Computer Aided Design
Prof. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi
Lecture No. #33
Modelling of B-Spline Curves (Contd.)

Yesterday, we had seen the modelling of a B spline curves and for the case when we say that N is equal to 5 and k is equal to 1 that means we have 6 points and the degree of the curve is 0.

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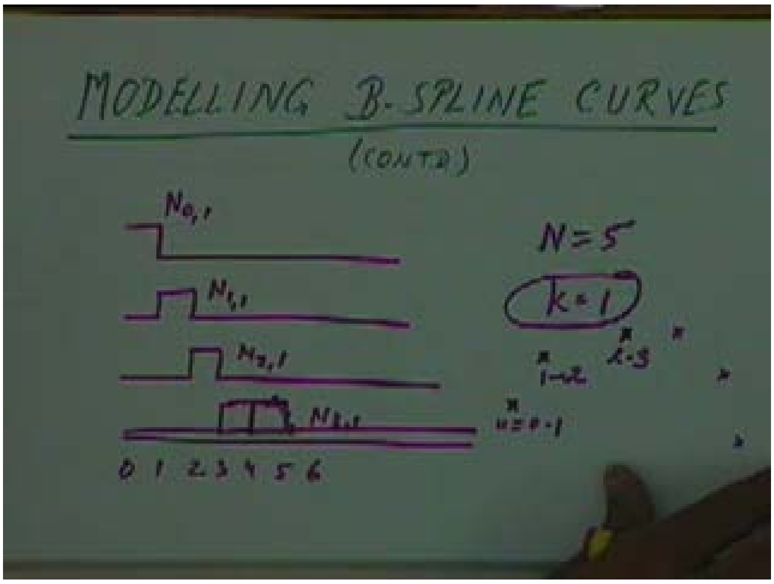

We had derived the expressions for the blending functions and we have said that the blending functions would look like this and so on actually, anyway which means that this is $N_{0,1,}$ this is $N_{1,1}$, this is $N_{2,1}$, this is $N_{3,1}$ and so on. And the curve we said if we have a set of 6 points, the curve is a discontinuous curve consisting of only these points. That means for the parameter u going from 0 to 1 the curve is concentrated at one point, from 1 to 2 it is concentrated at this point, from 2 to 3 is concentrated at this point and so on. These all in the case of B spline curves with degree 0 that means when k is equal to 1. And for the definitions of the B spline curves we had mentioned that any point on the curve will be given on this expression sigma p_i $N_{i,k}$ of u where $N_{i,k}$ is the blending function which depends on the parameter k where k minus 1 is the degree of the curve.

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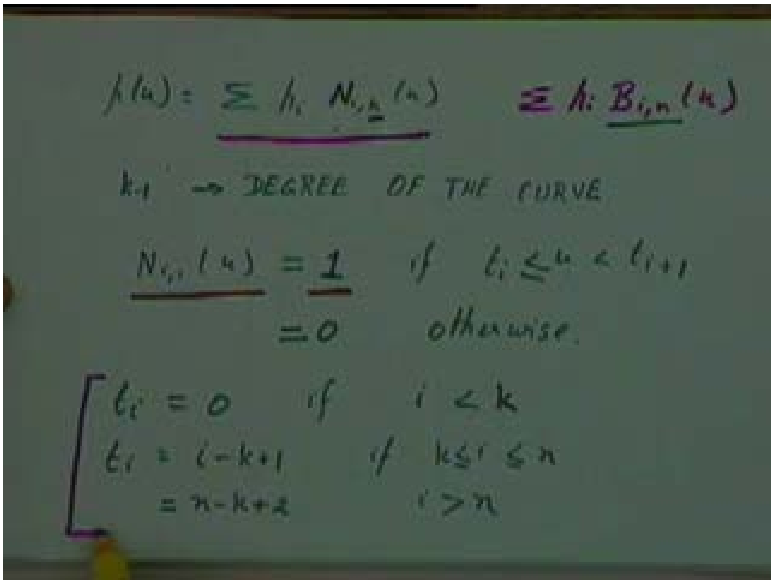

And for defining $N_{i,k}$ we defined it recursively. First defining $N_{i,1}$ of u to be equal to 1, if u is between two certain naught values where the naught values are defined by these expressions. And for defining $N_{i,k}$ for high values of k that is when k is greater than 1, we said we will view the recursive definition and Bezier curves is given by this expression where $N_{i,k}$ is u minus t_i times $N_{i,k-1}$ divided by this expression and so on.

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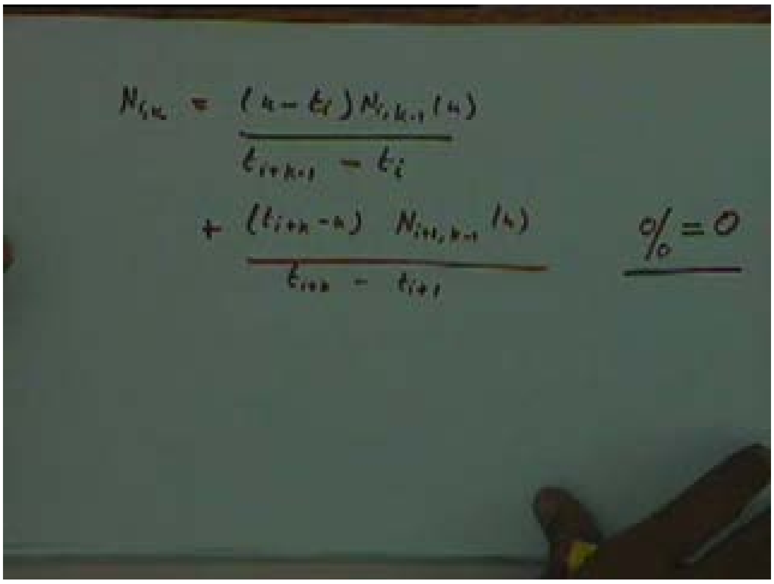

Essentially $N_{i,k}$ depends on $N_{i,\ k-1}$ and $N_{i+1,\ k-1}$. These are recursive definition we had given yesterday. Now let’s take the case when we say N is equal to 5 and k is equal to 2 that means the degree of the curve is 1, first degree curves.

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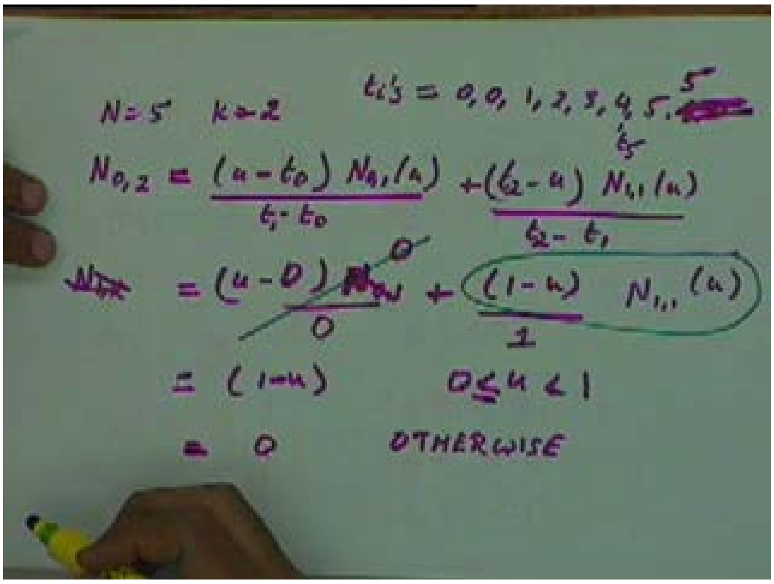

So we are eventually interested in defining $N_{0,2}$ $N_{1,2}$ $N_{2,2}$ and so on. So $N_{0,2}$ if you take up this expression I put i equal to 0 and k equal to 2. So $N_{0,2}$ would become u minus t_0 multiplied by $N_{0,1}$ of u divided by t minus t_0 sorry t_1 minus t_0 plus t_2 minus u multiplied by $N_{1,1}$ of u divided by t_2 minus t_1. This is directly obtained from this expression u minus t_i with i equal to 0; $N_{i,k-1}$ with i equal to 0 gives me this term, t_{i+k} minus u with i equal to 0 and k equal to 2 gives me this term and so on.

Now to be defined the naught values, in this case the naught values the t_i's we had seen them yesterday and these naught values will take values 0 0 1 2 3 4 5 0 and 0. Their naught value are defined using these expressions. So whenever i is less than k, in this case k is equal to 2. So t_0 and t_1 will be equal to 0 that's what I have written here, t_0 and t_1 are 0. After that when I is between k and n that means between 2 and 5, it will be i minus k plus 1, k is 2, it will become i minus 1. So t_2 will be 1, t_3 will be 2 and so on till i is less than or equal to n, n is 5 so up to t_5, t_0 t_1 t_2 t_3 t_4 t_5. At this point will be i minus 1, this is t_5 after that it will be n minus k plus 2, n is 5, k is 2 so it will be equal to 5 after that it will be 5 and 5. It can't be 0 after that sorry. So the naught values will be 0 0 1 2 3 4 and 5 5.

So whatever be my curve that is finally defined I have two naught at the starting point and two nauhgts at the end point, two coincident naught that is. So if I put these values over here, I will get u minus, t_0 is 0 so u minus 0 multiplied by $N_{0,1}$. If we have taken a definition of $N_{0,1}$ that is $N_{i,1}$ with i equal to 0 will be equal to 1 when u is between t_0 and t_1. So when it is between t_0 and t_1, so at u equal to 0 $N_{i,1}$ or $N_{0,1}$ will be 0 and it'll be equal to 0 otherwise. Yeah, between this, it will be 1, it will be 0 otherwise. So at u equal to 0 it will be 1 that is only at one point and all the other points it will be equal to 0. So if we look at the complete range, we put that equal to 0 and this term will turn out to be divided by 0. This plus t_2 is 1, so it will become 1 minus u divided by t_2 minus t_1, t_2 minus t_1 is 1 multiplied by $N_{1,1}$ of u and just get expression for $N_{0,1}$ and $N_{1,1}$ that will make it simpler to compute.

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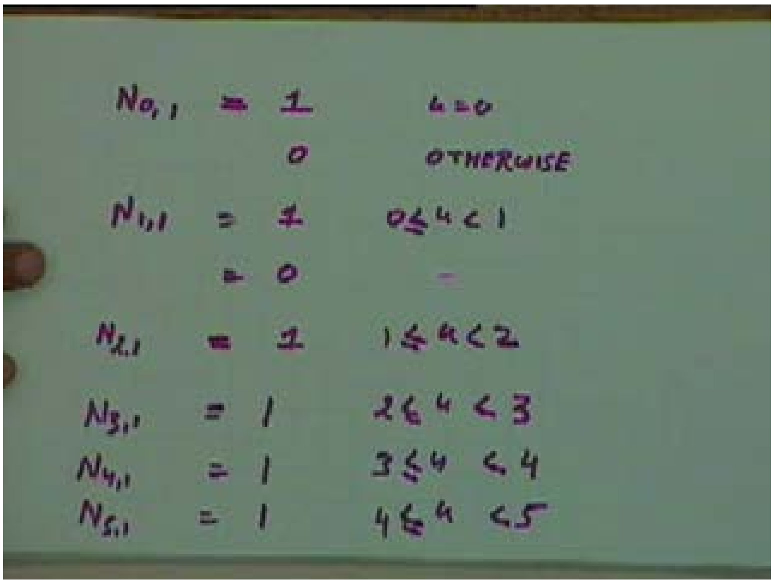

So $N_{0,1}$ is equal to 1 if u is equal to 0, is equal to 0 otherwise, $N_{1,1}$ will be equal to 1 when u is between 0 and 1, it will be equal to 0 otherwise. $N_{2,1}$ will be 1 when u is between 1 and 2, $N_{3,1}$ will be equal to 1 when u is between 2 and 3, $N_{4,1}$ will be 1 when u is between 3 and 4 and finally $N_{5,1}$ will be 1 when u is between 4 and 5. This is directly from the definition we had given earlier, this definition when u is between two consecutive naughts it is 1 otherwise it is 0. That's how we are getting all these relationships. So if I put it over here, this is I will put it over here again, this is $N_{0,1}$ and this is $N_{1,1}$.

Now $N_{0,1}$ at the point u equal to 0 it is 1 otherwise it is 0 throughout the curve. So I will take it to be 0 throughout curve right now and $N_{1,1}$ is 1 between 0 and 1 and is 0 otherwise. So this complete term $N_{0,1}$ is 0, so replace this by 0 so it will come to the form of 0 by 0. And if you remember I had mentioned that whenever we have 0 by 0, we will take that to be 0 or we will take the limit to be 0. So, this complete term will vanish and this $N_{1,1}$, N_{11} is 1 between 0 and 1. So between 0 and 1 this will be equal to, this will be equal to 1 minus u where u is between 0 and 1. And it will be equal to 0 otherwise which is when u is between 0 and 1 $N_{0,2}$ is 1 minus u otherwise it is 0. It is basically this term 1 minus u times $N_{1,1}$. Is that okay?

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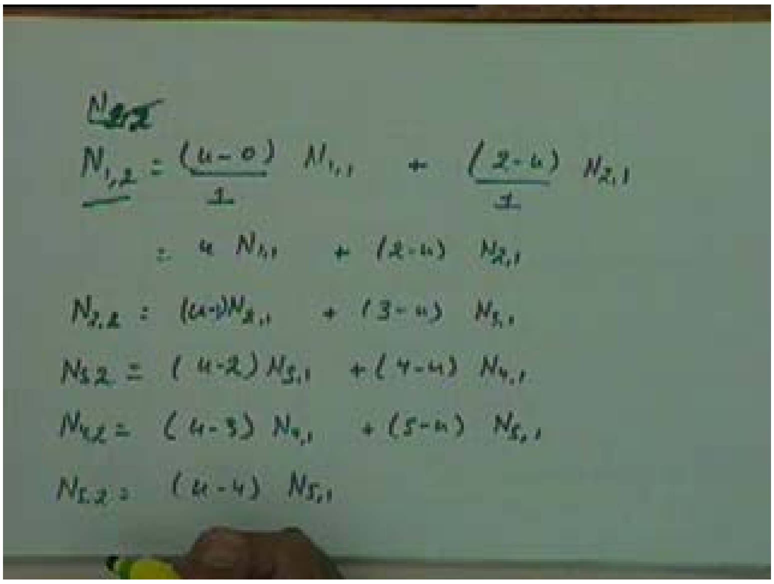

If we go to the next term that is $N_{0,3}$ sorry $N_{1,2}$, again from this expression $N_{1,2}$ will get in terms of $N_{1,1}$ and $N_{2,1}$.

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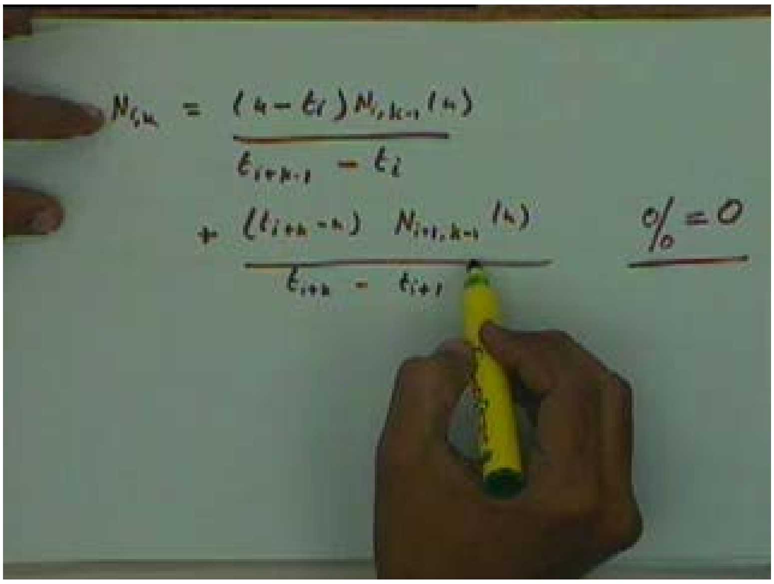

So this will be equal to u minus 0 divided by 1 multiplied by $N_{1,1}$ plus, this will be 2 minus u divided by 1 multiplied by $N_{2,1}$, u minus t_i with i equal to 1, t_i is 0 and t_{i+k} minus u so you will get t_3 minus u, t_3 is 2, 2 minus u and these terms will come out to be 1. So this is equal to u times $N_{1,1}$ plus 2 minus u times $N_{2,1}$. Now this $N_{1,1}$, $N_{1,1}$ is equal to 1 in this range and $N_{2,1}$ is equal to 1 in this range. So effectively what you will get will be that in this range when u is between 0 and 1 and 1 2 will be equal to u. And when it is in this range between 1 and 2 it will be equal to this expression because when $N_{1,1}$ is equal to 1, $N_{2,1}$ has to be 0 and when N_2 is 1, $N_{1,1}$ has to be 0. So $N_{1,2}$ will be given by this expression. Similarly other expressions I just write them down now. So $N_{2,2}$ will be u times sorry u minus 1 times $N_{2,1}$ plus 3 minus u times $N_{3,1}$. $N_{3,2}$ will be u minus 2 times $N_{3,1}$ plus 4 minus u times $N_{4,1}$ and $N_{4,2}$ will be u minus 3 times $N_{4,1}$ plus 5 minus u times $N_{5,1}$ and finally $N_{5,2}$ will be equal to u minus 4 times $N_{5,1}$. The second term will have a term of 0 by 0, likely N be taken as 0. So these are the expression we will get for $N_{1,2}$ $N_{2,\,2}$ 3 2, 4 2 and 5 2. So these are different shape function will get in this case.

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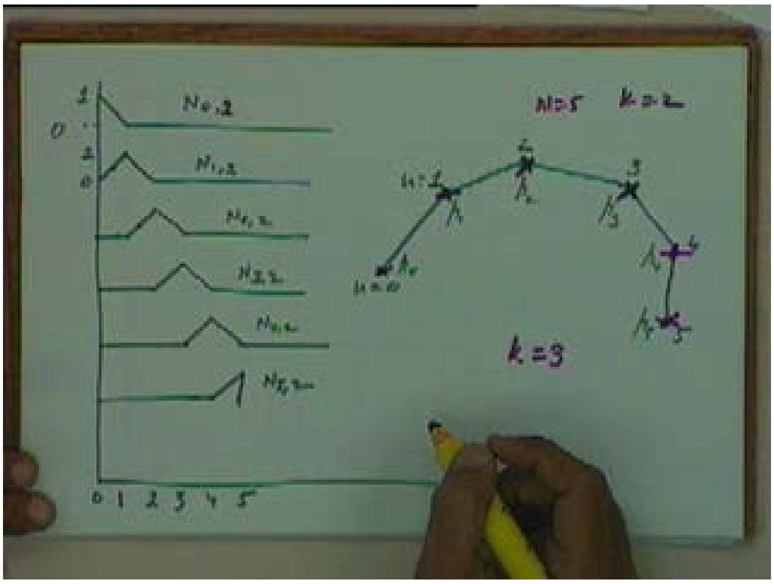

If we plot these shape functions, this will be $N_{0,1}$ changing between 1 and 0 sorry. The expression we had for $N_{0,2}$, so $N_{0,2}$ is 1 minus u when u is between 0 and 1.

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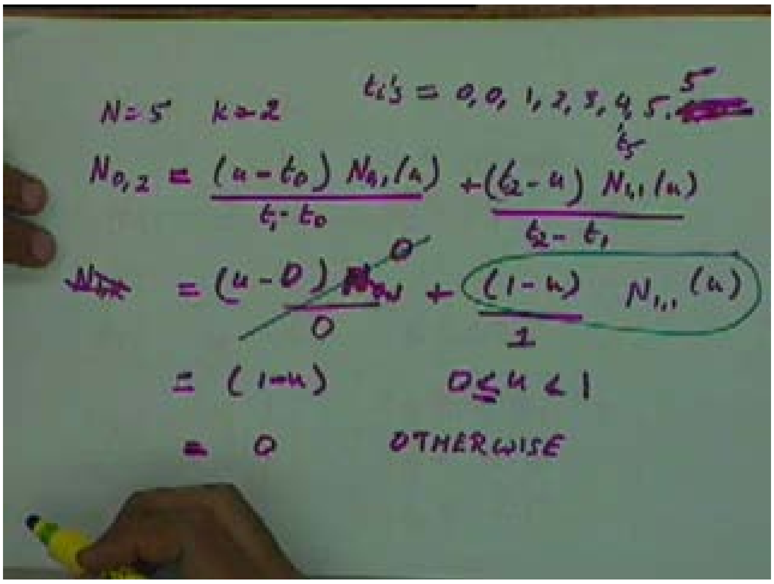

So between 0 and 1 it is equal to 1 minus u. So it is going to be a linear relationship with the slope of minus 1. So, you will get figure like this, this is $N_{0,2}$. If you take up $N_{1,2}$, $N_{1,2}$ is u times $N_{1,1}$ that means between 0 and 1 it will increase like this. It is again linear relationship in the range of 0 to 1 and between 1 and 2, it will be equal to minus u so it will be like this and the rest of it will remain 0. This is N_{12}, this is 1, this is 0. Again when I take $N_{2,2}$ I will get a curve like this, this is $N_{2,2}$. For N_{32} sorry N_{32}, so this is $N_{3,2}$, $N_{4,2}$ will look like this and 5 2 we have the curve is defined only up to u equal to 5, it will look like this. This is N_{52}, $N_{5,2}$ is given by this expression u minus 4 times $N_{5,1}$. $N_{5,1}$ is 1 in region and $N_{5,2}$ will be this slope. Is that okay? Yeah, now if you look at these blending functions at any point of the curve, you will find that only two points effective. That means in this range between 0 and 1 only P_0 and P_1 will have the effect, between u equal to 1 and 2 only P_1 and P_2, P_1 and P_2 will have their effect, between u equal to 2 and 3 only P_2 and P_3 will have the effect. What that means is if you have a set of points, what will the curve look like? This curve would look like a set of straight lines.

At this point you will say u is equal to 0. Here you will get u equal to 1, 2, 3, 4 and 5. This is the first control point P_0, this is P_1, P_2, P_3. So between these two points between these two points u equal to 0 and u equal to 1, these two points have their effect. So the curve is a linear combination of these two. Similarly between these two, the curve will be a linear combination of these two points. So you will get a set of straight lines. Just to compare when k was, this is the case when N is equal to 5 and k is equal to 2. When k is equal to 1, I had a set of points here like this. When k is equal to 2, my curve is continuous but it has only continuity. For k equal to 1, the curve was discontinuous. Now when we take k equal to 3, we will get a quadratic curve or second degree curve. For k equal to 2, I have linear relationship all my shape functions are linear functions, linear functions which are defined in a certain range beyond that range these functions are 0.

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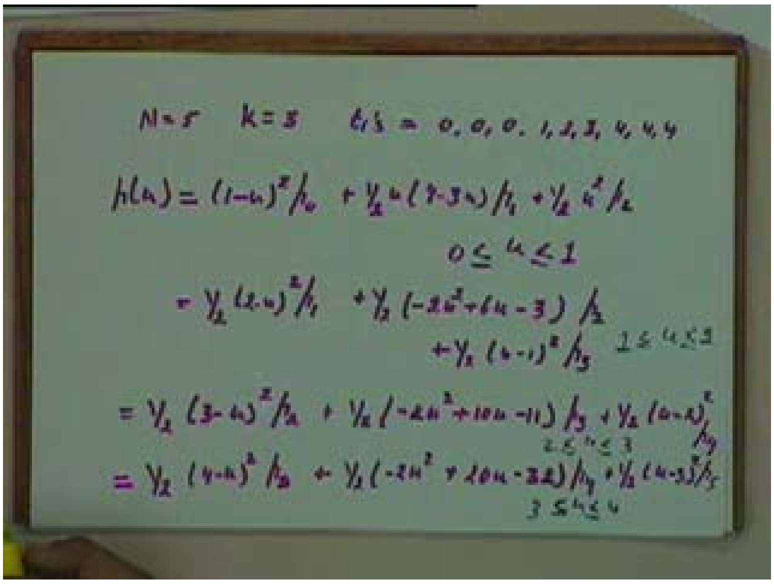

So now let's take the case when N is equal to 5 and k is equal to 3. If I say N is equal to 5 and k is equal to 3, the first thing we define are the naught values, so the t_i's, now the t_i's will be 0 0, 0, 1, 2, 3, 4, 4 and 4. Again this will have the same definition we have given for the naught values. If I carry out the complete, if I find out all the expressions we have did for N_{12} now I will have to find for N_{13} and so on. So I will get my definition the curve p of u that will come out to be, I am writing the expression directly now 1 minus u whole square times p naught plus half of u times 4 minus 3 u times p_1 plus half of u square times p_2 and this is in the range when u is between 0 and 1.

This will be equal to, the curves will be half of 2 minus u whole squared times p_1 plus half of minus 2 u square plus 6 u minus 3 times p_2 plus half of u minus 1 whole squared times p_3. This is when u is between 1 and 2, this will be equal to half of 3 minus u whole squared times p_2 plus half of minus 2 u squared plus 10 u minus 11 times p_3 plus half of u minus 2 whole squared times p_4. And finally it will be equal to half of 4 minus u whole squared times p_3 plus half of minus 2 u square plus 20 u minus 32 times p_4 plus half of u minus 3 whole squared times p_5 and again this is when u is between 2 and 3 and this is when is between 3 and 4. It doesn't make a difference because we have c_1 continuity because since their functions are continuous, it doesn't make any difference, we can show that they are the same. I can either put equality only on one side or in both sides in same material. Now again as we have expected since k is equal to 3, in any range of the curves we have three points which are effective, p_0 p_1 and p_2 are effective when u is between 0 and 1 and when u is between 1 and 2 p_1 p_2 and p_3 are effective between 2 and 3 p_2 p_3 and p_4 are effective and between 3 and 4 p_3 p_4 and p_5 are effective.

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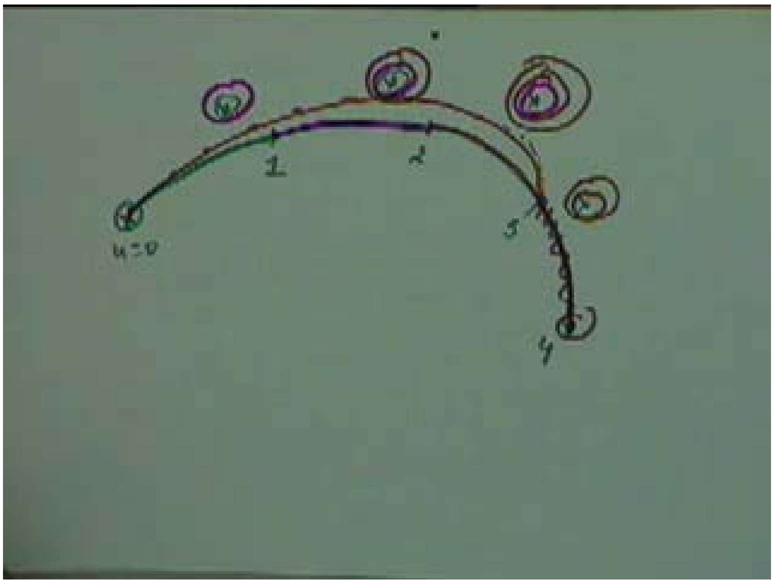

What that means is if we have 6 points like this, I have a curve which is defined like this. My naughts will be defined at 0 1 2 3 and 4. So this is u is equal to 0 1 2 3 and 4. From this part of the curve between 0 and 1, p_0 p_1 and p_2 are effective. So for this part of the curve, these three points are effective. If I change this point, this part of the curve will not get modified. For this part of the curve, for this part you will get p_1 p_2 and p_3 will be effective and then for this part we will have this and this being effective and finally for this part, we will get this point, this point and this point being effective.

So, again if I change this point, this portion of the curve will not get changed. Following the rest of the curve would change. So this point will move from here to here, my curve will probably get modified to something like this. It has to maintain c_1 continuity here, so my curve would look like something like this. This portion of the curve will remain unchanged if I move this point. This is the first thing that is important in terms of, to B spline curves. We will get a local control, the curve will get modified only in the certain range. This curve will be c_1 continuous. Yeah, the first derivative will be the same. It is as I said that slope would remain the same at this point.

Now the next thing that we will do, if you look at these expressions there are certain pattern in these expressions. We have p_0 p_1 p_2 here, p_1 p_2 p_3 here, p_2 p_3 p_4 here and so on. So what we will do is this is the case when N is equal to 5 and k is equal to 3. I will first draw the blending functions, in this case the blending functions, this is N_{03}, N_{03} is going to be a quadratic function starting from 1 going down to 0.

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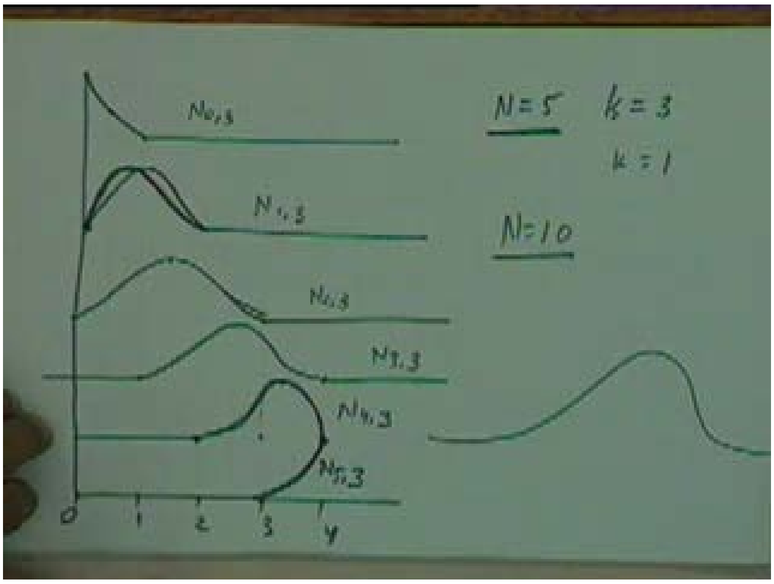

If I take $N_{1,3}$ let's say it will start from here. Actually the peak is slightly earlier, here this portion of the curve. This is $N_{1,3}$. If you look at N_{23}, this is $N_{2,3}$. Now you look at $N_{3,3}$ this will be $N_{3,3}$. So quadratic function symmetrical about the center just like $N_{2,3}$ which is also symmetrical about the center but $N_{1,3}$ it is not symmetrical $N_{0,3}$ is also not symmetrical, $N_{4,3}$ would be similar to $N_{1,3}$ now. So $N_{4,3}$ a peak slightly beyond this center, this is the center for N_{43}. A peak slightly beyond this and finally $N_{5,3}$ would look like this, this is N_{53}, $N_{5,3}$ is the quadratic function between these two points, zero before this. $N_{4,3}$ is a quadratic function between these two points. It's one quadratic function here, the second quadratic function here and it's 0 before this. $N_{3,3}$ is also set of quadratic functions but it is symmetrical about the center. $N_{2,3}$ and $N_{3,3}$ have the same shape. $N_{1,3}$ and $N_{4,3}$ are somewhat similar but mirror images and $N_{0,3}$ and $N_{5,3}$ are also similar. This is how the case when N is equal to 5 and k is equal to 3.

So far in changing the value of k, we said that when k was equal to 1, we got 0 degree curve. Now for k equal to 2, we got a first degree curve. For k equal to 3, we are getting a second degree curve and all are blending functions and quadratic functions. What happens if I increase the number of points? If instead of 5, N equal to 5 let's say N equal to 10, what you are going to expect? The shape function you will get more and more. In fact, no. What would happen if I change k, the geometry of the function will become more complex, it will become a higher degree curve but if I change N it will still remain a quadratic curve for k equal to 3. And since the degree of this curve is determined by the value of k, so it will always remain a quadratic curve.

So what will always happen is that at the starting on the curve, I will get one shape function like this. The next shape function will be like this, the last shape function will always be like this and the last but one shape function will always be like this except that instead of that being at u equal to 4 it will be some other value of u and the intermediate shape function will all be similar to this. So all my intermediate shape functions would look like this, symmetrical except at the ends.

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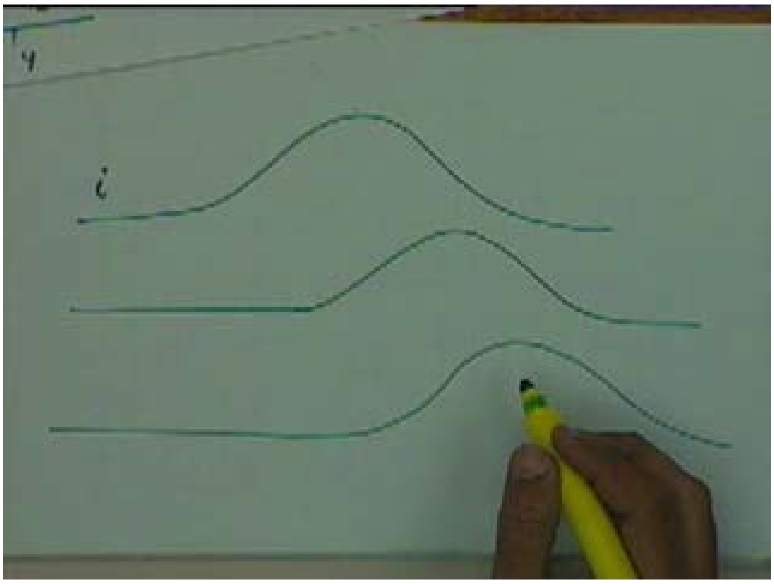

So if we consider a portion where as if we have many points, we have one shape function like this. The shape function for the next point is going to be staggered and it will be like this. The shape function for the next point will again be staggered and it will be like this. This is how we will get a three successive shape functions or blending functions, should use the term blending functions. So the three successive blending functions would look like this except at the ends. So what we can do is except at the ends we can write an expression for the curve in term of these general shape functions. I can have let's i points on this side and I can have any number of points beyond this and if you look at these expressions, these expression in some sense repeating p_0 p_1 p_2 here p_1 p_2 p_3 here and so on. Here 1 minus u_2 minus u_3 minus u and 4 minus u over here, u, u minus 1 u minus 2 and u minus 3 over here and even in this term there is a certain pattern.

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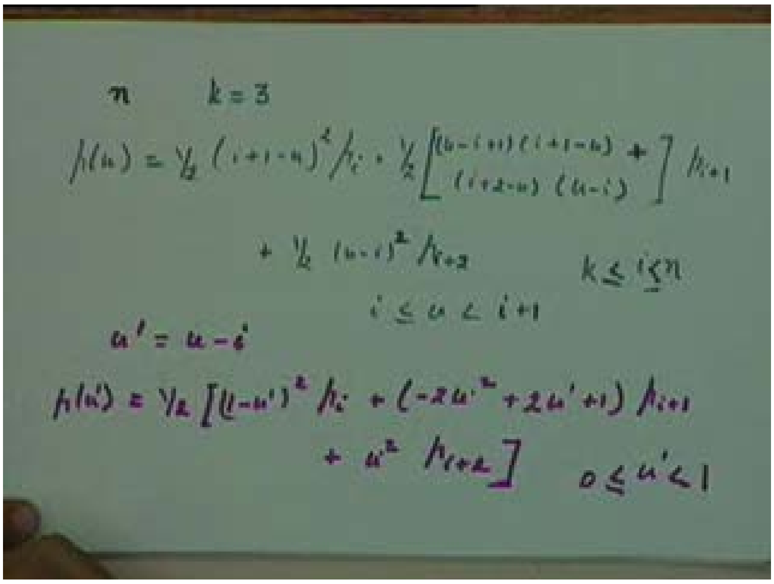

I will just write down the expression for the pattern, you can verify it later on. If I say particular case where n is any general value and k is equal to 3. Then we can say p of u will be equal to half of i plus 1 minus u whole square times pi plus half of this is u minus i plus 1 multiplied by i plus 1 minus u plus i plus 2 minus u multiplied by u minus i. This whole thing is multiplied by p_{i+1} and this plus half of u minus i whole square times p_{i+2} and this is the definition of the curve when u is between i and i plus 1 and when i takes any value between k and n. If i takes the value smaller than k then this is not correct because if i takes the value smaller than k then we will have these shape functions coming in, these shape functions which are not the same as the other shape functions. So we have this expression and now what we will do is in this expression we will put let's say u prime to be equal to u minus i.

If we do that the p of u will be equal to or let say p of u prime will be half of 1 minus u prime whole square times p_i plus minus 2, this is u prime squared plus 2 times u prime plus 1 times p_{i+1} plus u squared times p of i plus 2. All I have done is I put u prime equal to u minus i in this, so u minus i becomes u prime squared and for a u minus i will become, this will be u prime plus 1 and this will become 1 minus u prime. Now I can expand that to get this expression and this is valid when u prime takes any value between 0 and 1. As u taking a value between i and i plus 1, so u prime you will take a value between 0 and 1. Is that okay? What are we effectively doing? Firstly, why I am doing it?

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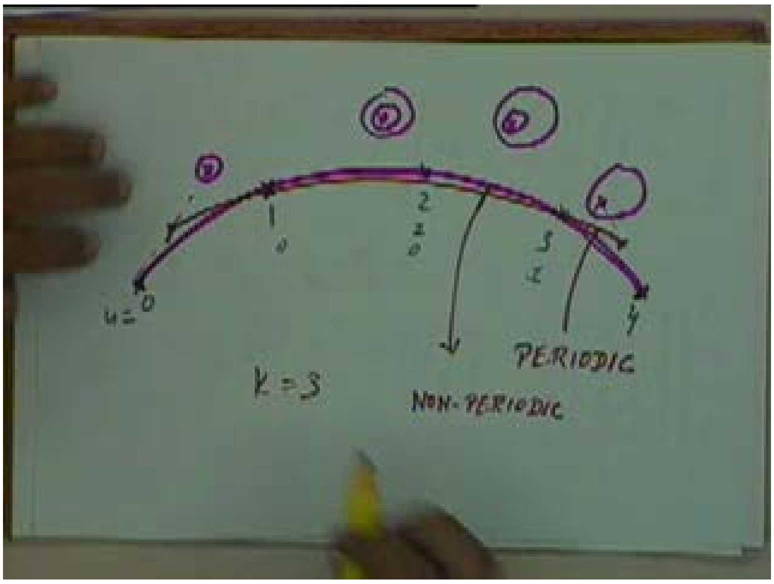

See if I have a set of points, 6 points let's say and I have a curve between this. My naught value is defined like this 0 1 2 3 4, so this is the value of u, u equal to 0, u equal to 1 2 3 and 4. Now I will define each portion of this curve, I will call it as a separate curve and I will say my u prime is changing from 0 to 1 here. It is also changing from 0 to 1 here and 0 to 1 here and so on. So in each portion I will have a different parameter which will be varying from 0 to 1 because I have said u prime is u minus i. So if in this case i is equal to 1 so u prime will vary from 0 to 1. Again u prime will vary from 0 to 1 here and so on. So I get this expression and we will rewrite this expression as and I can say u prime I will again go back to u just for the sake of convenience.

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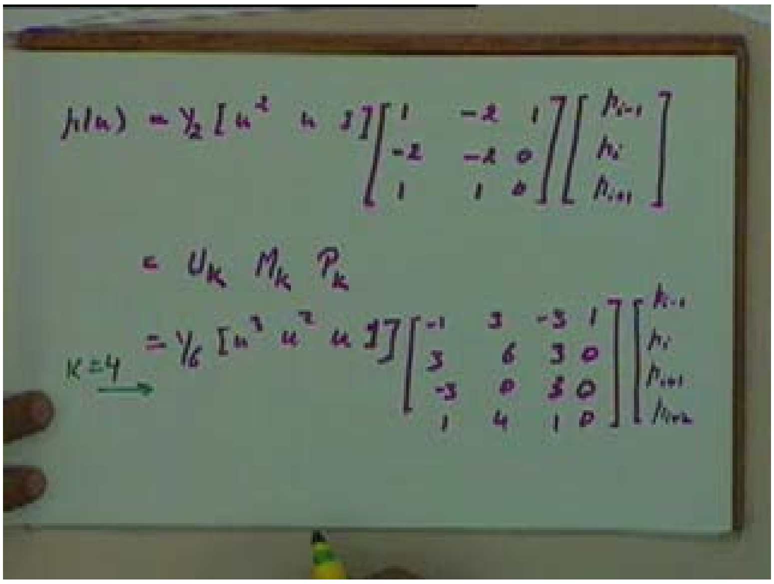

Let's say p of u will be equal to half of u square u 1, multiplied by 1 minus 2 1 minus 2 1 minus 2 0 1 0. For this expression I can rewrite it in this form or if this is equal to let say U_k, if you remember in the case of pc curves we had a matrix like this so u cube u square u_1. So in this case you have a second degree curve, so we have terms only up to u square. If we have a third degree curve we will have terms going up to u cube, so this matrix u depends on the value of k, size will be k, will be k elements in this. In this case k is equal to 3 multiplied by M_k. This matrix is a M matrix and this matrix is what I will call as P matrix which is P_k. In some sense this is a geometric matrix for the curves.

Let's say for this curve, my geometry matrix will consist of this and this point, for these three points. Similarly for this curve my geometric matrix will consists of this point, this point and this point and this is for the case where k is equal to 3. We can similarly derive expressions for the k's, where k is equal to 4. What we will get will be, this will be equal to one sixth of, this will be u cube u square u 1 multiplied by minus 1 3 minus 3 1 3 6 3 0 minus 3 0 3 0 1 4 1 0 and here we will get 4 terms actually p_{i-1} p_i p_{i+1} and p_{i+2}. This is this term, this is for k equal to 4 and this formation I have said we are using when i is between k and n, not for all the values of i. Only when i is between these two, can we use this formulation. What that basically means is that my curve has one formation in this range and this range but at the end the formation is slightly different. It is basically because my shape functions for this point and for this point, the blending functions are different. And if we want to use the same blinding functions then what happen is that my curve might not start from here but it will start from some other point.

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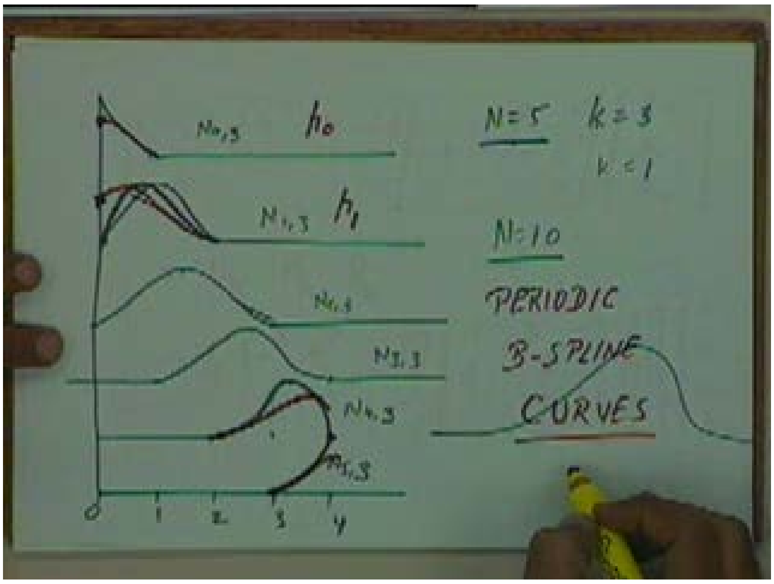

Let's look at the blending functions. This is one blending function we had, this is second blinding function which is similar in shape to this one but this blending function we said it has a different shape because this portion of the curve is totally different and then this portion, this curve is also different. Instead of that, if we take the same blending function and truncate it at this point, a blending function can look something like this.

Similarly these blending functions would look something like this. Now instead of having the blending function 0 that means $N_{4,3}$ was 0 here. Now $N_{4,3}$ is going to be non-zero. Similarly $N_{5,3}$ was 1 here, now it will be non-one it will less than 1 because the same shape of the blending function and truncating it.

If I do that then at this starting point, my curve will be a combination of this point and this point because this curve, this is the weightage to be given to the point p_0. This is the weightage to be given to the point p_1, I am giving a non-zero weightage to both these points. So the curve will start at some point which will be a combination of p_0 and p_1 and such curves are referred to as periodic curves, periodic B spline curves. Basically what to do in periodic B spline curves is you take the same shape of the blending function right throughout or this expression that we have got, we use this expression for all the parts of the curve. For all these parts of the curve, we use the same expression but our consequence of that, my curve will not start from the starting point but it will start from some other point.

In fact we can show that the curve will start for k equal to 3, it starts with the point which is a midpoint between these two points and ends at the midpoint over here. So my curve would become something like this. This is a periodic curve, this is a non-periodic curve, this is periodic, periodic B spline and this is a non-periodic B spline sorry. This one, this curve that one red is non-periodic, the one in brown is periodic. What we will do in the next class is that we will see the exact formation for the periodic curves and see how the properties are different from the non-periodic curves. Right now just mention; it will give the same shape function.

Essentially that means is we will use this formulation to be valid for all values, for all the curve segments. If I look at this curve, this curve, my like curve is starting here and ending here. Each of these from 0 to 1, from 1 2, 2 to 3 and 3 to 4 are referred to as curve segments and this formulation, this expression is valid only for a certain set of curve segments not for all. If we use this formulation for all the curve segments then my curve will look like this one in brown that means if I have a set of points, the starting point of the curve is here, ending point is here and my curve is like this.

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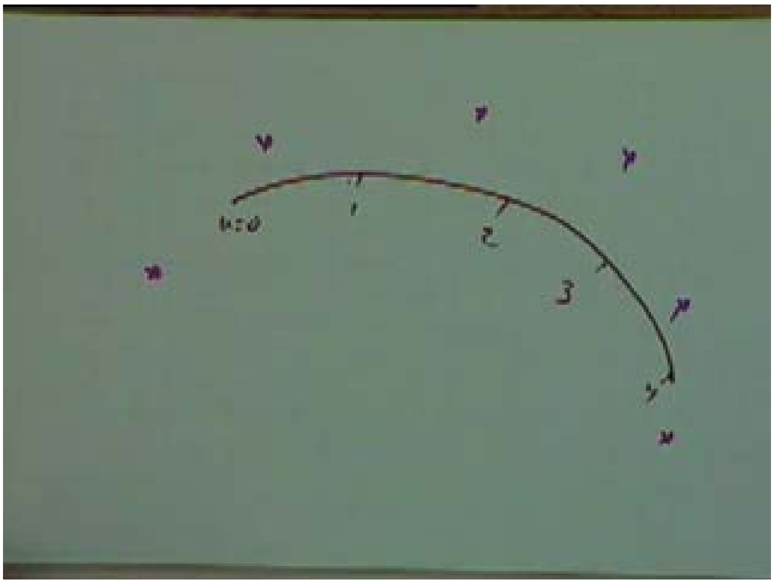

It doesn't start here, it doesn't end here and at u equal to 0 here 1 2 3 and u is 4. This part of the curve will be combination of these three. This part of the curve will be combination of these three and the combination will be given by this expression. That is why it is called periodic that means the same function is continuing again. We will see some more details on periodic and non-periodic curves in the next class.