e}_l \\
&= \sum_i \sum_l \underbrace{\left(\sum_k a_{ik} b_{kl}\right)}_{C_{il}} \underline{e}_i \otimes \underline{e}_l
\end{aligned} \tag{25}$$

As this is the expression for $\underline{\underline{C}}$ and any general second order tensor $\underline{\underline{C}}$ can be written as $\Sigma_i\ \Sigma_l\ C_{il}\, \underline{e}_i \otimes \underline{e}_l$ (using equation (13)) . Noting this in the above expression, we see that the expression within the bracket is nothing but C_{il}, i.e.

$$\begin{aligned}
C_{il} &= \sum_k a_{ik} b_{kl} \\
&= \underbrace{\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix}}_{i^{th}\ row\ of\ [\underline{\underline{a}}]}\ ,\ \underbrace{\begin{bmatrix} b_{1l} \\ b_{2l} \\ b_{3l} \end{bmatrix}}_{l^{th}\ column\ of\ [\underline{\underline{b}}]}
\end{aligned} \tag{26}$$

Writing this for all components together, we would get

$$[\underline{\underline{C}}] = [\underline{\underline{a}}]\;[\underline{\underline{b}}] \tag{27}$$

Thus, we see that when we multiply two tensors, their matrix forms multiply in the usual way. The matrix form of the resultant tensor is simply the multiplication of the matrix forms of individual tensors.

5 Rotation tensor (start time: 47:04)

Now, we will learn about rotation tensors. They are tensors that are related to physical rotation of objects. They can be used to rotate vectors as well as tensors. It should have the property such that after rotation, vectors and tensors do not change their magnitude but only direction. Let us consider two sets of orthonormal triads (see Figure 2: each set contains three vectors which are perpendicular to each other and also of unit magnitude).

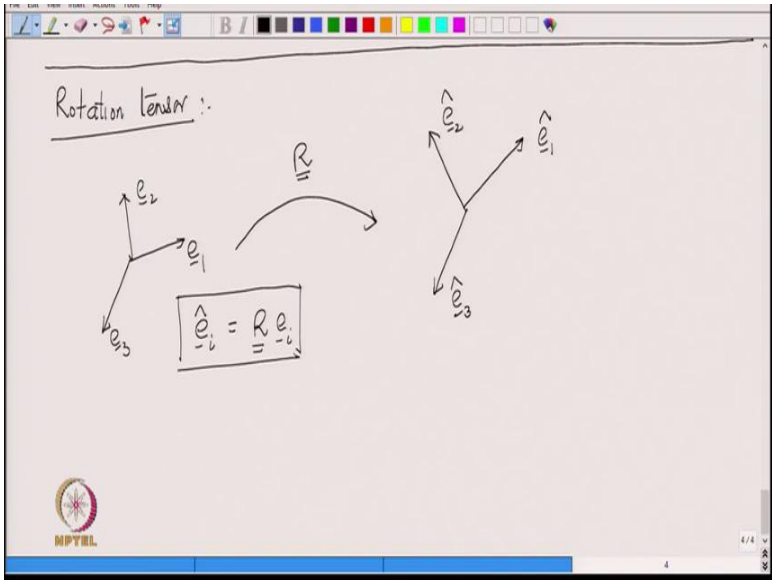


Figure 2: Two sets of orthonormal triads related through a rotation tensor

One can always transform a set of triad into another through a unique rotation or what we call a unique rotation tensor. Mathematically, we can write

$$\hat{\underline{e}}_i = \underline{\underline{R}}\,\underline{e}_i,\ \forall\, i = 1,2,3. \tag{28}$$

The matrix form of this rotation tensor turns out to be an orthonormal ('ortho' means perpendicular and 'normal' means normalized) matrix. The tensor itself is called an orthonormal tensor. Orthonormal tensors and their matrix forms have the following properties:

$$(a)\ \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}\ \text{(an identity tensor)}, \qquad (b)\ det(\underline{\underline{R}}) = 1. \tag{29}$$

$$\begin{aligned} &\Rightarrow\ [\underline{\underline{R}}]\;[\underline{\underline{R}}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(matrix form of (29a))} \\ &\Rightarrow\ \sum_k R_{ik}R_{jk} = \delta_{ij}\ \forall i,j = 1,2,3 \quad \text{(using indical notation)} \end{aligned} \tag{30}$$

This means that rows of $[\underline{R}]$ are perpendicular to each other and are themselves normalized of unit magnitude, i.e., orthonormal. One can similarly prove that its columns are also orthonormal. Let us consider a specific example and see how an actual rotation matrix looks like. In Figure 3, we have $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ coordinate system and it is rotated to get the new coordinate system $(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)$. The rotation is such that $\hat{\underline{e}}_3$ is same as $\underline{e}_3$ and the other two basis vectors are rotated by θ.

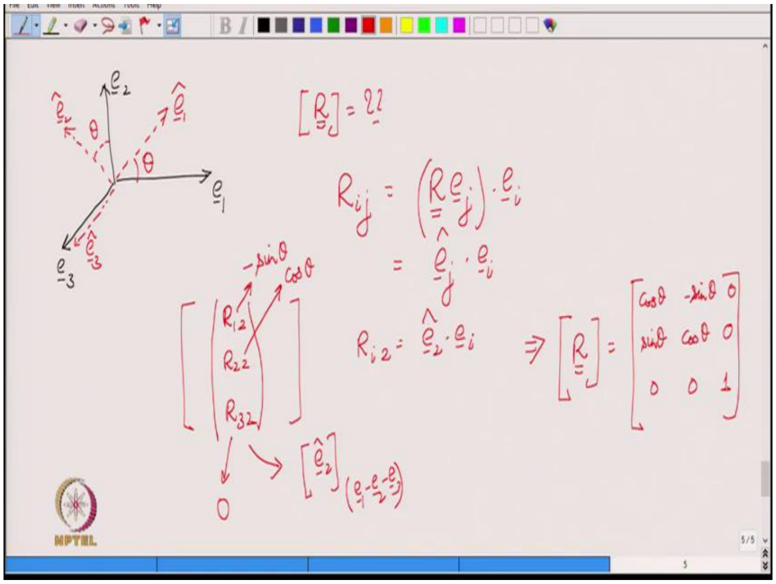


Figure 3: A basis triad rotated by an angle θ about $\underline{e}_3$

Let us determine the matrix form of $[\underline{R}]$ in $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ coordinate system. We know from equation (23) that

$$\begin{aligned} R_{ij} &= (\underline{R}\,\underline{e}_j)\cdot\underline{e}_i \\ &= \hat{\underline{e}}_j\cdot\underline{e}_i \qquad \text{(using (28))} \end{aligned} \tag{31}$$

So, the j^{th} column of the rotation matrix is column form of the vector $\hat{\underline{e}}_j$ expressed in $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ coordinate system. For example, consider j=2:

$$R_{i2} = \hat{\underline{e}}_2\cdot\underline{e}_i \tag{32}$$

We can find individual coefficients in terms of θ using dot product definition, i.e.,

$$\hat{\underline{e}}_2\cdot\underline{e}_1 = -\sin\theta, \qquad \hat{\underline{e}}_2\cdot\underline{e}_2 = \cos\theta, \qquad \hat{\underline{e}}_2\cdot\underline{e}_3 = 0 \qquad \text{(see Figure 3)} \tag{33}$$

Working out all the components in this way, we get

$$[\underline{R}] = \begin{bmatrix} cos\theta & -sin\theta & 0\\ sin\theta & cos\theta & 0\\ 0 & 0 & 1 \end{bmatrix} \tag{34}$$

Note that the third column is $[0\,0\,1]^T$ as $\hat{\underline{e}}_3$ is same as $\underline{e}_3$. We emphasize that the above matrix form is the representation of rotation tensor $\underline{R}$ in $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$ coordinate system.

In the next lecture, we will start with the concept of the traction vector.