Computer Aided Design
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Lecture No. # 31
Modelling of Curves (Contd.)

In the last class we had seen the definition of PC curves or parametric cubic curve and we had mentioned that, we get any point p as U times A where A is the coefficient of the algebraic matrix of the algebraic coefficients, U is the matrix consisting of the power terms.

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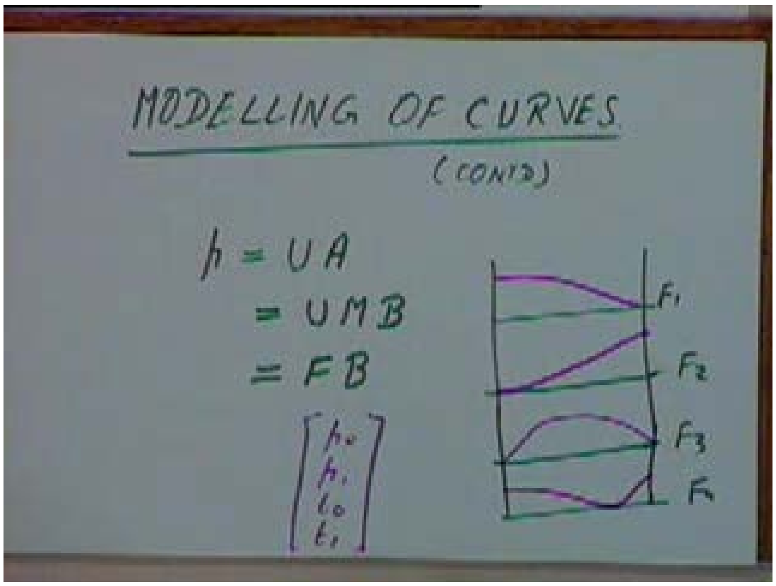

And the same thing can be expressed as U times MB where B is the standard geometric matrix and M is the matrix of coefficients of the curves. We had seen 4 curves F_1, F_2, F_3 and F_4, we also wrote this as F times B. This is what we had seen yesterday. And if we, the 4 curves, their shape was like this. They are 4 bending function curves. And this was for the case of a PC curve where the B matrix, the standard geometric matrix consisting of p_0, p_1, t_0 and t_1.

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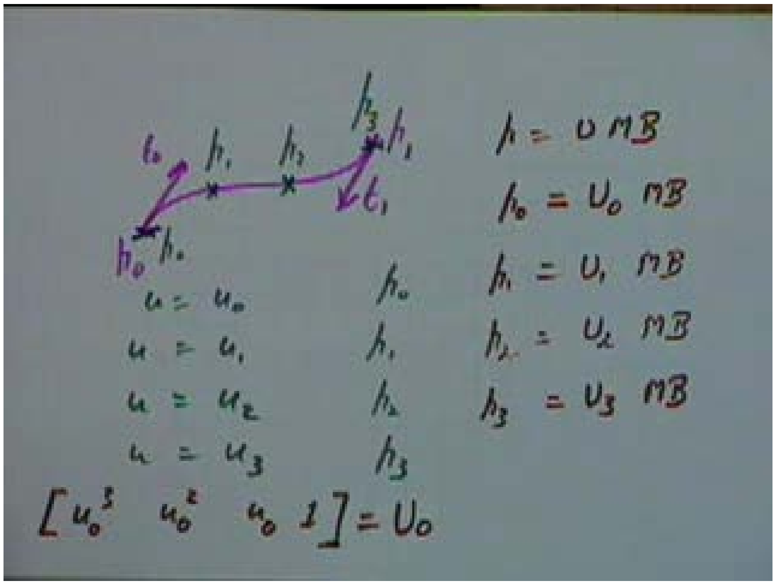

In contrast to this formulation, if we consider any cubic curve right now we are specifying this as using 4 parameters p_0 p_1 starting point end point, starting tangent vector and the ending tangent vector that is t_0 and t_1. Instead of that if we want the curve to pass through 4 points that means let's say I say this point, this point, this point and this point. Maybe I say that for u equal to u_0 equal to u_1 equal to u_2 and equal to u_3. My points are p_0, p_1, p_2 and p_3. Let's say this is p_0, this I say is p_1, this is p_2 and this is p_3. That means I want to get the equation of a curve which will pass through these 4 points. That means I am constraining my curve to pass through these 4 points, I want to interpolate between these 4 points using cubic using PC curves, using cubic interpolation.

If I consider the equation of my curve which is t equal to U times MB. At u equal to u_0, I have p_0, so this will become p_0 is equal to let's say I will write U_0 times MB where U_0 is a matrix obtained from u equal to u_0. That means this is my u matrix, So U_0 will be u_0 cube u_0 square u_0 and 1, this is U_0. Similarly I can write for U_1 and I will get p_1 is equal to u_1 times MB, p_2 is equal to U_2 times MB and p_3 will be equal to U_3 times MB. So I will get these 4 equations, for these 4 points p_0, p_1, p_2 and p_3. What I am saying is these 4 points are specified at 4 distinct values of the parameter u.

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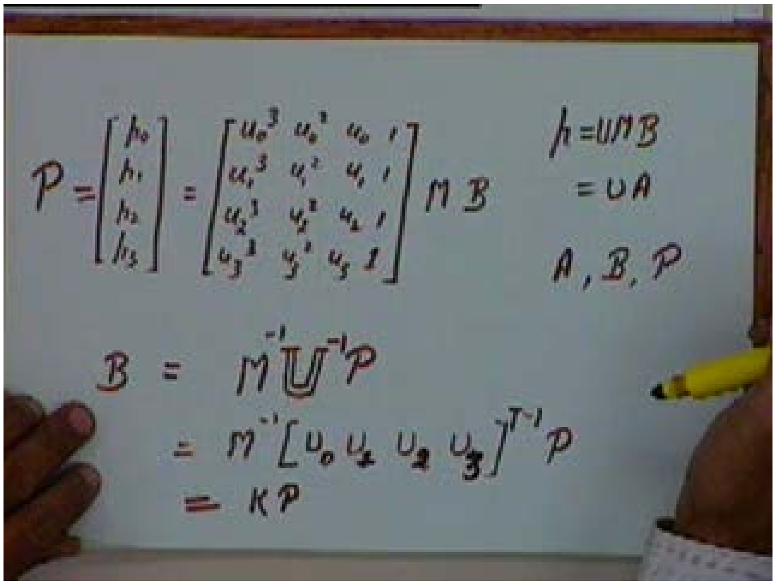

Again I will write these 4 equations in a matrix form and I will say that let's say this matrix p_0, p_1, p_2, p_3. Let this matrix be equal to… (Refer Slide Time: 06:12) multiplied by M multiplied by B. What I have basically done is I have taken these 4 equations where U_0, U_1, U_2 and U_3 they consist of 4 vectors like this. So I have taken these 4 equations and written them down as one single equation. Again the p_0 will be this vector multiplied by MB, p_1 will be this vector multiplied by MB and so on. So now if p_0, p_1, p_2 and p_3 are given and I want to find out this matrix B, all that I need to do is find out the inverse of these two matrices.

Let's say if I say this is my matrix capital P, now I will say B will be equal to P Pre multiplied by this matrix, the inverse of this matrix. So I just write this for the time being, inverse pre multiplied by M inverse. This is nothing but M inverse multiplied by U_1, U_2, U_3, U_4 transpose of that and the inverse of that multiplied by P where U_1, U_2, U_3 and U_4 are sorry U_0, U_1, U_2 and U_3 are these 4 vectors. U_0 is like this, U_1, U_2, U_3 are similar vectors. So we will get B will be a matrix which will be given by this expression. So what that basically means is that if I am given the 4 points p_0, p_1, p_2 and p_3, I can find out the standard geometric matrix by pre multiplying by these two. And this maybe I can say is equal to a matrix K times P. Student: P will also be different, isn't it? Different from? Student: the AB matrix is below every… t is, no no that was not a matrix, that was the small p I was saying which was equal to M times U times M times B, Student: it is a point that is a point vector, this capital P is a matrix of 4 points p_0, p_1, p_2, p_3 and M inverse multiplied by this U inverse, I am writing that as K, I am saying B is equal to K times P.

So if I am given this matrix P that means if I am given these 4 points through which my PC curve has to pass. I can easily get the B matrix that is the standard geometric matrix. And my curve is of course defined as UMB or it is defined as U times A. And basically it gives us a method of getting either the A matrix or the B matrix or the P matrix. No, sorry yeah or the P matrix if any of the three are given.

If I know the A matrix, I can get the B matrix and vice versa and if I know the P matrix, I can get the B and A and so on. The only thing is for getting this matrix K, I need to know u_0, u_1, u_2 and u_3.

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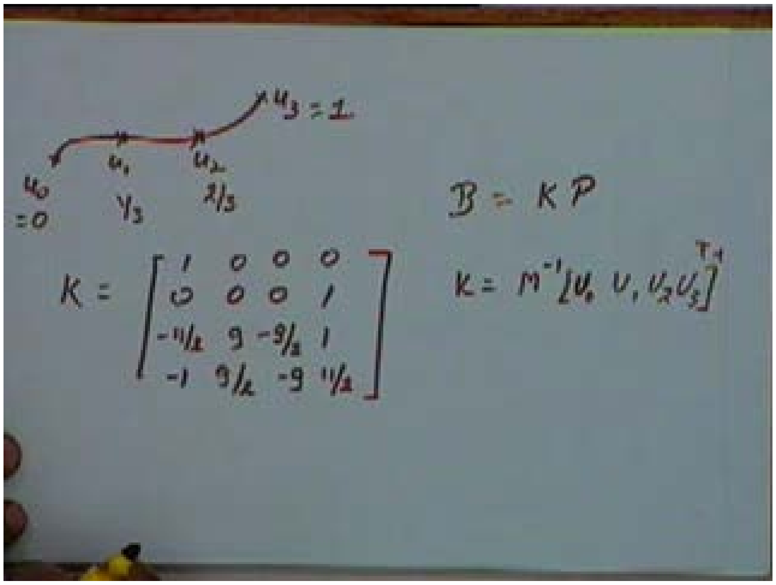

So if I take a curve like this and I say that it will pass through these 4 points. This is let's say u_0, u_1, u_2 and u_3. And I say u_0 is equal to 0, u_3 is equal to 1, u_1 is equal to 1 by 3 and u_2 is equal to 2 by 3, I say that these 4 points are equally spaced on the curve. So in that case, the K matrix that we will get that looks something like this. This is for the specific case where u_0 0, u_1 is 1 by 3, u_2 is 2 by 3 and u_3 is 1 where K is nothing but B is equal to KP and K is equal to M inverse multiplied by u_0, u_1, u_2, u_3 transpose inverse. This is for the specific case where u_0, u_1, u_2 and u_3 are equally spaced. Any question up to this point?

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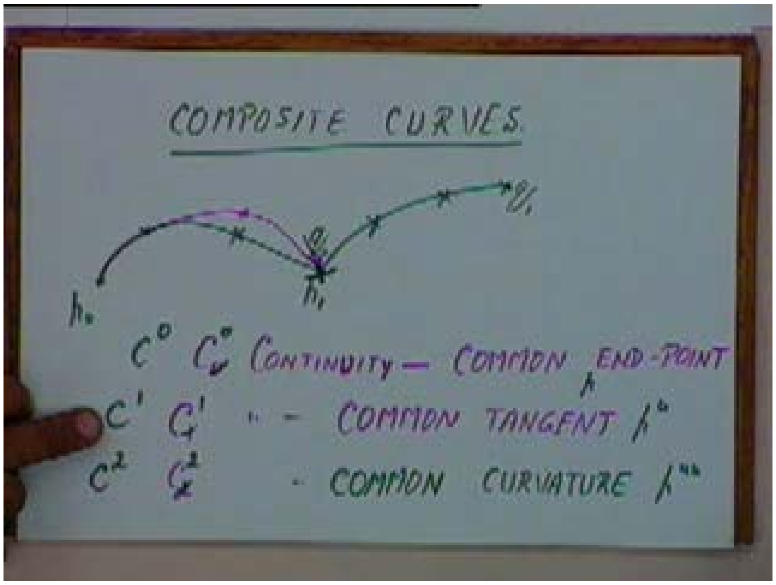

The next thing that we will see is that is a concept of what is called as composite curves. See, let's say if we have a curve like this, another curve staring from this point and continuing further. So far we have considered cubic curves, so let's say this is my starting point, this is my end point of that cubic curve and this cubic curve can be controlled through 4 points. And let's say if I want a larger curve and I want to control it through a larger number of points, in that case you can define composite curves this is the first curve, the second curve which is starting at this point and continuing further. So these are composite curves and this is one curve segment in that curve and this is second curve segment in that curve. That means the composite curves are basically a set of curves in succession. The question that comes up is in the case of composite curves at this point, what kind of continuity do we have? Because let's say this was a cubic curve like this, this is p_0 to p_1, this curve is let's say defined by q_0 to q_1.

Now if I modify this point I am going to change this point, this curve may be it will change to something like this. I am taking a 4 point form of the curve at the moment but then at this point, the slope of the curve is now possibly going to be different. So, at this point continuity between this curve and this curve might get disturbed. Then the first thing we see what type of continuity do we get at these curve segments. So we consider, the first step is continuity that we defined is what is called as C_0 continuity. In C_0 continuity we say that the curves have a common end point that means at this point, the end point of the first curve and the starting point of the second curve are the same. And when we talk of C_1 continuity, we say that the curve will have a common tangent.

Similarly C 2 continuity, it's normally C 0, C 1, C 2 we will normally write C 2 as C 2 continuity, C 1 continuity and C 0 continuity. In C 2 continuity it will have common second derivative or common curvature. Student: this is along with C 1. Yes yes, C 2 means C 1 and C 0 have to be there, only then you can talk of a common curvature. The slopes are not the same, there is no point in talking of curvature. If the end point is on the same then there is no point in talking of the tangent.

So in this case the points are the same, in this case the derivatives are the same and in this case the second derivatives are the same.

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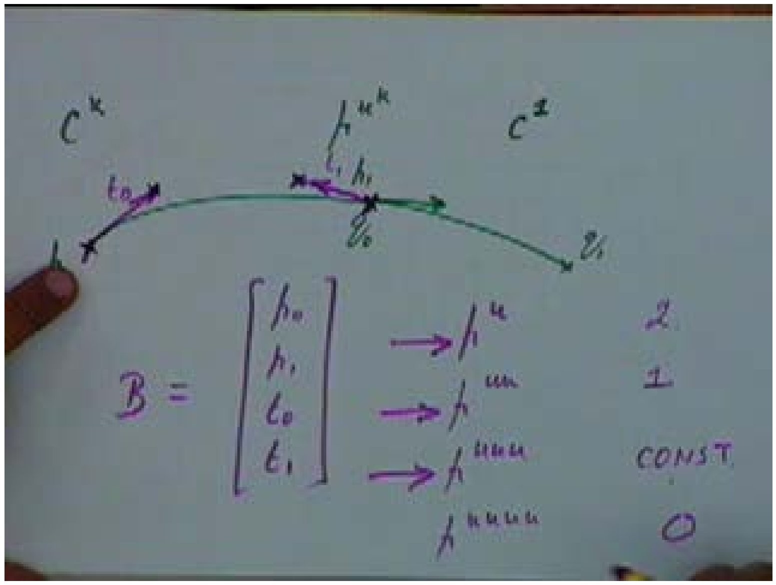

Similarly we can define a general kth order continuity. Let's say a C k, they we will say that the kth derivatives they are the same. So if you have any two curves, let's say this is if I say that these two curves have a C 0 continuity at the common end point that only means that the end point is the same but in terms of a slope there can be discontinuity. If I say that they have a C 1 continuity, the tangent vectors would be the same or will be in the same direction. Say, they have a C 2 continuity, the curvature will also be the same at that point. And if I have two curves which have the same tangent vector let's say then let's say this curve is defined through these 4 points this point, this point and these two are giving the slopes that means this is the slope at this point and this is the slope here.

So, now if I want to ensure C 1 continuity at this point, even if I change p_0 where C 1 continuity at this point is going to remain unchanged. If I consider the geometric form of this, this curve that will consist of p_0, p_1, t_0 and t_1. This is our definition for this first curve, B matrix for the first curve. This is, this vector is t_1, this vector is t_0. So even if I change p_0 and t_0, my continuity at this point doesn't get changed where the tangent at this point is given by t_1 and the end point is p_1. So we need to know, if we have a composite curve what kind of changes will disturb the continuity and what kind of changes will not disturb the continuity. If you are modeling a curve in any application which consists of a series of such curves, we need to ensure that continuity at the common end points is maintained because we don't want a discontinuous curve in most applications.

So let's take a simple case where we have two curves and you want to ensure that they have a C 1 continuity and the common end point.

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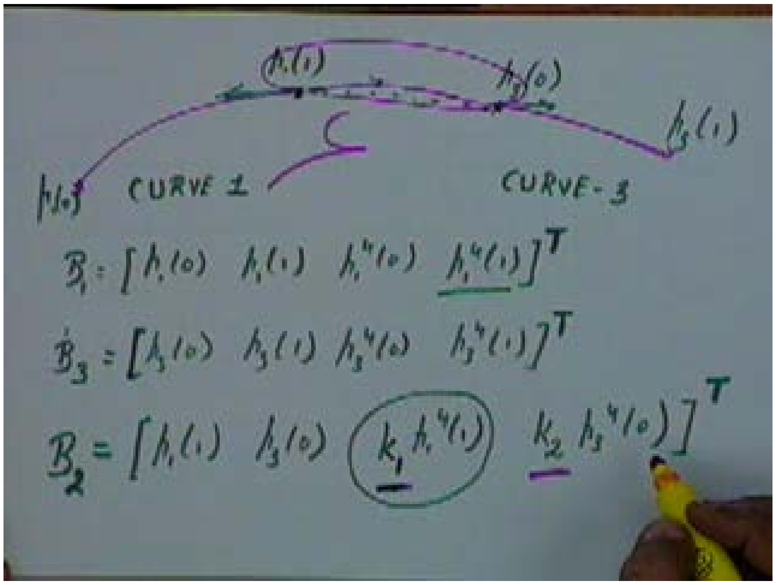

So, let's say if we have one curve like this. Yeah. What degree of continuity can you define for a curve defined by 4 points. In what length? In what level that is a good question. Student: By the 4 points here C 2 C 3. If we have the curve is defined through 4 points, it is basically cubic curve. So you have first derivative that is a second degree expression. The second derivative is a first degree expression and the third derivative is going to be a constant. So the third derivative is a constant and the fourth derivative is going to be a 0. Now if you have two curves, if you want instead of first C 1 continuity and tangent vector have to be the same. For C 2 this has to be the same, for C 3 continuity if I look at this term this is a constant for both the terms. If I can ensure that these two constant are the same, I will get C 3 continuity. And if I get C 3 continuity all higher orders of continuity are always there because all those derivatives are 0 anyway but defined at continuity because all those derivatives are 0. The only up to the third derivative, we will get non-zero terms.

So continuity can be defined, we can define even higher orders of continuity but once you get the C 3 continuity, higher orders of continuity don't make any sense to that. So now if we consider a set of successive curves let's consider the case where we have let's say this is my first curve and this is another curve and I want to join these two curves by another PC curve. This is the curve one, this let's say I call it as curve three and I want to join them by another curve such that C one continuity will maintain at both these points. Now this is a PC curve, so let's say it's a geometric matrix is given by p_1 of 0 and p_1 of 1 are the starting points and the end points. For this curve the starting points is p_3 of 0 and p_3 of 1. The starting tangent vector is let's say p_1 u at 0 and p_1 u at 1, so this is a B matrix for the first curve.

The B matrix for the third curve is given by p_3 of 0, p_3 of 1 p_3 u at 0 and p_3 u at 1. This is the B matrix for the third curve. And if I want to define a PC curve between these two points, so that C 1 continuity is ensured. The B_2 matrix, what will be the starting point for that? P_1 of 1 will be the starting point. So starting point will be p_1 of 1, the end point will be p_3 of 0.

So this end point is p_3 of 0. The starting tangent vector here that has to be in line in the same direction as this tangent vector. So we are saying that let's say it is some k_1 times, this ending tangent vector here is p_1 u at 1, so this will be p_1 u at 1. The tangent vector here is p_3 u at 0 so that let's say we have some constant times p_3 u at 0. Yes, I am just coming to that. I can have this tangent vector as this tangent vector multiplied by some constant. The direction of the tangent vector will be the same, so we will say C 1 continuity is there. So the unit tangent vectors have to be the same in that direction.

Student: since we got k_1 and k_2, we have to define another degree of continuity, for summing k_1 and k_2 we have to go down to C 2. No, now depending on the values of k_1 and k_2, we will get different curves. So if I change k_1 and k_2, I will get different set of curves that means we will get a family of curves. My curve can be like this or it can be like this or it can be like this or may be it can also be like this, as long as the tangent at this point is the same. My first curve can come like this and second curve can go like this, so I will get a family of curves depending on the value of the parameter k_1 and k_2.

Student: Sir, we have models in which continuity is ensured, C 1 continuity basically. Yeah. But even at the end there are models, certain models in which continuity is not ensured. Then you don't bother about that's all. Student: because then why do we use our models just there is no, if you cannot ensure continuity what is the use of those models like if we because… yeah, Student: in that continuity is not ensured. See it basically depends on the application whether you want to have continuity in that case or you don't want to have continuity.

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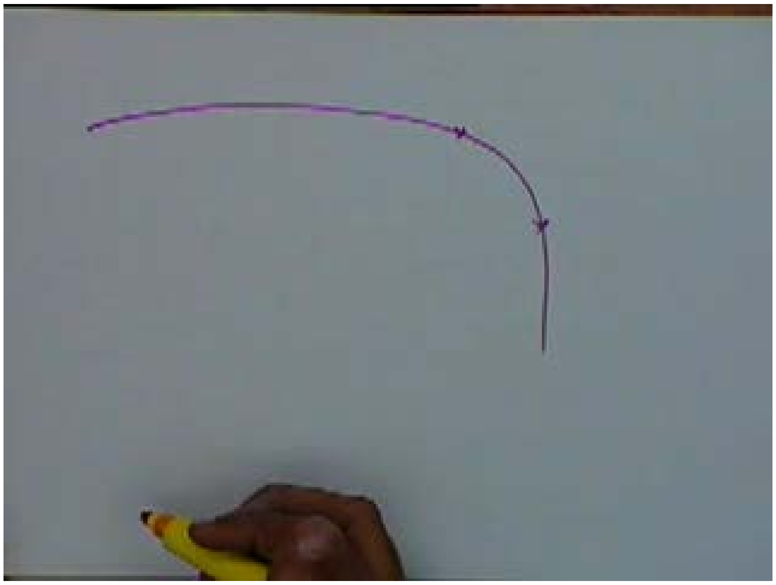

For example if you come, so right now modeling curves and if I say that I want to model let's say this is the front of my cup. For purely from aesthetics point of view or for my sake this is one curve, this is second curve, this is third curve and I want to have a C one continuity purely from aesthetics point of view.

For some of the application i might not need this continuity then I won't bother. Only when I need this continuity will I go for a formulation like this. In most cases of where one is talking of let's say aesthetics, either in situation like this or may be in a … where you don't want to have extra impact because of the ways and all that. We will like to have some continuity between the successive curves but if there is an application, you don't need that continuity you don't bother.
So in this particular case if you have two curves then you want to get a curve in between, a cubic curve in between to ensure that continuity. You can use this formulation and get two parameters k_1 and k_2, a modifying which you can get your C 1 continuous PC curve between these two curves. Any question up to this point?

Student: sir direction of that opposite curve is not exactly the same. You mean opposite curve, this curve. No, when you multiply it by minus 1 its direction get changed effectively. Yeah, so that the tangent vector that you are giving… Student: that is 180 degree out of phase. It's 180 degree out of phase. Student: is not the same tangent vector. Unit tangent vector it's the same on the opposite direction. So depending on this parameter k one to give it negative value, you can get results like that. It's eventually the question of application whether you want common data or not. It all depends on the application you are considering. Any question up to this point?

Student: these tangents for the two curves will be the same point in degree of the phase, will it still ensure C 1 continuity. Yeah, mathematically you can say it is, it will ensure C 1 continuity. Yeah, because even if you are giving it a negative direction you can mathematically say that but whether the application accepts that or not is a different issue. Student: for any that n derivative. For any n derivative. Now, we will go to another type of curves which are called Bezier curves.

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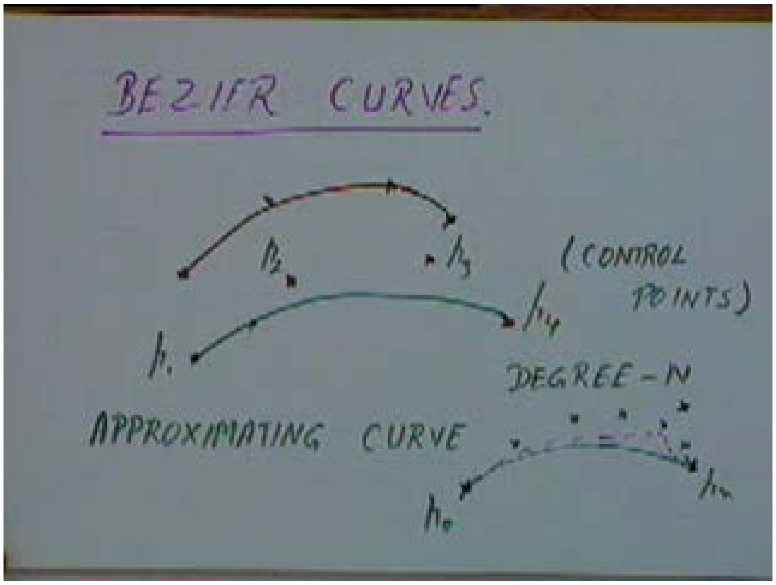

See so far when we say that we are talking of a 4.5 of a curve, our cubic curves are passing through these points like this. That is we are doing an interpolation between these points. Now we will define a curve which does not pass through these points but is only in approximation amongst these points. So we say that it is an approximating curve and not an interpolating curve.

The curve will look like this. These points let's say p_1 p_2 p_3 and p_4, these points are called control points. And this curve is an approximating curve and not a interpolating curve, is an approximating curve not an interpolating curve. Any point on this will only be approximated by these 4 points. This curve will never pass through any of these points except the end points, except maybe the end points. Yeah, again. Student: it has the least square with end points. No, no that is a different kind of curve where we will define Bezier curve which will have a different definition, we will give the definition for that. That is one way of approximating, we are not doing that right now. I will give the definition for that. The basic need for basic properties that needed from these curves, the first major property is let's say we say that the curve should start and end at the first and the last points at p_0 and p_n.

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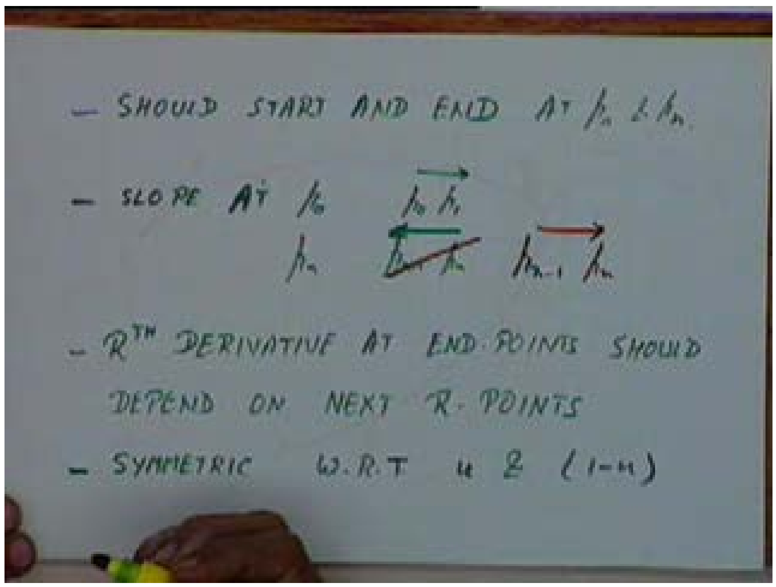

Right now I mentioned p_1, p_2, p_3, p_4 but typically this curve need not be just a cubic curve, it can be a higher degree curve it can have a degree N. As the degree is N, for defining an N degree curve we will have to have N plus 1 points. So the N plus 1 points will be ranging from p_0 to p_n, this is the point p_0 and this is the point p_n. And what we say is that we would like in the curve to start from p_0 and end at p_n. We want to define a curve which will go like this. It will be approximated by all these points and it will go from the starting point to the first from the first control point to the last control point. We do not want it to pass through any of these points.

The idea is that if you have curve like this and I make and I move this point from here to this point to this place. My curve, my point end will move from here to here, my curve might get modified somewhat like this. But I do not want my curve to pass through all these points. Student: each of the point sort pulls the curve. Yes, each of the points will sort of pull it towards itself. Student: and what's the relation regarding… I will just define the mathematical relation. Student: but this curve would be of very low degree than actual curve which can be made to pass.

What you do you mean by low degree? a low degree of the, see if you 6 points so we can have a curve of 5 degree passing which can be made to pass through in 6 points. Student: yeah so this curve will be of much lower degree. No, this curve will be nth degree curve. If I have 100 points that will be in a curve of degree 99. Student: it would be for that all the… yes, so this curve will have same degree as the number of points minus 1. That is for Bezier curves but in general these types of curves will not have that property. We will see other kind of curves which do not have that property also.

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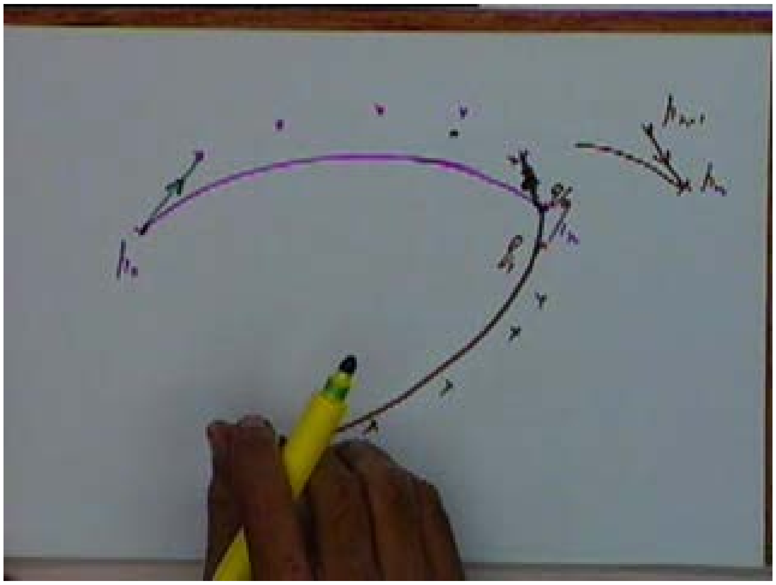

So if we have a set of points, the first property is my curve should start from here and end at the last point. This is only what we desired from the point of view of application. So the starting point should be p_0 and the ending point should be p_n. The other properties that we desire from these curves is that slope at p_0 should be p_0 p_1. That means at this point, the slope should be given by this vector and at this point the slope should be given by this vector. I haven't drawn it properly but anyway. So I have this starting point, the slope should be in this direction, end point the slope should be in this direction. So this is a slope at p_0 should be this and at p_n which should be given by p_n minus 1 p_n which is the other way around but… Actually at this point, the slope I have again drawn it wrongly.

Let's say my point is here, I will draw it here. This is my curve, this is one point, the second point is here and the slope is in this direction. Now if I consider derivative, my u is changing in that direction. So this is t_n minus 1, this p_n and generally this will also be… So slope at p_0 should be this and p_n should p_n minus 1 to p_n. The reason why we need this constraint is that again if we have composite curves, this is one curve and the next curve is starting from here is a set of points like this, so the curve goes like this. At this point let's say we have ensured C 1 continuity. So at this point if we have C 1 continuity then I know for sure that the tangent vector here will depend only p_n and p_n minus 1. The tangent vector of this curve will depend only on q_0 and q_1. So even if I change any of the other points, my continuity at this point is still ensured.

You know basic idea is that if I have a curve which I am using in any application, I want to have as much flexibility in modifying the curve as possible. Just one minute. So at this point a tangent in this direction, if I change this point that means let's say I change it from here to here. The slope at this point will still remain the same. So if I have two curves and I have C 1 continuity that continuity does not get disturbed by changing this point or by changing this point or any of the other points. You are saying something? Student: like in composite curve we will ensure C 1 continuity by fixing the like p_n minus 1 and q_1. Yes, so we will ensure the C 1 continuity here by fixing these two points. Student: Collinear. If they are collinear, that C 1 is in continuity. That C 1 continuity. That is the simple way of ensuring C 1 continuity. Student: and because we modify those points, we won't modify those points or if we modify them, we will modify them such that they will remain collinear.

So in order to ensure C 1 continuity easily, it gives this constraint that at this point the tangent vector should be given a p_n minus 1 to p_n and at this point the tangent vector should be given a p_0 to p_1. The next condition that we give is that the Rth derivative at the end points should depend on next R points. next Yeah, I will just explain what I mean. At this point, if I want to find out the second derivative that I should depend on p_0, p_1 and p_2. At this point the second derivative should depend on p_n, p_n minus 1 and p_n minus 2 again for the same reasons. So I want to have C 2 continuity, I can modify all the other points without bothering. Yes sir. And this condition, second derivation is derivation of the third one. Yes I just mentioning this, this condition is the simple sort of this condition.

So if the first derivative at the end point will depend only on the next point, the second derivative will depend on the next two points, third derivative will depend on three points and so on. This is again purely from the point of view of easy manipulation of the curve. And the next condition that we or the next property that we want from these curves is that they should be symmetric with respect to u and 1 minus u. That means if I take these set of points, whether I specify these points in this order or in that order, my curve should remain the same. Whether I give this as the point and this as the last point or the other way round, my curve should not get changed. This is another property that we need from these curves and all these properties are purely from the point of view of easy manipulation, easy definition and so on.

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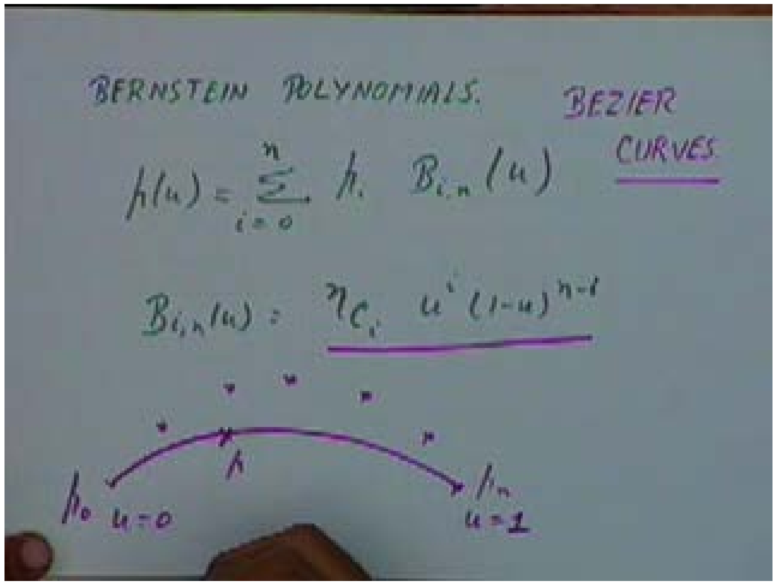

Now, so this type of curves or number of definitions can satisfy these properties. One of those is what is called as Bernstein polynomials where we define a curve as p of u is equal to summation of i going from 0 to n p_i multiplied by $B_{i,}$ n of u where $B_{i},\ _{n}$ of u is equal to n_{ci} multiplied by u to the power i into 1 minus u to the power n minus i. So if we have a set of control points like this this is my point p_0, this is my point p_n and we will get a curve between these points like this. What we are basically saying is that this, that the coordinates of any point p are given by the sum of the p_i into B_{in} terms where B_{in} is the weightage given to the point i. So, to each of these end points, we will give them separate weightage and that weightage is given by this expression and u is varying from 0 to 1. This point we will say u is equal to 0 and this point we will say u is equal to 1. And these B_{in}'s are the weightages given to each of these distinct points. These are called the Bernstein polynomials and this definition is a definition given to Bezier curves.

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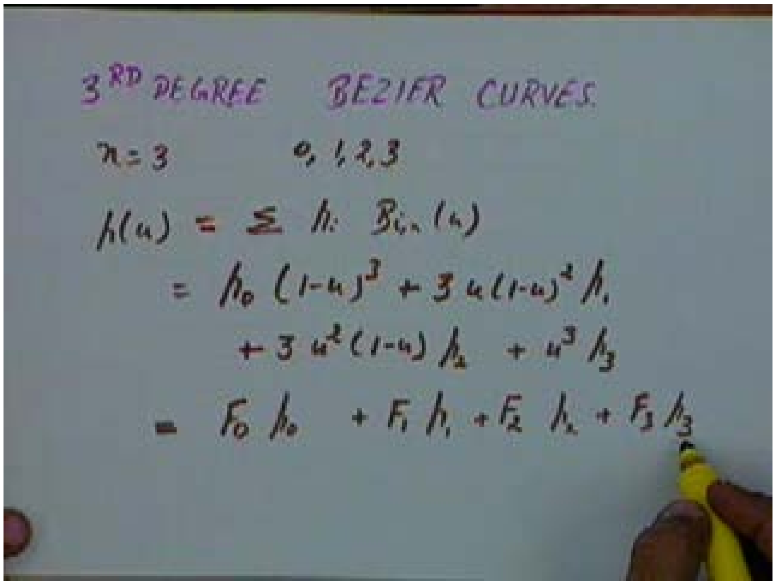

So let's take a simple case for third degree curves. So if we consider third degree Bezier curves for these third degree Bazier curves n is equal to 3, total number of points are 4 p_0, p_1, p_2 and p_3. And we will say p of u equal to, I first write the same expression sigma p_i B_{in} of u. For p_0 we will say this will be p_0, of this term (Refer Slide Time: 44:55) for n equal to 3, i equal to 0 we will get 3 C_0 multiplied by u to the power 0 into 1 minus u cube. So this will be 1 minus u whole cube plus coefficient of p_1, n equal to 3, i equal to 1. So n_{ci} will be 3, u into 1 minus u square, i is 1, n is 3, so I will get 3 u into 1 minus u squared multiplied by p_1 plus the next term will be 3 u squared into 1 minus u into p_2 plus u cube into p_3. So this will be a simple equation for the third degree Bezier curve where p_0, p_1, p_2 and p_3 are the 4 control points.

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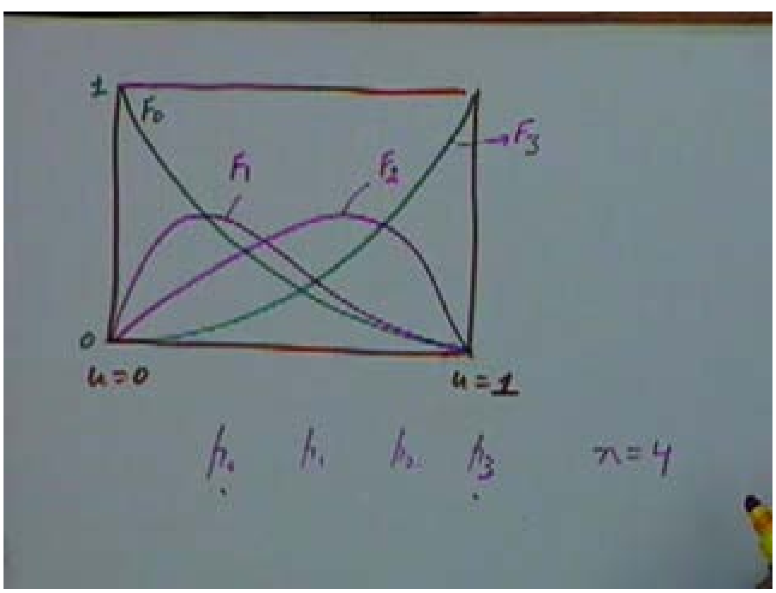

And again maybe we can write this as F_0 p_0 plus F_1 p_1 plus F_2 p_2 plus F_3 p_3 where F_0, F_1, F_2 and F_3 are the 4 blending functions. And in this case if you plot the blending functions, the bending functions, this is u equal to 0, this is u equal to 1. This is the blending function F_0. F_0 is 1 minus u whole cube, so u equal to 0 it is 1, u equal to 1 it is 0 and it is a cubic variation, so this is 1 and this is 0.

Similarly if I take F_3, F_3 is u cube so F_3 would look this and F_1 is 3 u into 1 minus u square. So F_1 looks something like this and F_2 would look something like this. This is F_1, this is F_2 and this is F_3. So by looking at this, one can make out that all these 4 blending functions if I consider F_0 and F_3, they are symmetric with respective u and 1 minus u. If change u to 1 minus u, F_0 will become F_3 and F_3 will become F_0 and F_1 will become F_2 and F_2 will become F_1. That means instead of specifying points in the order p_0, p_1, p_2 and p_3 I specify this to be the first point and this will be the last point, I will still get the same curve. One can make that out by looking at these blending functions.

Instead of n equal to 3, if I take n equal to 4 that mean if I take 5 points then I will get 5 blending functions and my curve will be of degree 4 that will be a bi quadratic curve. This is how one can get higher order curves, higher order Beizer curves by using this formulation.

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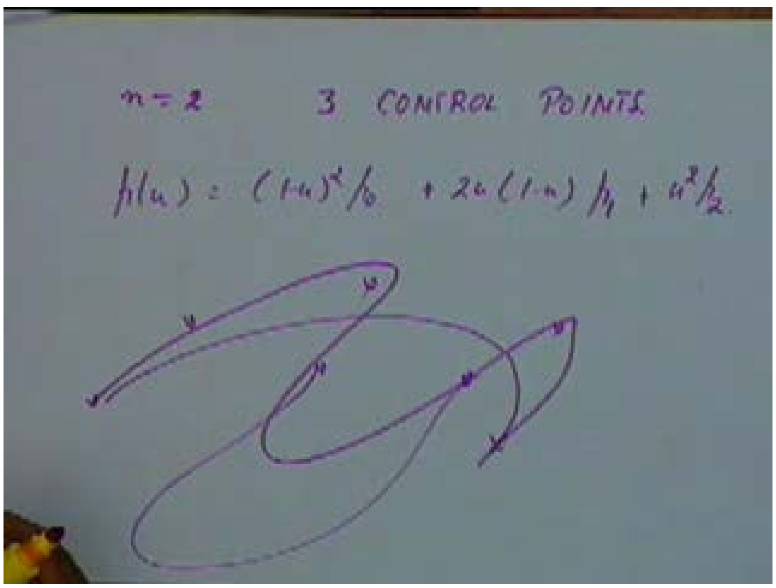

And we had a simple second degree curve, this is the third degree curve, a simple second degree curve that means you will be having only three control points where n is equal to 2 and 3 control points. We will get p of u will be 1 minus u whole squared times p_0 plus 2 u into 1 minus u times p_0 times p_1 plus u square times p_2. This is for simple second degree curve and similarly we can get higher degree curves. Any question up to this point? Student: why do we use Beizer curves? Why do we use Beizer curves? The first thing is to get higher degree curves because in PC curves, you are getting only curves up to third degree. Specify more points in the… Student: specify more points even in like, if you say you specify more points as specified but that's… you will have to take the higher degree curve in that case, it won't be any cubic curve, you have to take higher degree curves.

Now other reason is in this you get a better control. For example if you have a lot of points like this and you say that the curves should pass through all these points. You will get a very complicated shape of the curve something like this. In applications you might like to approximate between these points, these kinds of certain other properties which will be covering in the next class. The outer limits of this curve are bounded. Yes, it can't go beyond a certain limit. It will always remain within a closed polygon. For example a curve like this can go let's say these points are positioned wrongly, if the curves can go like this because the curve has to pass through all those points that even happens in the case of Bezier curves.

In a case of Bezier curve I am trying with Bezier curves for the same points. It has to remain within a certain area. So the curve would probably looks something like this or it might go within it but it will remain within a closed polygon. So it has some properties, you see some of properties in the next class but in order to get those properties these curves are defined. And then we will see how these curves are further modified to get maybe the b-spline curves, we have some additional properties are needed. We will be going to that in the next class. Any other questions? In that case I will stop here now. In the next class we will see some more properties of these types of curves and then go on to b-spline curves.