Computer Aided Design

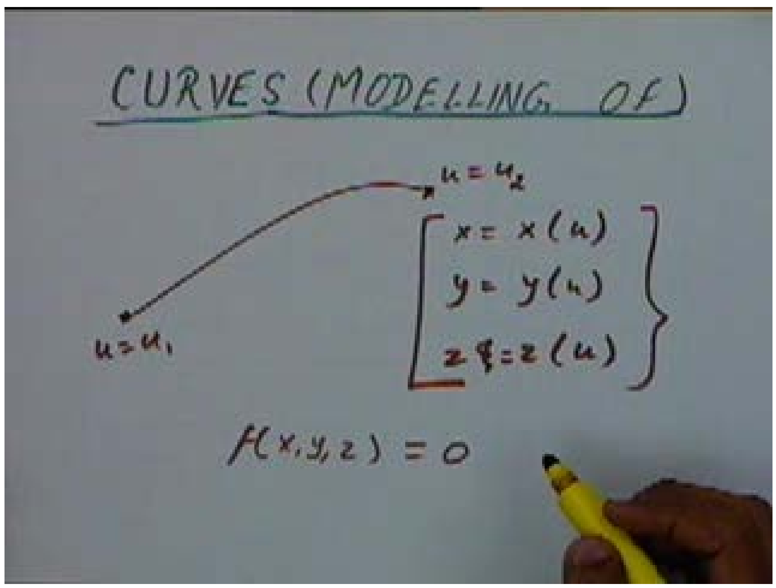
Dr. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi

Lecture No. # 30

Modelling of Curves

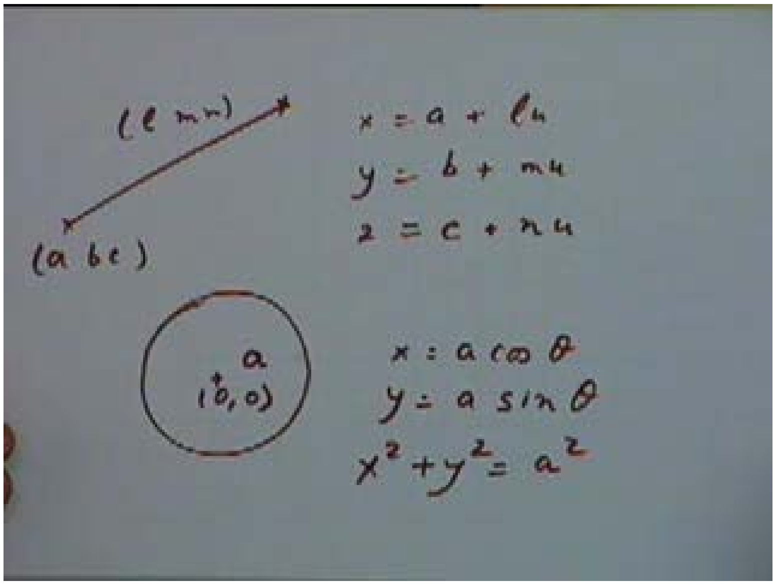
We will be studying a new topic that is modelling of curves and I hope you are probably aware that if you have any curve, any general three dimensional curve we can normally express the equation of this curve in a parametric form. And that is if you say x is equal to some function of u, similarly y will be some function of u and z will also be some function of u where u is a parameter which will vary within a certain range from this point to this point, from starting of the curve to the end point of the curve.

(Refer Slide Time: 00:01:10 min)



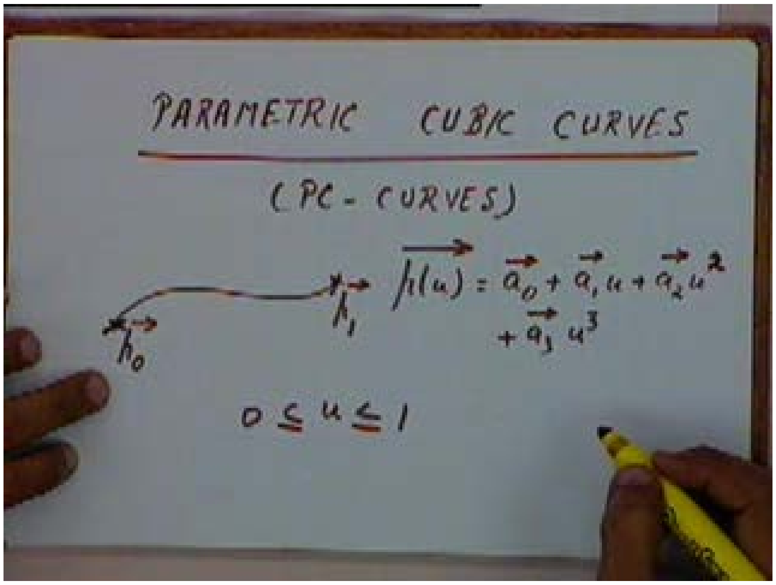
I said typically if we say u is equal to u_1 here and u is equal to u_2 here, so curves can normally be expressed in a parametric form in this manner. And typically we will be dealing with only the parametric representation of the curves. They can also be represented in an implicit manner as f of xyz is equal to 0 but we will normally be dealing with curves when they are expressed in a parametric form like this. And by convention we take the range of this parameter to be varying from 0 to 1 that means we will say u will be between 0 and 1. So if between u_1 and u_2 without loss of generality, we can say that it is between 0 and 1. And some typical examples of parametric curves if you have straight line, we can express the equation of this straight line as x is equal to a plus l times u, y will be equal to let’s say b plus m times u and z will be equal to c plus n times u where the starting point is abc, the direction cosines are lmn.

(Refer Slide Time: 00:02:45 min)



Similarly if you have a simple circle, this is a closed curve. Again we can write it in a parametric manner. This is the center let's say the center is at 0 0 and the radius let's say is a. We can say x is equal to a cos theta, y is equal to a sin theta. This is a parametric representation, an implicit representation will come out to be x squared plus y squared equal to a squared. So this way we can represent curves in a parametric manner. A circle is typically a case of a second degree curve, quadratic curve but we can also have higher degree curves and the first type of curves that will be dealing with in detail are what are referred to as cubic curves. So if we call, if we talk of parametric cubic curves or we will refer to them as PC curves.

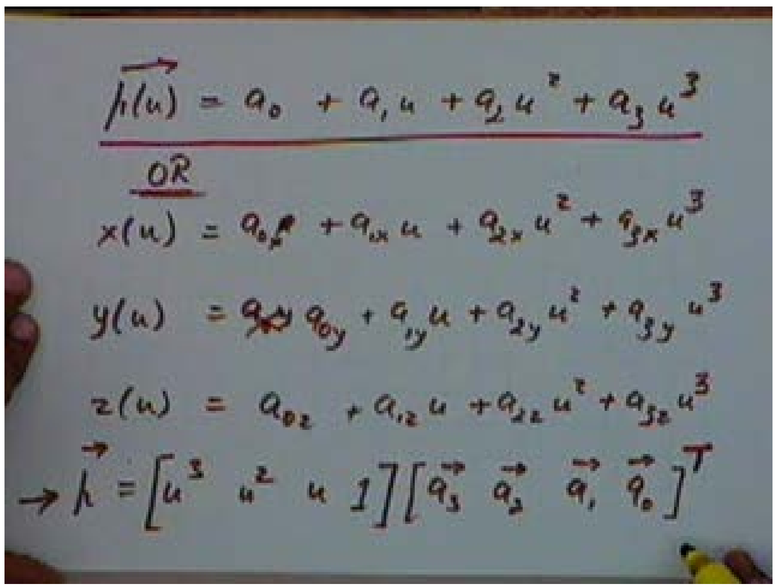
(Refer Slide Time: 00:04:11 min)



Now as we have, as the name suggests we are talking of cubic curves. So the parametric equation of these curves, the equation in this form will be a cubic expression. And if you write it lets say if you have a curve like this, we will write the equation of this curve as is let's say a_0 plus a_1 u plus a_2 u squared plus a_3 u cube. That means this is a vector, u is a parameter in the range of 0 to 1 and a_0 a_1 a_2 a_3 are constants.

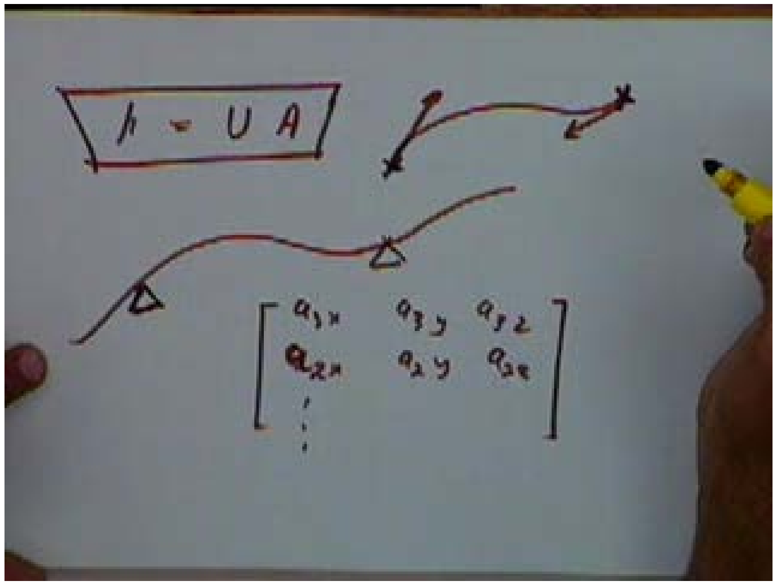
Typically in fact a_0 a_1 a_2 a_3 all four of them will be vectors because we are talking of the vector equation in three dimensions, all we are talking of space curves that is curve which are in three dimensions. So let's say the starting point here will be some p_0 which is the starting point. At this point u is equal to 1, so we will call this as p_1. Again these are the position vectors p_0 and p_1. So this is a vector equation and we can write it as, I will again write it here p of u a_0 plus a_1 u plus a_2 u squared plus a_3 u cube.

(Refer Slide Time: 00:06:16 min)



We can write this as… The x component of this equation will give us this scalar equation. Similarly the y component will give us (Refer Slide Time: 07:05) and similarly for the z component we will get a... Now the x y and z components, all three together can be expressed as one vector equation like this or alternatively what we can say is with this vector p will be or is equal to this matrix multiplied by… So this equation can also be written as this where p is equal to u cube u squared u_1, this matrix multiplied by a matrix of the four vectors, not the transpose, the transpose. I want this to be a column vector, this row vector multiplied by a column vector.

(Refer Slide Time: 00:08:50 min)



And this we will say p, this we can say p is equal to U times A where U, this matrix U is a matrix of the four cubic the four terms in a cubic polynomial and this matrix A is a matrix of these 4 vectors. Now these 4 vectors are the coefficients of this equation. And if you are talking of 4 vectors of this type then these 4 vectors have, each vector has got 3 components in it, so these 4 vectors together have got 12 components. Each vector is a 3 tuple xyz, so these 4 vectors together have got 12 components and this matrix A will also have 12 components. So if we specify these 4 vectors, we will get a unique PC curve.

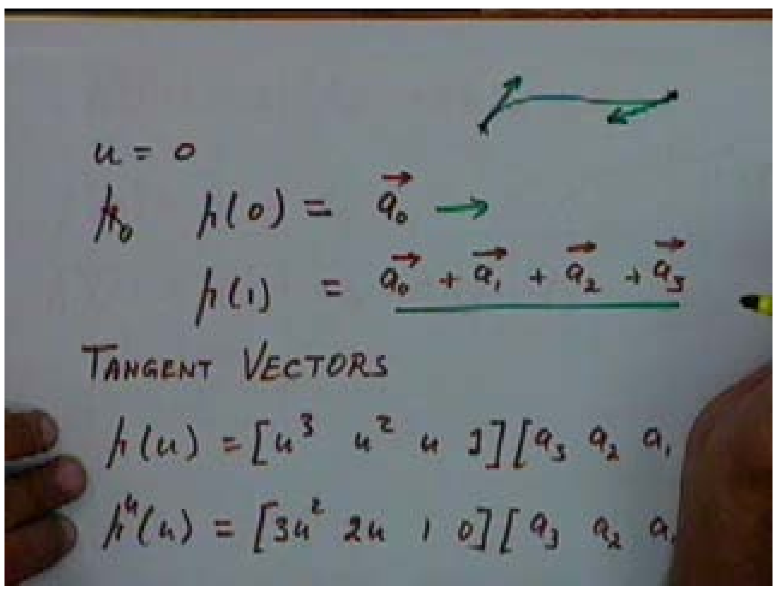
So if you want to specify any cubic curve between two points or any general cubic curve, we need to specify these 4 vectors. Now, before we go further very briefly about the history of a cubic curves or history of how the definition of these curves have come about. You will be aware of the concept of splines. Are you aware of splines? No. The splines are basically let us say you take a wire, a steel wire or some wire like that and in the ancient days for modeling surfaces we should take wires of this shape and attach weights at different points. If you hang a slightly stiff wire and then attach weight at different points, you will get different kinds of curves in that. Those curves will typically be cubic curves, you can write down the deflection equation and prove that. So, and for getting smooth curves, especially good aesthetics may be in a case of say pulse and so on these kinds of instruments have been used which are basically wires with weights at different points, weights, physical weights.

You take weight and hang it over there. So you take a slightly stiff wire which is let us say fixed even from the top, even from the side and so on and then you hang weights or give a tension from one side and so on. Then even get a general curve and in order to get a smooth curve, people have used instruments of that type to get different kinds of curves which are used in different places specifically in the case of shapes and boards and so on.

An extension of that is a spline set that you might have used in a graphic courses. You get a set of spline sets which are basically instruments like set squares and so on with different kinds of curves. And those curves are typically curves that are generated by that kind of process. It can be shown mathematically that if you are using a set of weights on a wire, if the equation that you will get will eventually be a cubic equation, so that is how you we come to splines. And typically this parametric cubic curve, pc curves that we are talking of is cubic spline which we are going to deal with and an equation of this type, now coming back to pc curves, an equation of this type I have said that A, this matrix A consists of 12 numbers or 4 vectors. So if I write it down, it will consist of a_{3x} a_{3y} a_{3z}, similarly a_{2x} a_{2y} a_{2z} and so on.

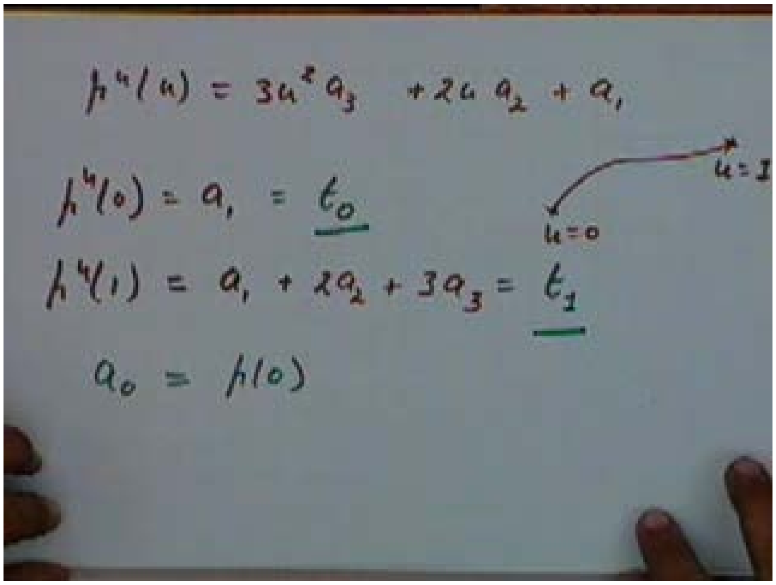
So now if I want to specify a cubic curve, I have to give these 12 numbers or these 4 vectors. Typically what I like to specify a cubic curve by specifying starting point, end point and may be some intermediate points. It is very difficult to give numbers and then get a physical feel of the curve. So we are working on a editor, I like to specify these curves so these end points or may be I like to specify the slopes here. Only then one can get a physical feel of this curve and one can modify this curve freely. So if you take a curve of this type, what are its end points to start with.

(Refer Slide Time: 00:14:29 min)



If I consider this equation, the starting point will be given by u equal to 0. So if I put u equal to 0, I will get the starting point p_0 or we can write it as p of 0 to be u_0, these three terms become 0, this is one, will be a_0. p_1, p_1 will be, these 4 terms will become 1 so we will get a_0 plus a_1 plus a_2 plus a_3. If I consider tangent vectors, tangent vectors for this curve will be given as a derivative with respect to u. And if I differentiate this expression with respect to u, what will I get? Is that okay? This is a constant matrix, so I can just differentiate this with respect to u or I can take this initial definition and differentiate that with respect to u. So I will get expression. So now this is the same as 3 u squared times a_3 plus 2 u times a_2 plus a_1.

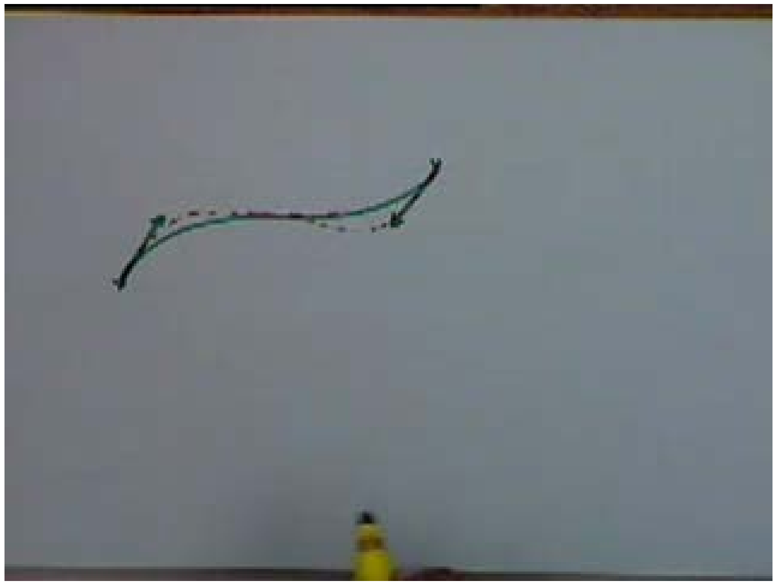
(Refer Slide Time: 00:16:41 min)



Now if I consider a cubic curve and I want to find out the tangents at the starting point and at the end point, at u equal to 0 and at u equal to 1. So p u at 0 will be equal to a_1 and p u at 1 will be equal to a_1 plus 2 a_2 plus 3 a_3. Of this let us say if I call this as my tangent at 0 and this may be I will call it as the tangent at 1, tangent at u equal to 0 and tangent at u equal to 1. So now I have got 4 equations, this is one equation, this is a second equation, starting point, end point, starting tangent vector, ending tangent vector. So I have got 4 equations and my aim is to find out the 4 variables a_0 a_1 a_2 and a_3.

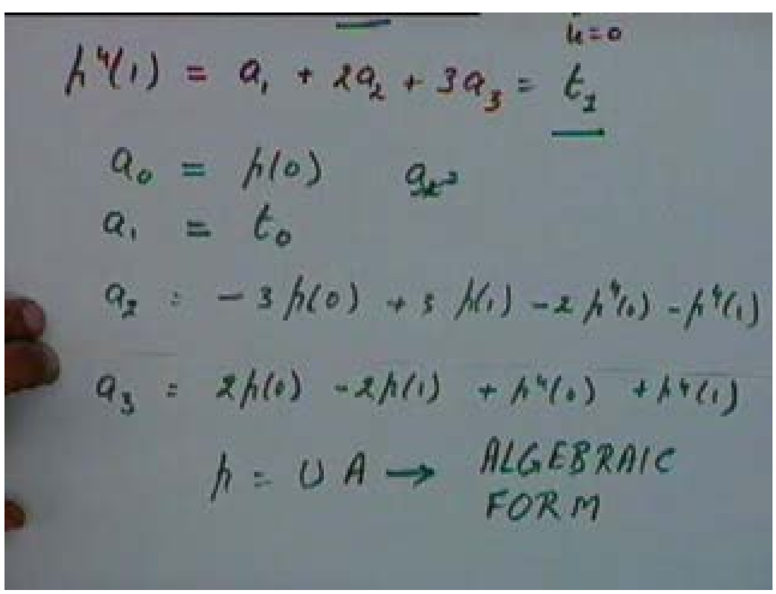
As I said for any general cubic curve, I like to specify the starting point, end point and may be the two tangent vectors to specify this curve. So if I want to get the values of these 4 variables in terms of starting point, end point, starting tangent vector and the ending tangent vector, I can solve for a_0 a_1 a_2 and a_3 in terms of these 4. If I solve it out, I will get a_0 is equal to p of 0. This is the first equation that we have. a_0 is equal to p of 0, a_1 will turn out to be equal to from here t_0. Student: sir, yeah no no. When I am taking the derivative, that is a unit tangent vector you are talking off. (Refer Slide Time: 19:45) What?

(Refer Slide Time: 00:19:51 min)



See typically what happens is if you take a curve like this, you have taken a unit tangent vector here and a unit tangent vector here, we get one curve. (Refer Slide Time: 20:08). No, what one can do is at this point you change the magnitude also. If you change the, if you keep the direction the same and change its magnitude, this curve from this position will change to something like this. Even by changing the magnitude, we can change the shape of the curves. Similarly if I change this tangent vector also, may be the curve would become something like this. Typically when you are specifying the 4 vector in question, you have to specify the magnitude as well as the direction. You just specify the direction as a unit vector, we can still get other cubic curves by changing the magnitude of the tangent curve.

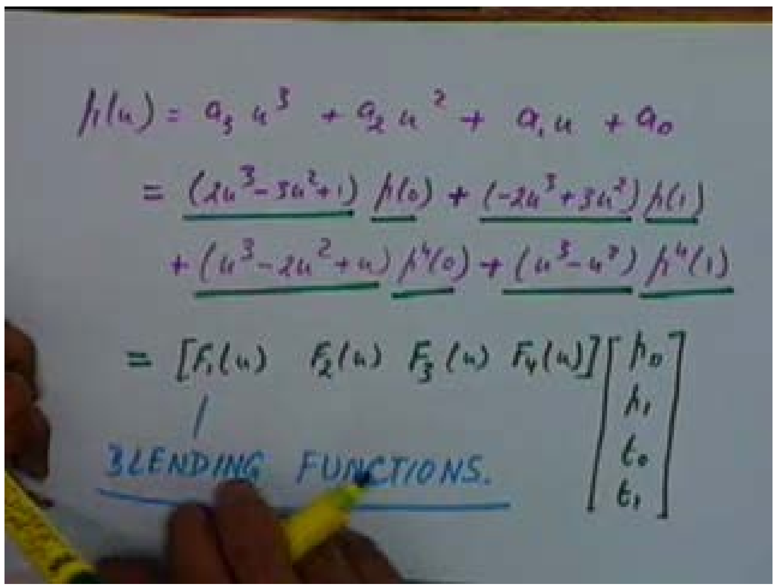
(Refer Slide Time: 00:21:30 min)



So this tangent vector that we take typically t_0 and t_1, we always deal with the complete vector and not just a direction. So we get a_0 and a_1 and if we want to get a_2, we will get a_2 to be equal to minus 3 times p_0 plus 3 times p_1 minus 2 times p u at 0 or t_0 minus p u at 1 or t_1. This you can verify from the 4 simultaneous equations. Similarly, a_3 you will get will be 2 times p_0 minus 2 times p_1 plus p u at 0 plus p u at 1. So this way we can get a_0, a_1, a_2 and a_3, the 4 vectors in question.

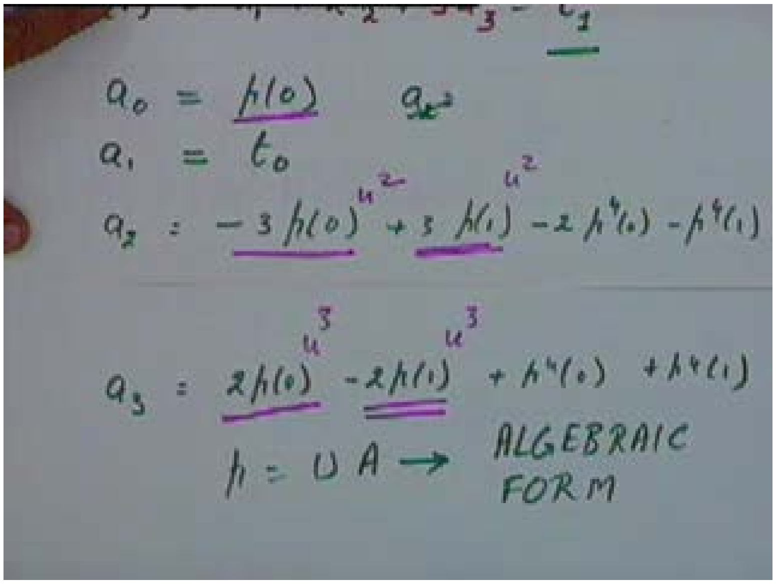
And again if you take, initially the definition that we had p is equal to u times a. now this matrix a consists of a set of algebraic coefficients. There is no direct physical significance for the 4 vectors a_0, a_1, a_2 and a_3. Then we defined a cubic curve using this equation. So this expression is referred to as the algebraic form of the PC curve. This is called algebraic form of the PC curve because the coefficient of this matrix A are the algebraic coefficients.

(Refer Slide Time: 00:23:18 min)



And if you take a_0 a_1 a_2 a_3 in this form and put it back over here, what we will get? I will just… our equation is initially p of u is equal to a_3 u cube plus a_2 u squared plus a_1 u plus a_0. Now if this, if I take this expression for a_3 and similarly I take these expressions of a_0, a_1 and a_2 and I assemble them together, what we will get will be an expression of this type. If we look up p_0, it is appearing in a_0, p_0 is appearing here also and similarly p_0 is appearing here also. The coefficient of a_3 is u cube, the coefficient of a_2 is u squared and a coefficient of a_0 is 1. So when I complete out, when I write down this equation the terms of p_0 will be obtained from a_0, a_2 and a_3. So the term of p_0 will contain 2 u cube minus 3 u squared plus 1. Is that okay? Because p_0 is contained in a_0, a_2 and a_3, coefficient of a_3 is u cube, a_2 is u squared and a_0 is 1. So I will get 2 u cube minus 3 u squared plus 1 and that is what I have written over here as a coefficient of p_0. Similarly I consider coefficient of p_1, p_1 is contained here and it is contained here. This is u cube, this is u squared.

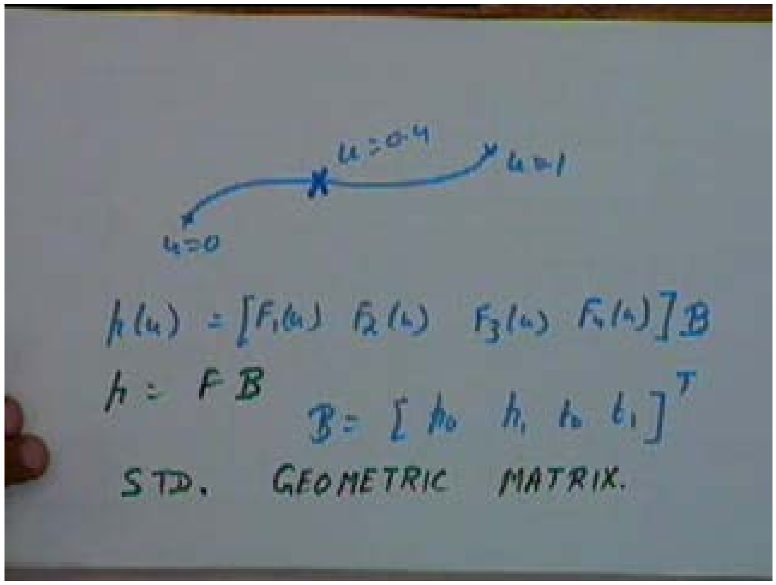
(Refer Slide Time: 26:06)



The coefficient of p_1 will be minus 2 u cube plus 3 u squared. This is this will be minus 2 u cube plus 3 u squared times p_1. Similarly if I consider coefficient of p u at 0, this is one term and p u at 0 this is the second term and this is the third term. So if I assemble them, I will get this plus u cube minus 2 u squared plus u times p u at 0. And the remaining term that I will get will be u cube minus u squared times p u at 1. Is this all right? The coefficients of p_0, p_1 p u of 0 and p u of 1 will be these 4 coefficients.

Now this I will write that as, now this is one function of u, this is the second function of u, this is another function of u and this is also a function of u. So I will write this as F_1 of u, F_2 of u, F_3 of u and F_4 of u. This multiplied by the vector p_0, p_1, t_0, t_1. Now these 4 vectors, expressions F_1 u, F_2 u, F_3 u and F_4 u are basically cubic expressions in u and I say u is varying from 0 to 1. So, these 4 are referred to as blending functions.

(Refer Slide Time: 00:28:58 min)

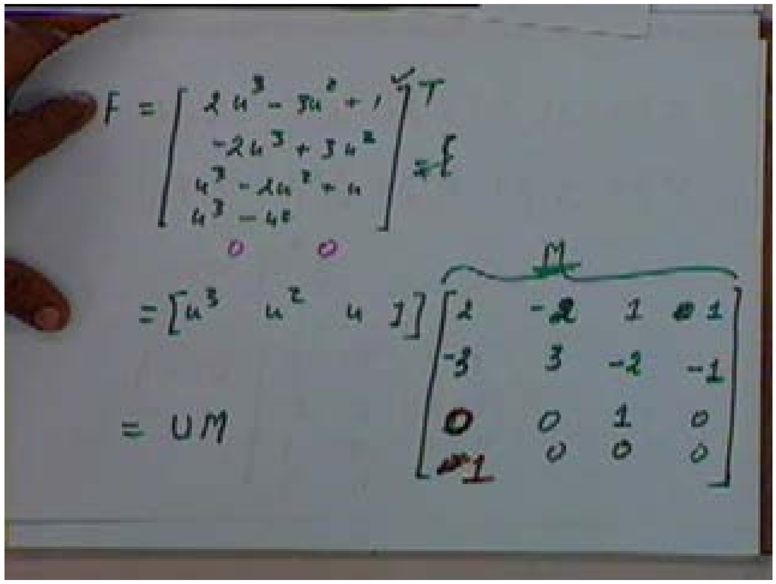


Essentially what we are doing is along the length of the curve, if I consider my curve again like this, we say u equal to 0, we say u equal to 1. Along the length of this curve at any specific point let's say at this point, if I take the point here which is u equal to 0.4 and at this point if I want to find out this, coordinates of this point I will find out the value of these 4 functions and then I will get some weightages, I will some get numbers. I will take a weighted sum of these 4 tangent vectors, these 4 vectors. So these 4 functions are basically giving how much weightage is to be given to each of these 4 vectors as we move along the length of the curve. So along the length of the curve these 4 weightages will change.

The way, in the manner in which these 4 weightages will change that can be seen by our blending functions. If I consider these four blending functions, this is F_1 u, this is F_2 u, F_3 u and F_4 u. So these are the 4 blending functions which define this cubic curve. If I have any general kind of curve I give these 4 vectors, the shape of the curve will be decided by the blending function of that curve. For a cubic curve, for a parametric cubic curve these are the blending functions we used. For some other kind of curve, we might be we might use different kinds of blending functions. If we use different kinds of blending functions, we will get a different shape of the curve, we will get a different characteristics of the curve.

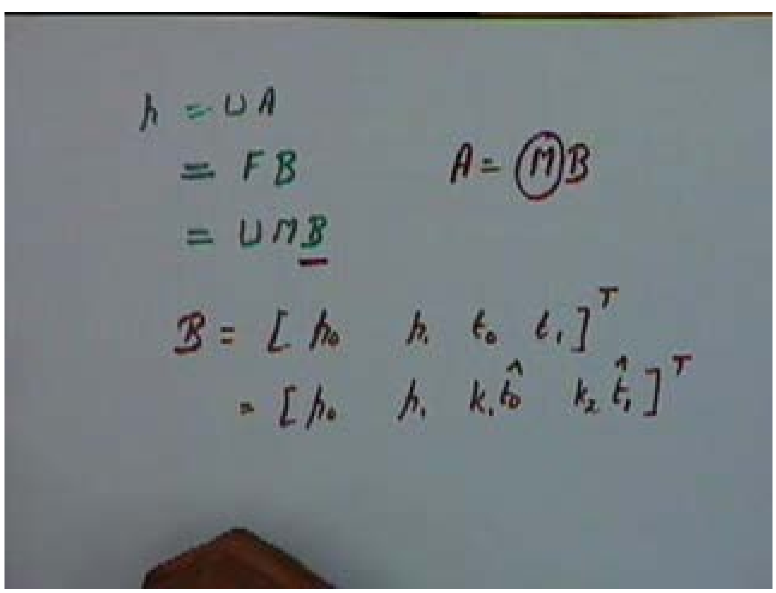
So this equation for the cubic curve, for the PC curve would become p of u will be equal to F_1 u, F_2 u, F_3 u, F_4 u multiplied by this matrix which we call as the matrix B. This matrix B is p_0 p_1 t_0 t_1 transpose. This B is the geometric matrix or the standard geometric matrix consists of starting point, end point, starting tangent vector and the ending tangent vector. Yeah. Yes, for cubic curves blending functions, this thing. If we define some other kind of curve, may be a the Bezier curve or the Bezier curve, again on the same degree then the blending functions can be different. For a parametric cubic curve, for the PC curve these are the other blending functions we choose. And this matrix, this equation we will write that as p is equal to F times B where F is the matrix of the blending functions, B is the standard geometric matrix, standard geometric matrix.

(Refer Slide Time: 00:32:52 min)



And this matrix F consists of the four blending functions which we can write down as 2 u cube minus 3 u squared plus 1. This is minus 2 u cube plus 3 u squared u cube minus 2 u squared plus u and u cube minus u squared. And again we will write that as… (Refer Slide Time: 33:33). This set of 4 expression we can write down in this matrix form where this is u cube u squared u_1 same in the standard u matrix that we had earlier, multiplied by this set of coefficients. This row multiplied by this column will give us the first equation and so on. And this again we will say will be equal to U times M. So F will be equal to U times M where M is the matrix of, is this matrix, this is the matrix of the coefficients of the blending functions.

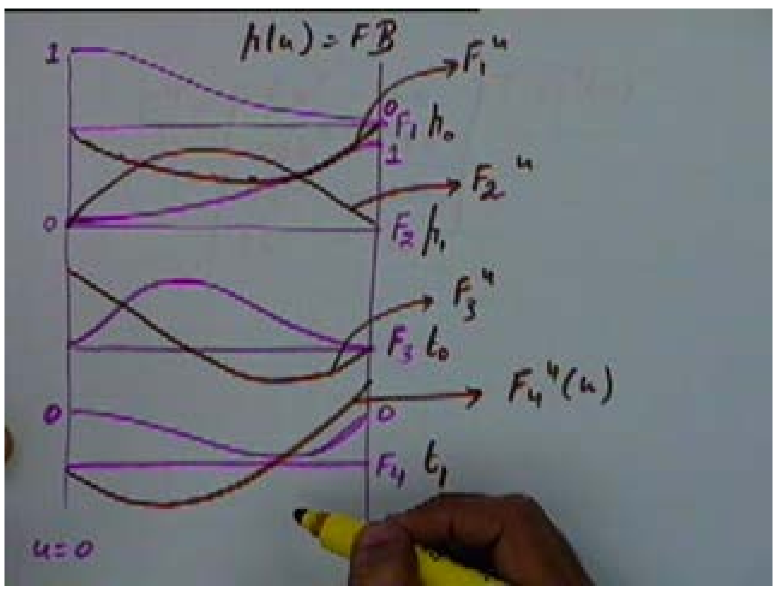
(Refer Slide Time: 00:35:04 min)



So the definition of the cubic curves which was earlier p equal to UA, we change that F times B where B is the standard geometric matrix, F is the matrix of the blending functions and this is equal to U times M times B. So what we will get will be from this A will be equal to M times B, A is equal to M times B. So if you have given the standard geometric matrix that is the matrix B and we want to find out the algebraic coefficients, we only have to pre multiply it by M where M is the matrix of the coefficients of the blending function. And B is the standard geometric matrix which is given by p_0 p_1 t_0 t_1 or we can say p_0 p_1 some constant let's say k_1 times unit normal vector, unit tangent vector and k_2 times unit tangent vector. Any questions up to this point?

Now if you look up the definition of the PC curves in terms of the standard geometric matrix, we have these 4 blending functions. And when I say these blending functions will be varying from or will be varying in the, as we go from 0 to 1, as we go from the starting point to the end point of the curve. As we go from u equal to 0 to u equal to 1, these 4 blending functions will change. Let's just see in what manner do these functions change?

(Refer Slide Time: 00:37:41 min)



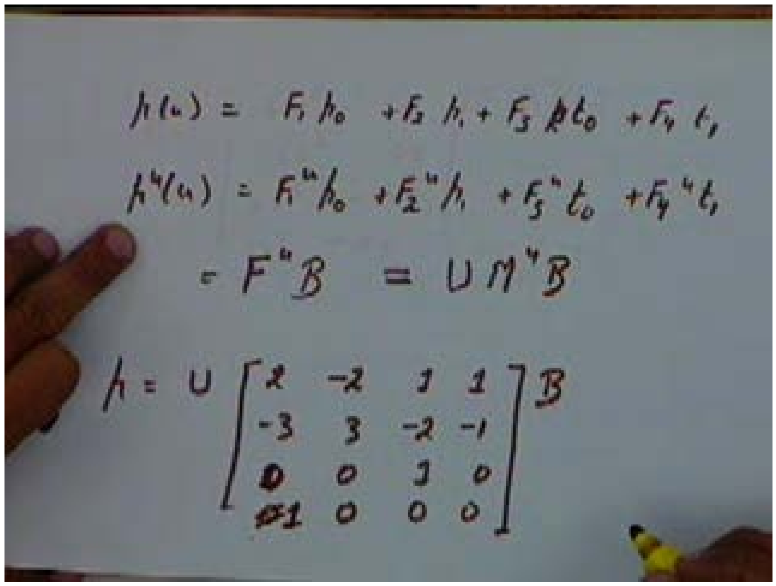
So if we consider this is u equal to 0, this is say u equal to 1 and if I want to plot let's say F_1, my expression for F_1, this is my expression for F_1. So at u equal to 0 it will be equal to 1 and at u equal to 1, it will be equal to 0, 2 minus 3 plus 1. So F_1 will change and will change in a cubic manner, it will change something like this. This is 1 and here it is 0.

Similarly if I want to draw F_2, if you look at F_2 at u equal to 0 it is 0 and at u equal to 1 it will be 1, so F_2 will change something like this. Here it will be 0, here it will be 1. And this should be expected because F_1 and F_2 are the weightages given to the starting and the end points. So to the starting point we have to give it a weightage of 1 at the starting point and 0 at the end point. Similarly at the end for the end point the weightage has to be 0 at the starting point and 1 at the end point. If we consider F_3, at u equal to 0 it will be 0 and at u equal to 1 also it will be 0, 1 plus 1 minus 2. So F_3 will take a shape which will be something like this. And F_4 again at u equal to

0 it will be 0, u equal to 1 also it will be 0 and the shape of this curve is something like this. (Refer Slide Time: 40:25) It's negative.

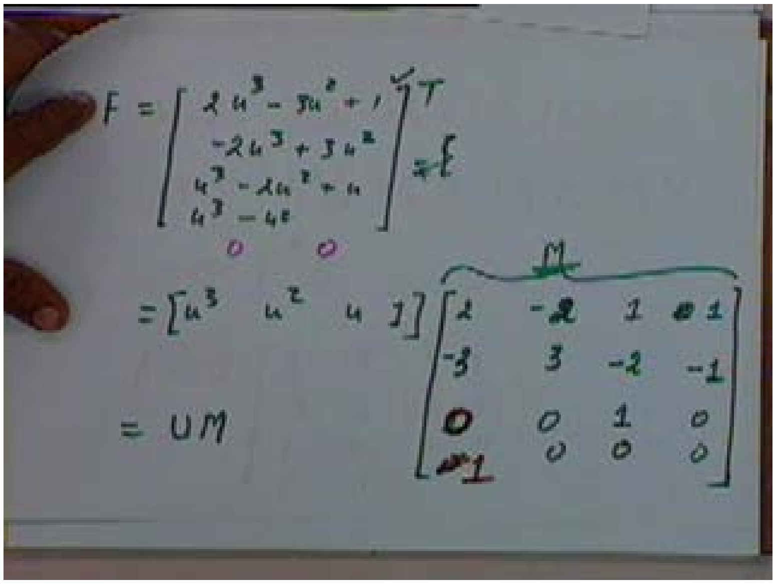
If you take any intermediate values, you will find out that it is negative. So the 4 blending functions will have a shape like this. Our equation is given by, this is the weightage for p_0, this is the weightage for p_1, this is for t_0 and this is for t_1. When I repeat t_0 and t_1, the way I have defined them are not unit tangent vectors but they are the other tangent vectors.

(Refer Slide Time: 00:41:26 min)



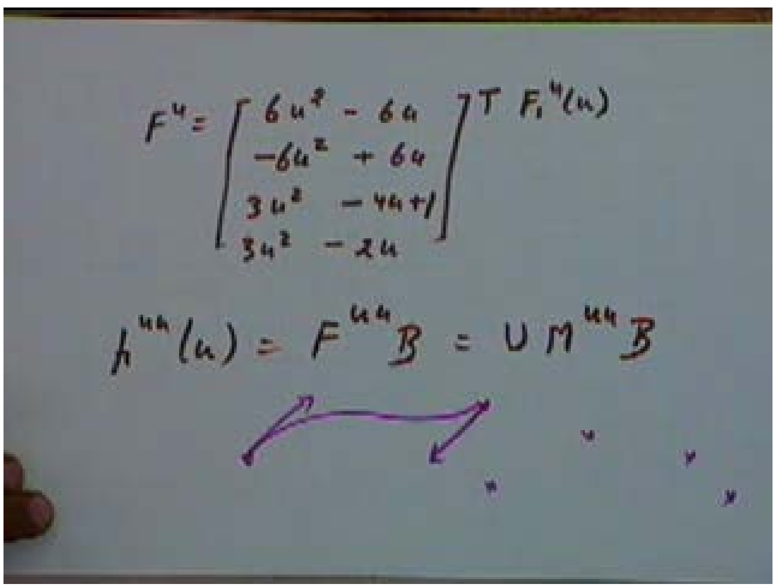
And if I, my equation is p of u is equal to F_1 p_0 plus F_2 p_1 plus F_3 sorry F_3 t_0 plus F_4 t_1. If I want to find out the tangent vector at any point on the curve, I will get p u at u to be in the differential of F_1 with respect to u times p_0 plus differential of F_2 with respect to u times p_1 plus F_5 u times t_0 plus F_4 u times t_1. And this again I will write that as F u times B where F u is the matrix of these 4 vectors, of these 4 expressions sorry. And this I can again obtain by looking at p is equal to u times the matrix of coefficients that we had 2 minus 2 1 1 minus 3 3 minus 2 minus 1 1 0 1 0 0. You can make this change.

(Refer Slide Time: 00:43:01 min)



I want 2 u cube minus 3 u squared plus 1, this term has to be 1 this will be, this 1 will be 0. Yeah, so this one is 0, this times B. So if I want to express this, I can look at these coefficients and carry out my differentiation and directly get a matrix for F u.

(Refer Slide Time: 00:43:54 min)



So my Fu just by simple differentiation would be 6 u squared minus 6 u, this will be minus 6 u squared plus 6 u 3 u squared minus four u plus 1 and 3 u squared minus 2 u. And again these are the 4 blending functions for getting the tangent vectors, we can plot these also. If I plot them, if I superimpose on this, my first blending function that is F_1 of u that will be 0 at u equal to 0, 0 at u equal to 1 and if I plot it here, it takes a shape something like this because these blending functions are quadratic, they are not cubic. F_2 of u at u equal to 0 it is 0, otherwise it is 1 but it is the complement of this. So, F_2 of u that comes out to be like this. F_3, we will get that to be I think something like this and F_4 will be something like this. So these are the blending functions for the tangent vectors. This one is F_3 u, this is F_2 u and this is F_1 u and typically by looking at these blending functions, one can make out a lot about the nature of the curve because you can make out that at the starting point, the weightage is 1 to the tangent to the vector p_0. That means my curve is starting from p_0. Similarly the curve is ending at p_1. And then from the shape of these curves, one can normally make out the nature of the curve. We see that let us say when we go into the base curves, the shape of these blending functions will be totally different and so on. So normally when we are talking of modeling of curves, a lot of stress is given on these blending functions. Any questions on whatever I have covered today?

Just one small thing here. The same p u of u is equal to F u times B. This again we can write that as U times Mu times B where Mu a set of coefficients obtained from this matrix. And similarly if I want to find out the second derivatives at a curvature, I will get p uu will be equal to which will again be equal to U times M uu times B. This will be for the second derivatives or the curvature.
Any question on whatever I have covered today? I will stop here, next time we will see how to specify these cubic curves using 4 (Refer Slide Time: 0:48:07 min) one has to pass. Right now we have specified the curve by specifying the two end points and the tangent vectors, we will get a curve let's say something like this. If instead of this we want to specify 4 points through which the curve should pass like this, through any 4 points you should be able to specify a cubic curve. If you want to do that then how do you get the equation of PC curve? We will see that in the next class.