Computer Aided Design
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Lecture No. # 34
Modelling of B-Spline Curves (Contd.)

With respect to B-spline curves, last time we had derived this expression that if we consider any curve segment then for second degree curve then that can be given by this expression.

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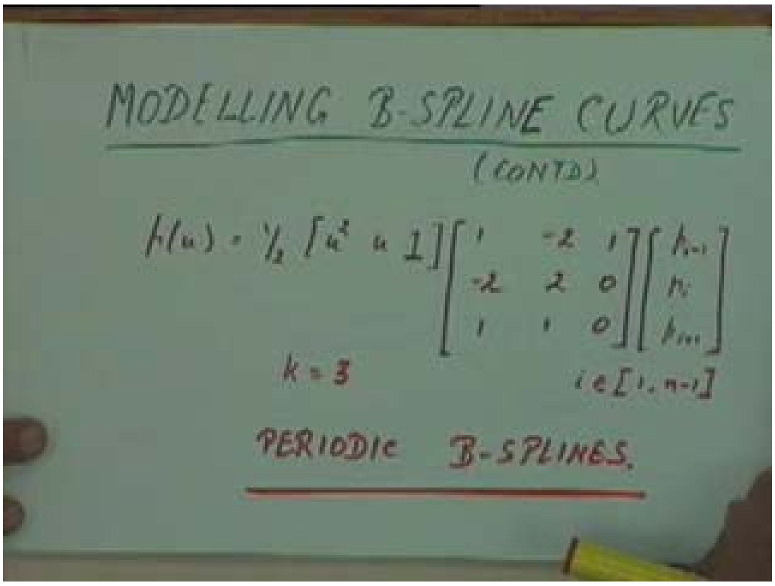

This means that if we have a set of control points and a curve is defined using these control points then for some portion of the curve will have three points let's say this i minus 1, this is p_i and this is p_{i+1}.

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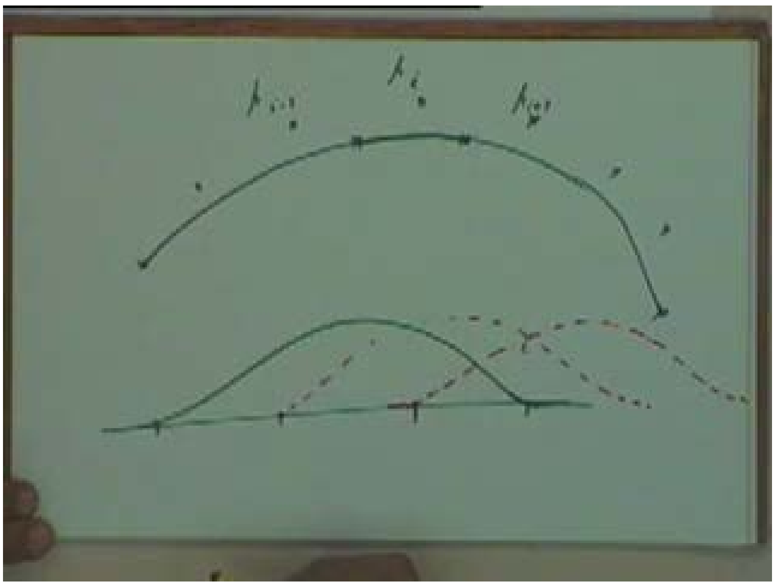

These three points will together define this curve segment using this kind of relationship. And this essentially means that if we consider the shape function for these three points or for any of these points for that matter, they will look like this where these are different naught values. Let's say this is a shape function for the i'th point. If I look at that the next point that will have a shape function which will be effective from here but will have an identical shape to this. The shape function after that will look something like this. All the shape functions are periodic in nature and that is this formulation is called a periodic B-spline curve. And if we define B-spline curve in this manner, the total number of curves for a given end points will be n minus 1, total number of curve segments. So we will have n minus 1 curve segment defined by this kind of relationship. First one will be for i equal 1 that means P_0 P_1 and P_2. The next one will be between P_1 P_2 and P_3 and so on.

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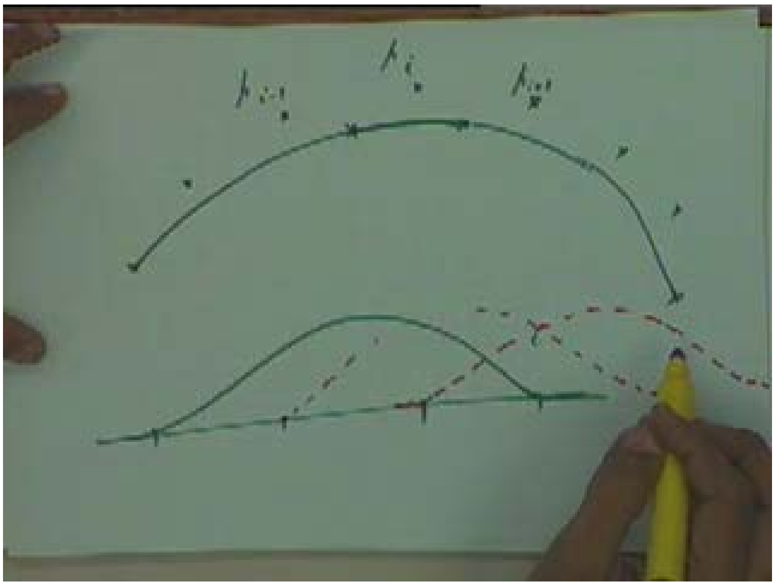

So this is for the case of B-spline curves with k equal 3 and so far we are considering only those curves where the two end points are not the same. That means we are taking of open B-spline curves, we haven't yet talked of closed B-spline curves.

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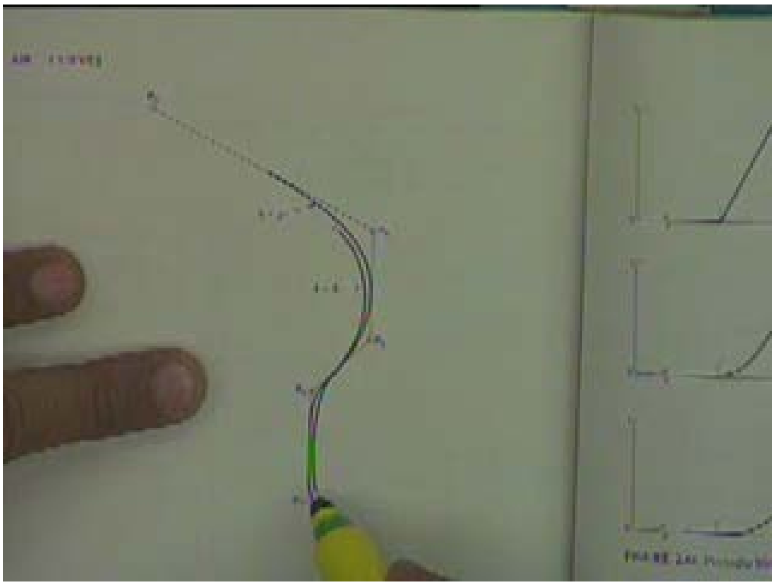

For periodic B-spline curves if we look at a figure, if you look at this figure the given 6 points p_0 p_1 p_0 p_1 p_2 p_3 p_4 and p_5 and if you look at this curves which I mentioned with k equal to 3 that is a periodic curve for k equal to 3. And I had explained last time, a periodic curve doesn't start from the end point. It will, in this particular case it will start from a mid-point between p_0 and p_1, this is for second degree curves. This curve for which is mentioned with k equal to 3 that is a curve we have to look at.

Another property that is seen for curves specifically with k equal to 3 is that if you have to consider line joining let's say p_4 and p_3, the curve will be tangent to that curve. You see at this point, this line joining p_3 p_4 at the midpoint this B-spline curve is tangent to it. Similarly, the line joining p_2 p_3 at the midpoint, the curves crossing with but tangentially. This happens only in the case of k equal to 3 curves that means second degree B-spline curves. If we consider third degree B-spline curves they don't start from the starting point, they don't from the midpoint but they start from some other point totally. This point, it is a starting point for k equal to 4 curve with the same set of control points and similarly here this point is the end point for this curve.

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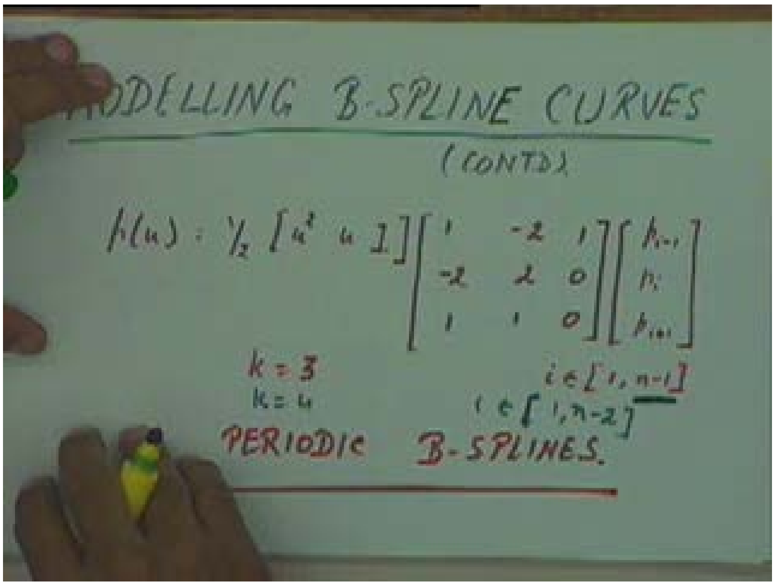

And the number of curve segments, we will see the number of curve segments is n minus 1 when k is equal to 3 and when k is equal to 4, the number of curve segments will be n minus 2. The formulation for k equal to 4 case would be p of u is equal to one sixth of (Refer Slide Time: 06:36) and i exist between 1 and n minus 2 because the last point that we are saying is i plus 2 and the last point is p_n. So number of curve segments will go up to n minus 2.

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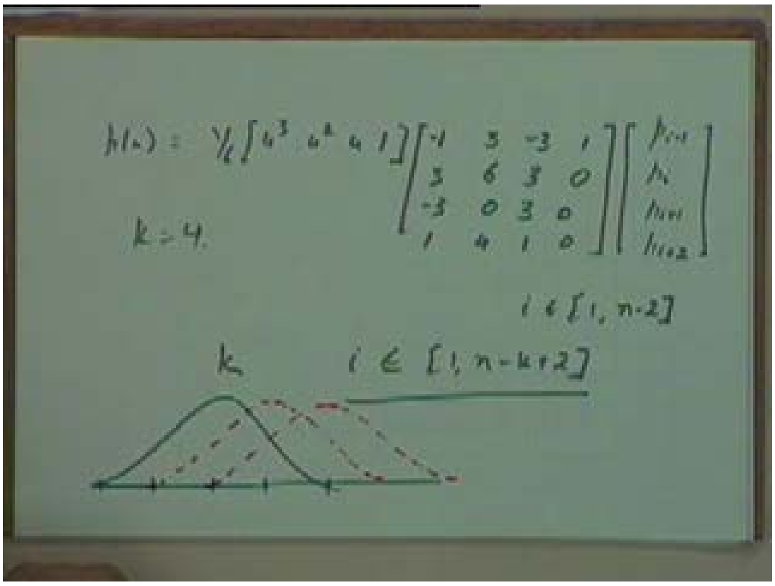

Similarly for a general k degree curve, the number of curve segments will be n minus k plus 2. This is for general curve with degree k minus 1, this will be the total number of curve segments in the case of a open periodic B-spline curve. Is that okay? And for the k equal to four case, the blending function look like a cubic look like a cubic curve spanning four naught ranges and the next one will of course again be, it will have the same function but with the stagger and so on.

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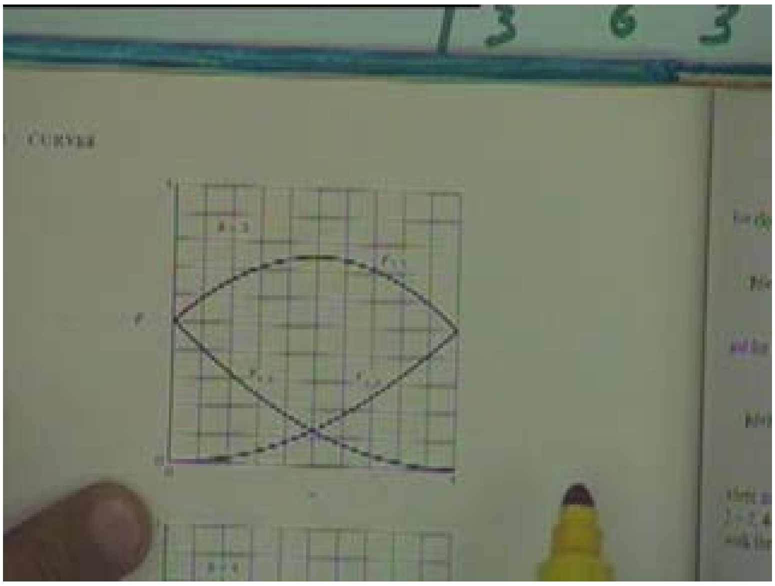

If we look at the shape functions, this is giving the four shape functions for k equal to 3. This f_{13} the first shape function, f_{23} is second shape function and f_{33} is third shape function, three shape functions for k equal to 3. This is all within one single range that means I have drawn this figure.

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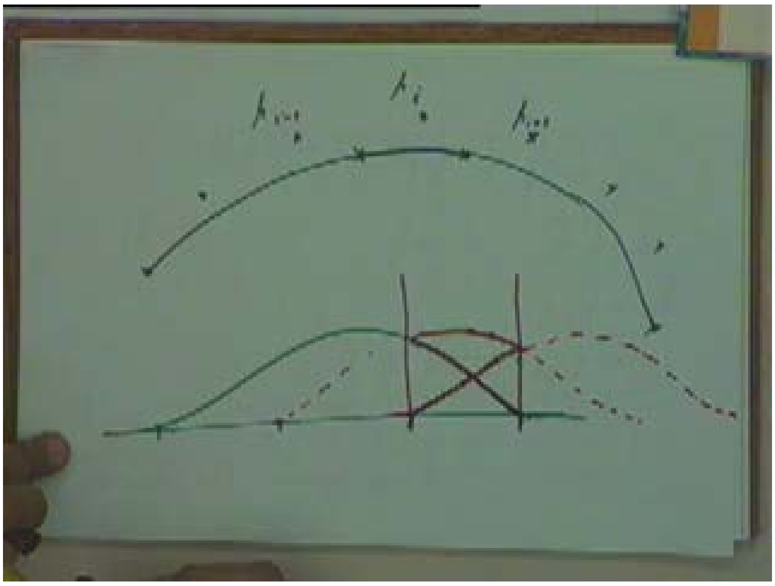

In this figure if you look at this particular range, this is where three points are effective. One shape function looks like this, one looks like this and the third looks like this. The exact shape is like this and this is for k equal to 3.

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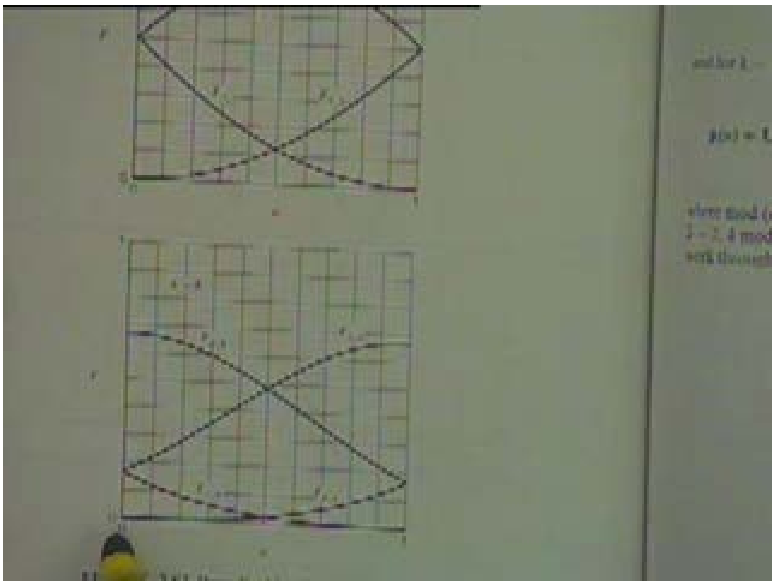

And for k equal to 4, the shape functions will look like this, this is 1 4 and 4 4 is symmetric like this, 2 4 is like this and 3 4 is like this again they are symmetric. And for any curve segment we have 4 points which are effective and again the starting point can be computed by looking at these weightages. So this is the, let's say if we look at the first curve segment which is between p_0 p_1 p_2 and p_3 the weightages to be given to each of the 4 points will be given by the values at the first, at the starting on the curves. Similarly the end point will be computed by these 4 weightages. This is in the case of open periodic B-spline curves and I have already mentioned that the kind of continuity that will be there at the curve segments will be of the order of k minus 2 or C k minus 2.

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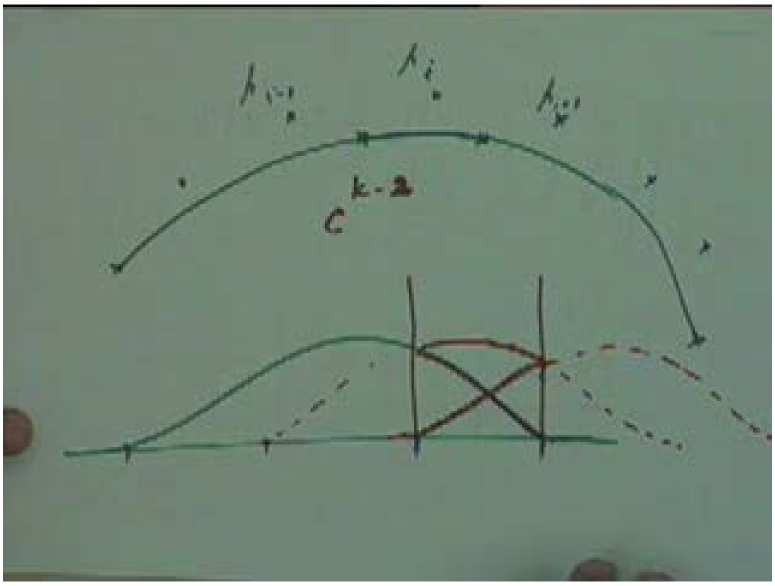

That means when k is equal to 3, we will have C_1 continuity at these points and when k is equal to 4 we will have C_2 continuity at these points. Any question up to this point?

In that case let's take the next formulation in case of B-spline curves that is if we have a set of points and we want to have the first and the last point the same.

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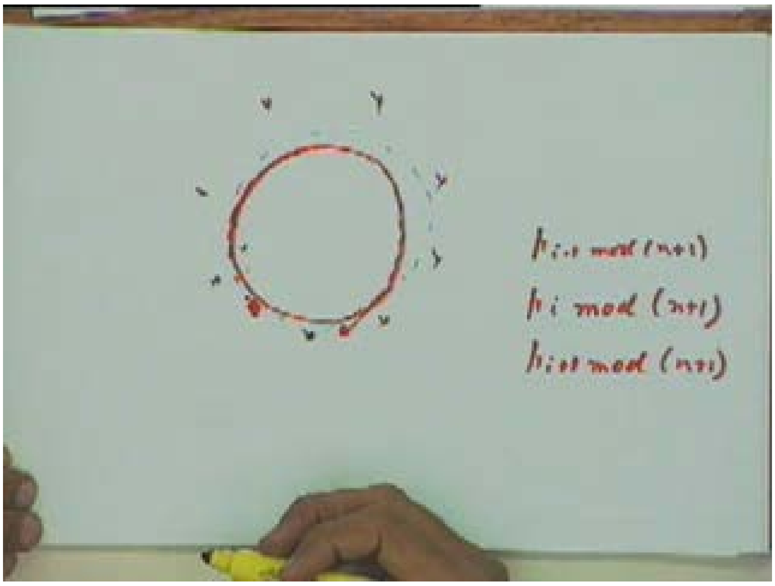

Then if we remember the case of a Bezier curves, we have defined the closed curve in which the first and the last points are same and we got a curve which are something like this. Now if we do a similarly thing in the case of B-spline curves, let's say again if we start with k equal to 3 we have one curve defined by these 3 points that will look something, that will be let's say this curve segment. The next curve defined by these 3 points, the next one defined by these 3 and so on. The last curve will be defined by these 3 points but we can go a step further and then define another curve by these 3 points, if we are talking of periodic curves.

If I define a closed curve like this which is let's say to start with non-periodic then the non-periodic curve let's say for k equal to 3 will start from this point and this point and it will look something like this, this is how it will look like. It's not going to start from this point, it will be starting from a midpoint of this edge and the midpoint of this edge. So even though my first and the last point are the same, my curve is not a closed curve. So if I have to define a closed curve, the way to define that it would be that I will take these 3 points and if I another curve segment and so on.

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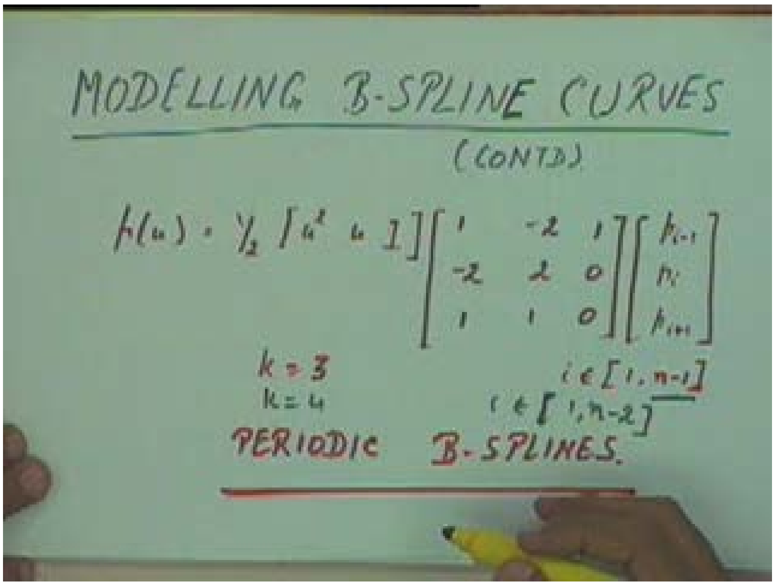

So when I am defining my curve using p_{i-1}, p_i and p_{i+1}, I will actually have to take these three points in a cyclic order. And what we will do is we will define every curve segment between p_{i-1}, p_i, p_{i+1} and we will take a mod n with n plus 1, so that my first curve will be between these 3. Next curve will be between these 3 and so on and there will be another curve which will be defined by these 3 points. As a result of that we will get a continues curve between these, between all the control points and the actual curve instead of starting from here or here, it will look like a closed curve which might be something like this.

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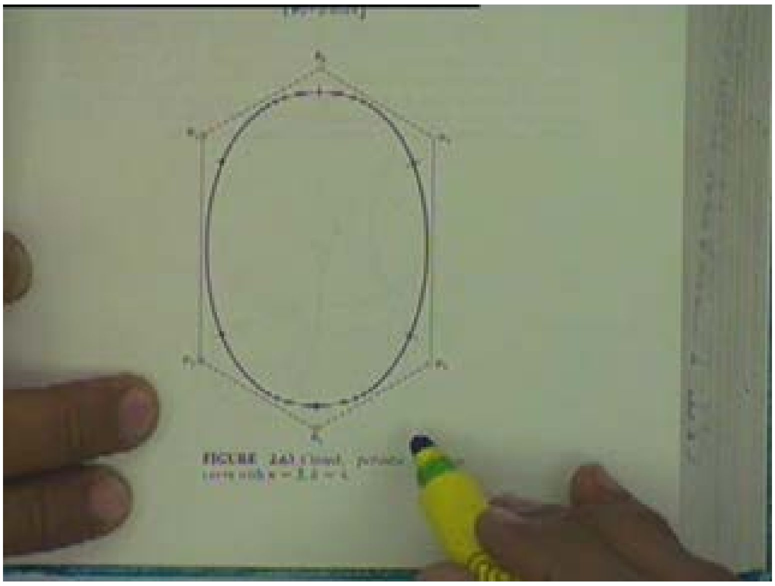

Let's have a look at it and then we will see the details. So look at this curve, this is a closed curve, closed periodic B-spline curve with n is equal to 5 that means we have total of 6 points and k is equal to 4. That means it's a cubic closed B-spline curve. Now we have this curve segment, all the marks which we are seeing these are the different boundaries of the curve segment. So we have one curve segment here which is controlled by 4 points which are P_5 P_0

P_1 and P_2 probably. Next curve segment is controlled by the next 4 points and so on. It won't touch any of the lines that was for k equal to 3, only for k equal to 3. So this won't touch any of the lines, it won't pass through any of the points but it will be within the polygon formed by or within the convex hull of the control points. This is a start from a point which you can find out as I said by looking at the value of the blending functions at the starting point. So it will start from a point which will be given by let's say this is about 0.75 and this is about 0.15.

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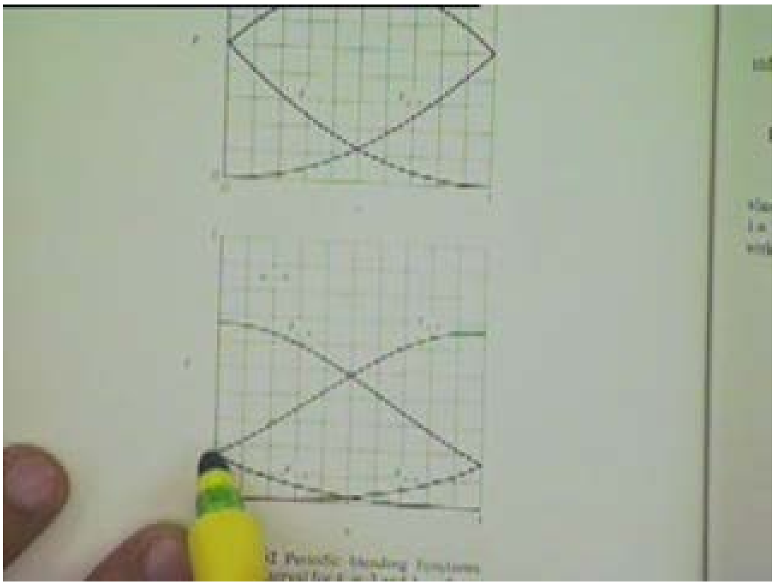

So 0.15 times p_0 plus 0.15 times p_2 plus 0.7 roughly times p_1. You are taking the combination of these three points that is what will give you the starting point which is not the midpoint, which is not a control point. Looking at this, so we will find that each of these, we will find at each of these curve segments are controlled by four points. The four points will now go on in a cyclic manner, there will be one curve segment controlled by P_4 P_5 P_0 and P_1. In other curve segment will be controlled by P_3 P_4 P_5 and P_0 and other curve segment will be controlled by P_5 P_0 P_1 and P_2. The number of controlled segments will be the same as the number of points. If you have n plus 1 points, you have n plus 1 control segment, n plus 1 curve segments.

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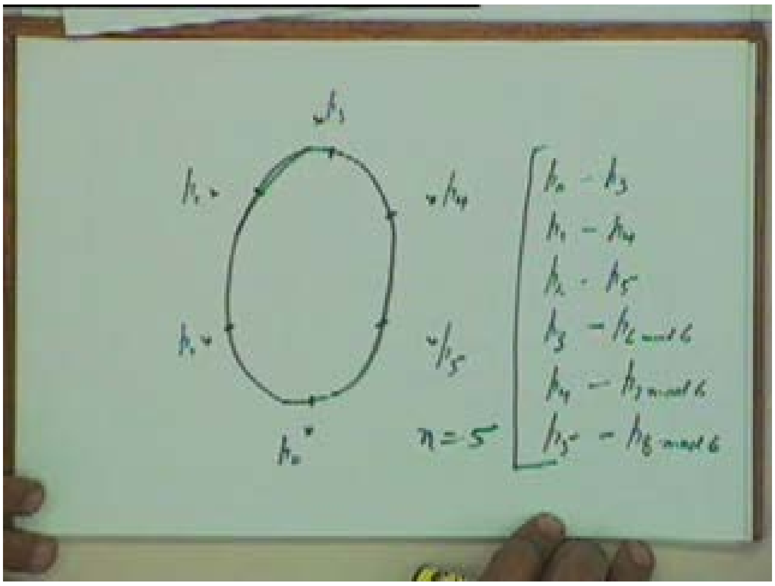

I will just, so our control points are something like this. The curves are something like this. Now these are the curves segments. If you look at let’s say the first curve segment that is between P_0 P_1 P_2 and P_3 let’s say, so one is defined between P_0 to P_3. The next one will be defined between P_1 to P_4, P_2 to P_5. Now when we say P_3 to P_6, we are taking the mod with n plus 1. In this case we have taken n equal to 5, it will be up to P_6 mod with 6 which is the same as P_0. The next one will be P_4 to P_7 mod 6. All the number that we are taking are mod with 6 and P_5 to P_8 mod 6. That is how we will get the 6 control segments, 6 curve segments sorry.

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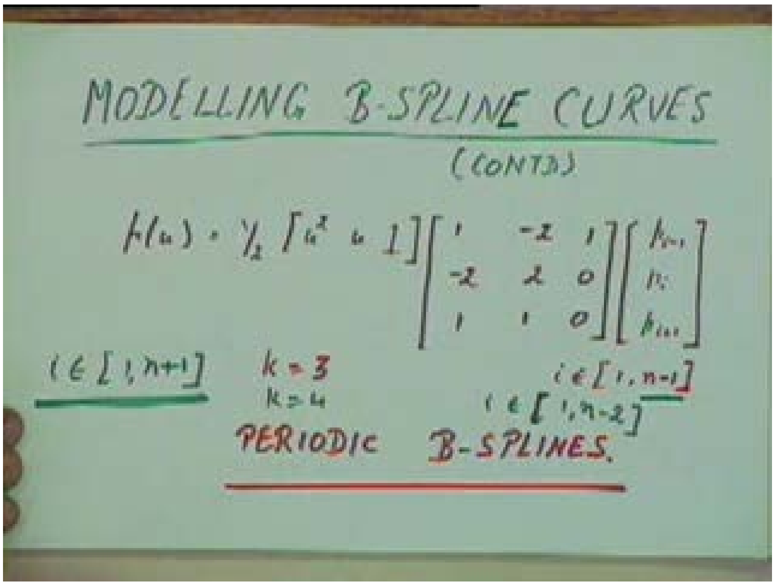

And if we look at this formulation, this was of course for k equal to 3. For k equal to 3 then earlier the number of curve segments that we had was from 1 to n minus 1. Now it will become i will be between 1 and n plus 1, even when k is equal to 4 the number of curve segments will remain the same. The number of curve segments will be same as number of points. Any questions up to this point?

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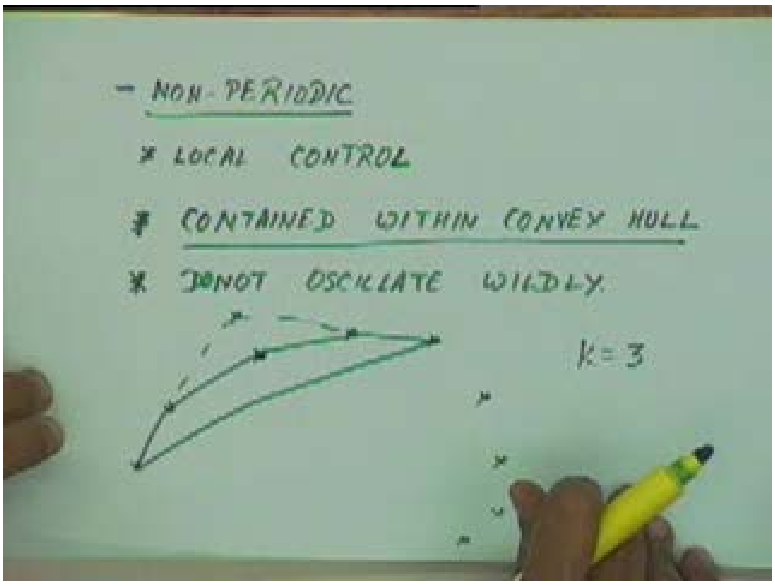

Then let's see some of the properties that we have seen of the B-spline curve so far. First is let's start with non-periodic B spline curves. First is they gives us local control and this is true for all B-spline curves whether they are periodic, non-periodic closed or open. They will always give us local control which is governed by the degree of the curve. Then they are contained within the convex hull. And as we had seen for Bezier curves they don't oscillate. In fact when we said that they are contained within the convex hull for B-spline curves, this property is much stronger than that for Bezier curves. What that means is if we have a set of control points like this and we are considering let's say k equal to, for the sake of argument we start with 3. If we consider any point and we consider k points to one side, k points to the other side whatever convex hull will get from these points.

The curve segment corresponding to this will lie within this convex hull, it won't go beyond this convex hull even if you have other points. That means the effect of this will remain within this convex hull. Even if I move it from here to here, the convex hull will change like this. So my curve segment corresponding to these points will remain within this, it won't go beyond this convex hull. So this convex hull property of B-spline curves is stronger than that of Bezier curves and this property of course shared by periodic as well as closed B-spline curves.

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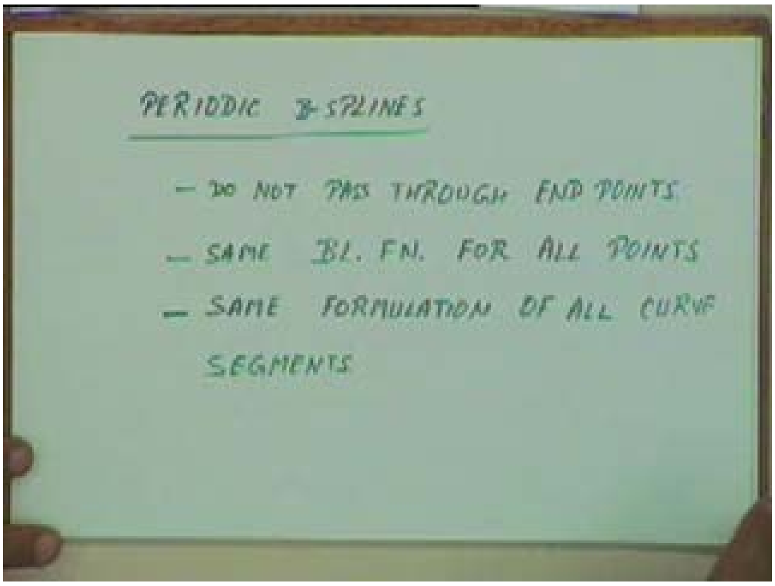

Then if we come to periodic curves, these do not pass through end points. You have a same blending function, I am using an abbreviation blending function for all points and our same description or same formulation of all curve segments.

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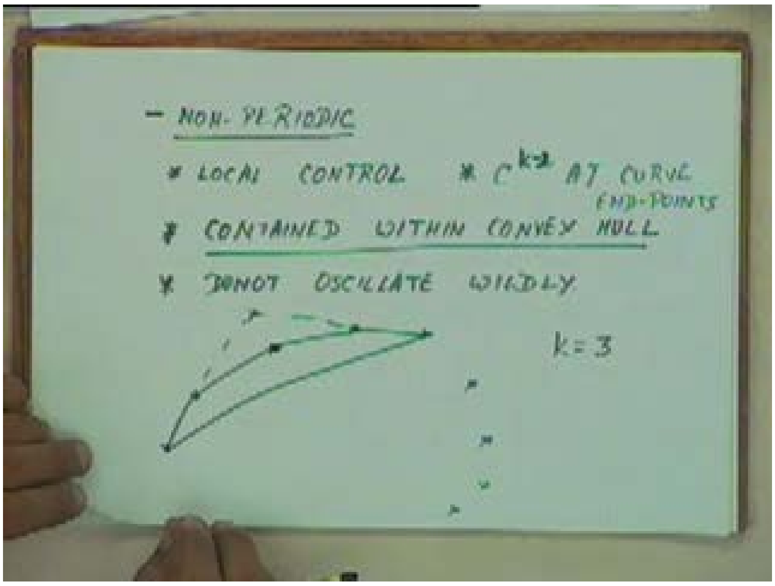

And of course I should have mentioned here itself that I am talking a non-periodic curves and of course for all other types B-spline curves, the degree of continuity is C k minus 1 continuity, C k minus 2 continuity sorry. So if my k is equal to 3, I will have C_1 continuity, k is equal to 4 I will have C_2 continuity and so on at curve end points, curve segment end points and degree of continuity will be C k minus 2. Then same formulation for all the curve segment this is for periodic B-splines.

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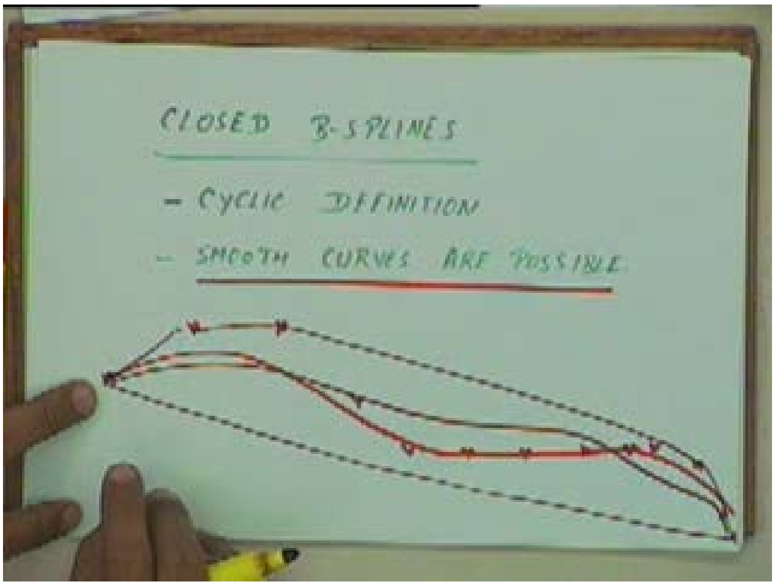

If we look up the properties of closed B-splines, the first thing that is done is we have a cyclic definition and we say smooth curves are possible. In case of Bezier curves or closed curves are not smooth. In this case we are getting smooth curves as closed B-spline curves. So this is the main difference as far as closed curves are concerned with respect to Bezier curves. And of course it can be shown that all these B-spline curves they are the generalization of the Bezier curve, in fact these are a generalization of Bezier.

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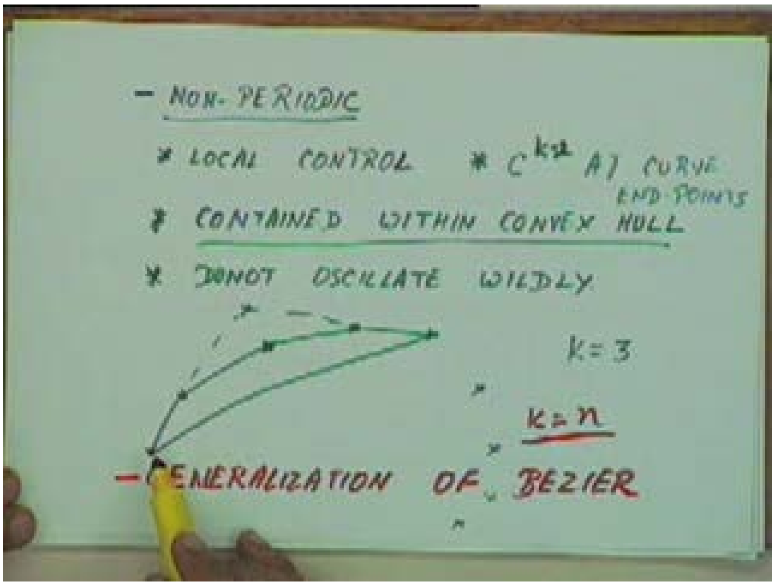

That means if we take k equal to n, that means if we take the degree of the curve to be same as the number of points in that case our B-spline curve will become similar to the Bezier curve. Your local control property will be lost and we will get almost a global control. So a non-periodic curve will be a generalization of the Bezier curve. And of course non-periodic curves for lower values of k, for k equal to 1 and k equal to 2 they can also give us line segments and a discrete points because even the lines segments can be modeled as B-spline curves.

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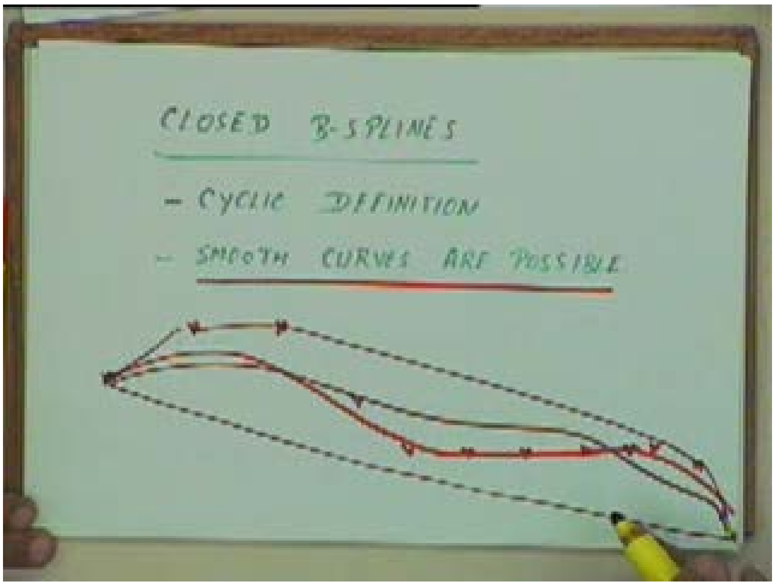

Just one more point, that is I have mentioned that B-spline curves follow or fall within a convex hull which is something like this which means that if I have set of control points which are let say I have a set of linear points here and then a set of control points after this. My curve let say might look something like this but in this portion, it will become a straight line and then again it might go like this. The portion of a B-spline curve can be straight line. It can pass through some of control points in that case, not otherwise. This kind of thing cannot happen in case of Bezier curves. No, it has to pass through all these points because my convex hull of these points is a straight line, it has to fall within the convex hull. This is with respect to B-spline curve. Any questions with respect to B-spline curves?

In this case what will happen to the Bezier curves? A Bezier curve within these points will lie within the convex hull, the convex null looks like this. So my curve might look something like this, this is my Bezier curve. This is the complete convex hull, my curve will lie within these bounds but the B-spline curve that can vary depending on the degree. There is a lot of local control.

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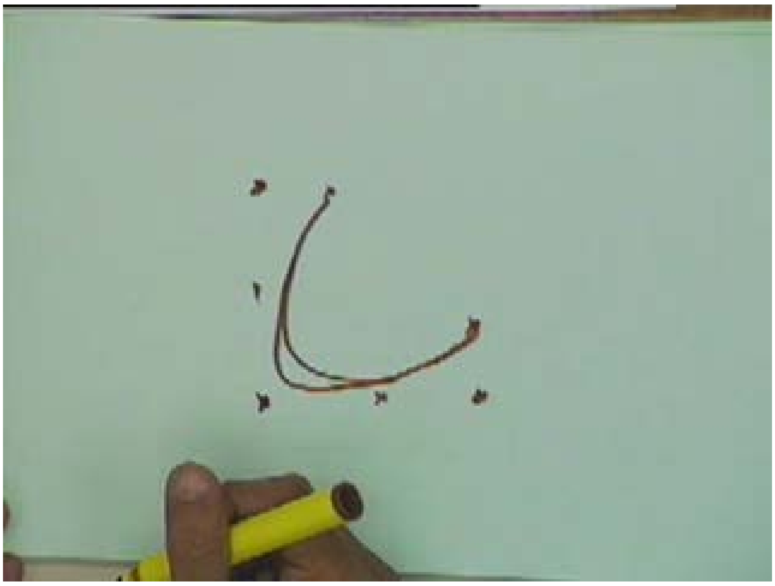

Of course, other property that if we have number of points coincided at the same place, my curve let say if it is initially like this and I have multiple points here my curve will get pull towards it. That will happen even in the case of B-spline curves. If I have too many points in the same place, my curve might even pass through to that. That is again because if I have multiple points here, my convex hull becomes a single point, so my curve has to go through that point. Any questions on anything regarding modeling of curves? In that case I will initiate the next topic that is modeling of surfaces and we will go in details of surface modeling only in the next class.

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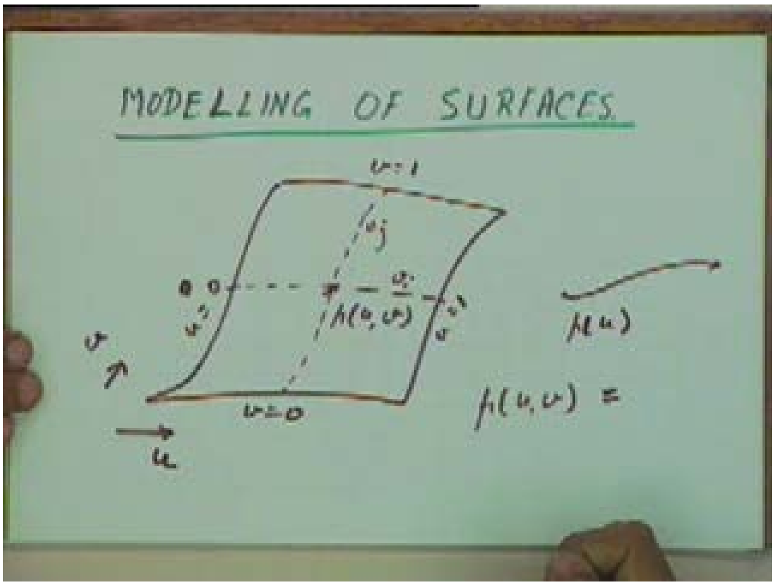

We are talking on modelling of surface patch. Let’s say this is a general three dimensional surface. Typically when we had curves, we are modelling them by a single parameter. So if we have a curve like this, we are saying that P of u or p will defend only on one parameter that is u.

Now since we have a surface, we cannot model by one single parameter. For a parameter definition, we need two parameters and typically we will take two parameters one which vary in this direction, the other varying in a orthogonal direction or varying in the other direction. So let's say if we say at this point, we take this curve to be a u equal to 0 curve and for this patch if we take this to be a u equal to 1 curve that means my parameter u is varying in this direction and similarly I take a parameter v varying this direction. So at this point I get v equal to 0 and at this point, at this curve I get v equal to 1. So if I take any arbitrary curve like this and a curve like this, my this curve let say has some value of v_i and let's say this curve has some value of u which is let's say u_j. My u is changing from 0 to 1 for successive curves. Let's say this particular curve has u equal to u_j.

So essentially I am looking at this surface as a family of curves in this direction and a family of curves in this direction. And any arbitrary point here can be expressed as a function of u and v. So typically we can write p of uv as and an equation of any point here can be written as some function of the different control points or whatever. If we look upon some standards surfaces and I express them in a parametric form.

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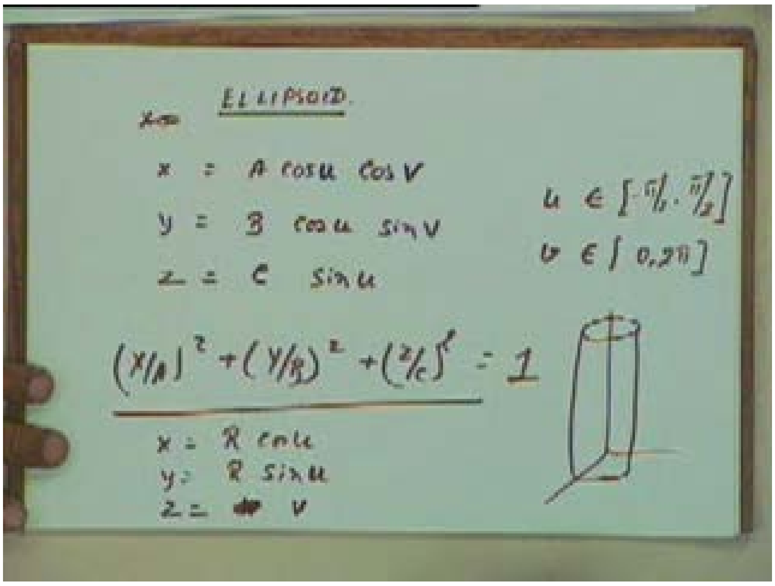

Let's say, we can say we think of an ellipsoid, we will say X is equal to A cos u cos v, Y will be equal to let say B cos u sin v and Z will be equal to let's say C sin u. This is the equation of an ellipsoid where u lies between minus pi by 2 to Pi by 2 and let say v lies between 0 and 2 pi. If I write it in an intrinsic form, we get X by A whole square plus Y by B whole square plus Z by C whole square will be equal to 1. So this is the equation of a ellipsoid.

This way similarly you can write equations for a cylinder and for other standard forms of the curves. If we just talking of simple cylinder, the equation of a cylinder in a parametric form can be written as X is equal to let's say R cos u, Y is equal to R sin u and Z equal to let's say w or Z is equal to v, whereas the second parameter just gives the height, the first parameter is controlling the a point on the circumference. This is a simple equation for the cylinder. Similarly you can write for sphere and so on. So parametric forms for standard surfaces can usually be written like this.

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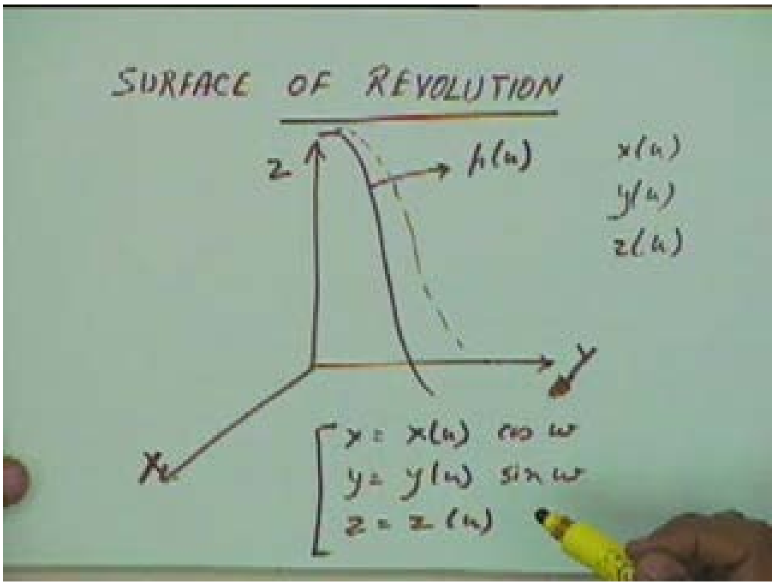

In other very standard type of surface is what is referred to as a surface of revolution. That means if you have any arbitrary curve like this and we revolved it about let's say about the Z axis. As I revolve this curve about the Z axis, I will get a swept surface. So the equation of that swept surface how do we write that? Let say the equation of this is given by some p of u. This is an arbitrary curve which can define using any formulation. And what will be the equation of this surface? p of u means let's say the X value is given by X of u, Y value is given Y of u and Z value is given by Z of u. So if you look at the surface of revolution that we can write down as X will be equal to X of u times let's say cos w, Y will be Y of u times sin w and Z will be same as Z of u. I am rotating it about the Z axis. So this is a simple parametric equation and any arbitrary curve is rotated about Z axis. If it is rotated about any other axis, you can write the transformation and get the equation and so on. So, this is with respect to surface generated by revolving a given curve.

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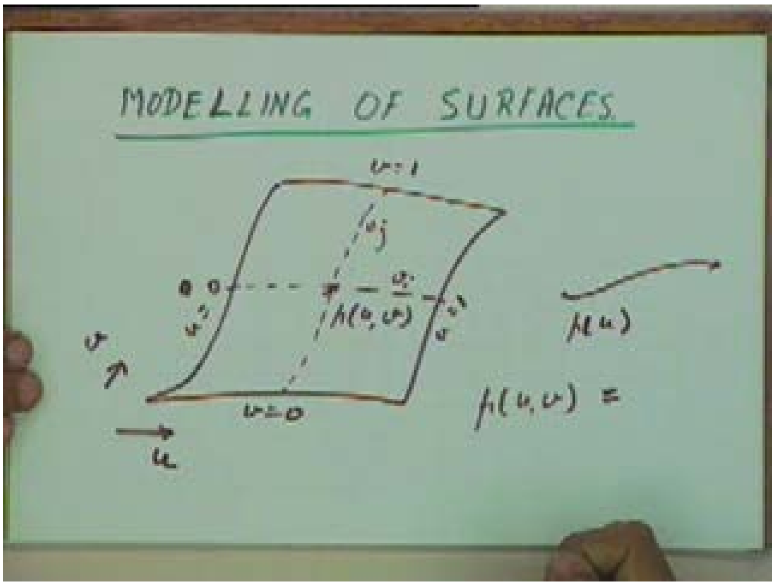

And if we are talking of surface patch which is of this form then the surface patch is not being or need not be generated by a set of curves in this manner, by taking a curve and revolving it or standard form like that. So what we will see now that if we have let’s say or if we want to define cubic surface patch that means you want to define a surface patch such that each of these curves is a parametric cubic curve or of each of these curve is a Bezier curve or a B-spline curve. That means this surface we will call that either a cubic surface or a Bezier surface or a B-spline surface. How can we formulate the surfaces in that case? So in order to formulate cubic curves or bicubic surfaces, we’ll just extend our earlier definition that we had given. If you remember for curves we have said let say p of u we said is equal to a_0 plus a_1 u plus a_2 u square plus a_3 u cube or if we write it in a long form or we can also write it as sigma a_i u to the power i where i goes from 0 to 3. This is how we have defined a pc curves.

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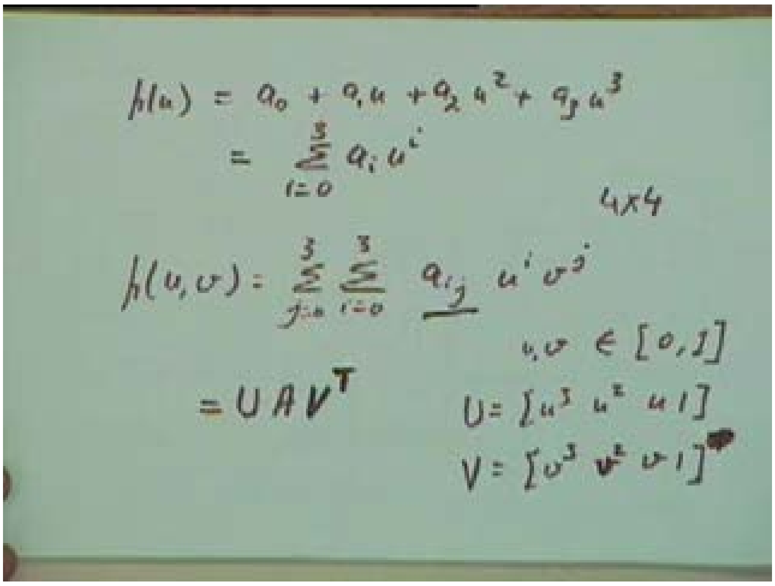

So for defending bicubic surface patch, we will do a similar thing p of uv, we will say will be equal to sigma a_{ij} u to the power i, v to the power j where i goes from 0 to 3 and j also goes from 0 to 3. And of course u and v they both lie between 0 and 1. And again this will say will be u times of matrix A multiplied by v, UAV where U is equal to u cube u square u 1 and V is equal to v cube v square v 1. And this matrix A is a set of coefficients that we have, this matrix obtained by these a_{ij}’s. So this way we will find that this matrix consists of 4 by 4 terms, 16 vector terms where i is varying from 0 to 3, j is also varying from 0 to 3. So we will get of set of 4 by 4 terms that is 16 terms and all and they are all vector terms. Again? Student: a would be a transpose of this. Right thanks. V will be a column vector and U is the row vector or for the sake of uniformity we will put a transpose here. So this will be UA times V transpose.

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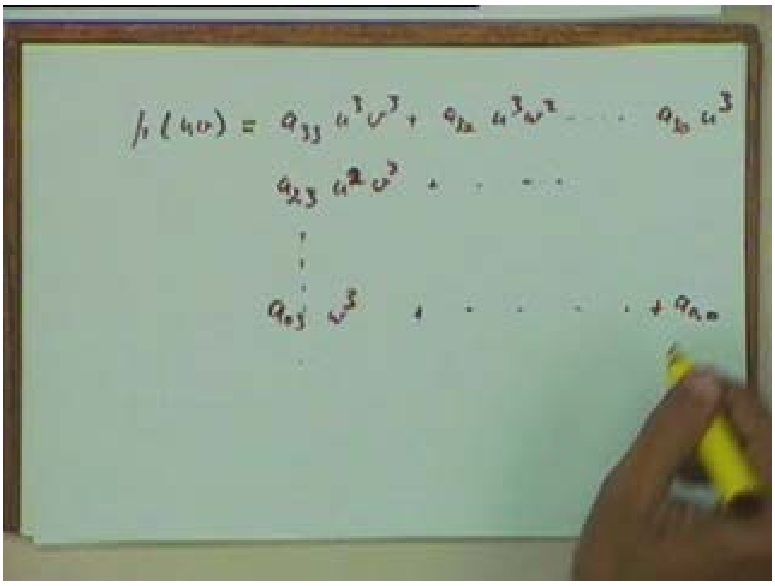

And if we expand out the same thing, what we will be getting? It will be p of u sorry p of u v will be a_{33} u cube v cube plus a_{32} u cube w square so on. The last term will be a_{30} u cube, here we will get a_{23} u cube v cube u square v cube and so on. We will have total of 16 terms. The last term here will be a_{03} w cube and the last term here will be a_{00}. We will get total of 16 terms in this when we expand out this single expression, this summation.

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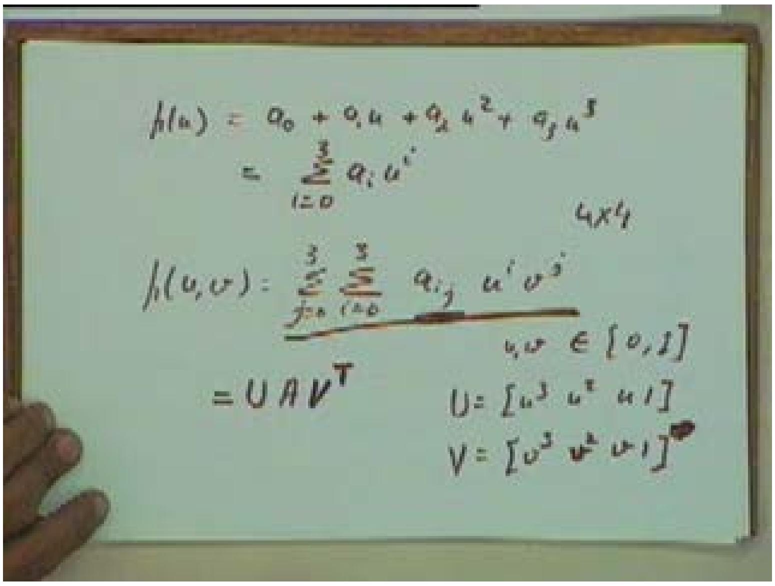

In this thing we can write it in the matrix form in this manner where A is equal to this. So this how we will get the definition of a bicubic surface patch.

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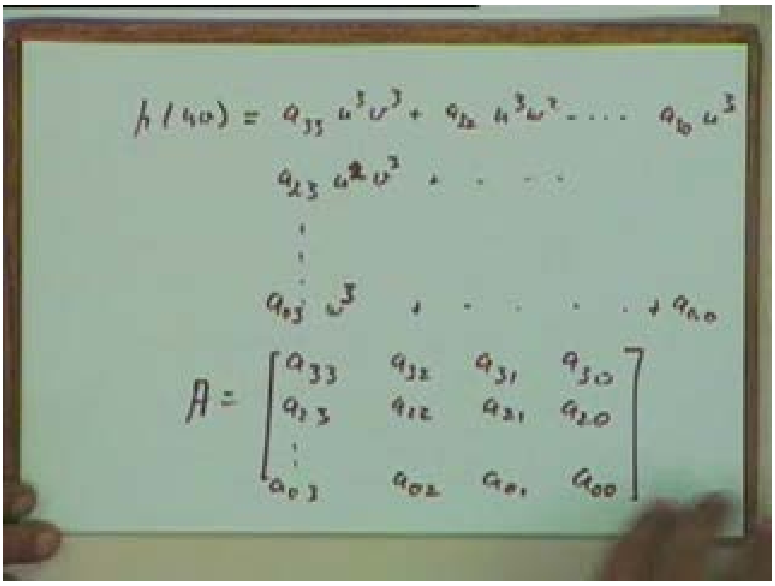

The geometric property of this surface patch and how do you get it in a geometric form and so on, we will see that in the next class. We will go into detail of this modelling only in the next class.