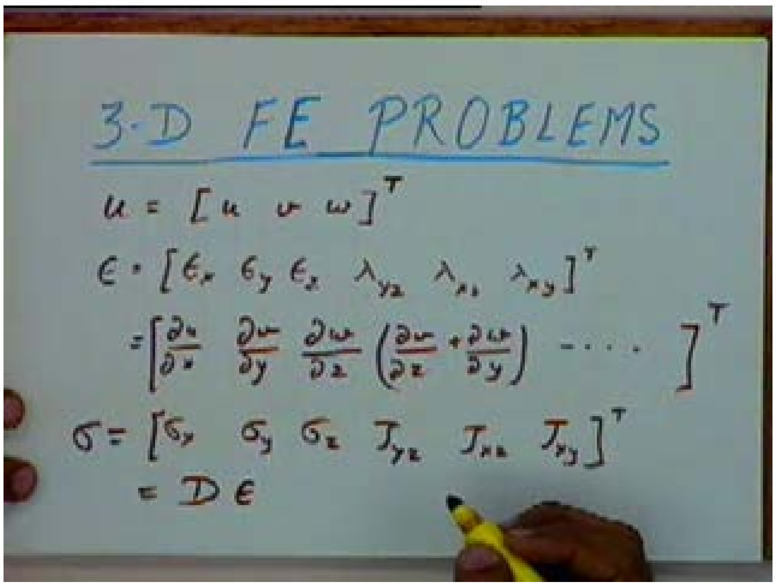
Computer Aided Design
Dr. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi
Lecture No. # 27
3D-FE Problems

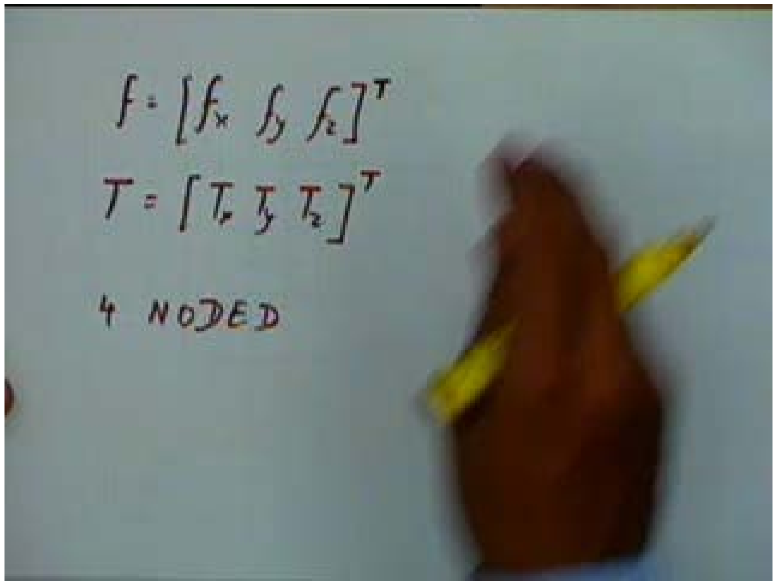
Today we will see how to do the analysis for three dimensional finite element problems and for three dimension problems the deformation will take that to be a vector given by 3 elements so u v and w where u v and w are the deformation in the x y and z direction respectively.

(Refer Slide Time: 00:01:08 min)



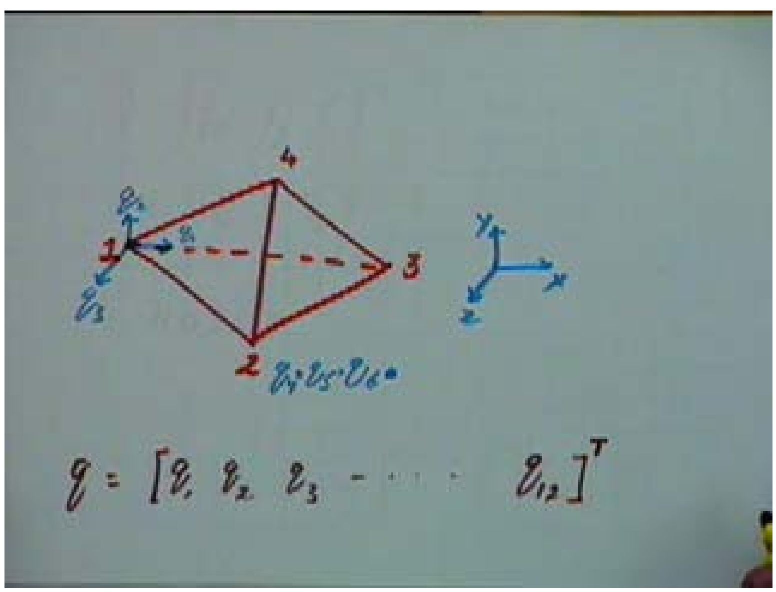
The strain vector will be given by this vector epsilon $_x$ epsilon $_y$ epsilon $_z$ lambda $_{yz}$ lambda $_{xz}$ lambda $_{xy}$ and epsilon $_x$ is nothing but del u by del x, epsilon $_y$ will be del v by del y, epsilon $_z$ will be del w by del z and by lambda $_{yz}$ will be del v by del z plus del w by del y. This will be lambda $_{yz}$ and so on and sigma which are the stresses they will be given by again 6 triple vector that will be sigma $_x$ sigma $_y$ sigma $_z$ tau $_{yz}$ tau $_{xz}$ and tau $_{xy}$ transpose of that and this will be equal to d times epsilon. The same notation as we had for two dimensional problems and the body force vector that is f will represent that as f_x f_y f_z.

(Refer Slide Time: 00:02:28 min)



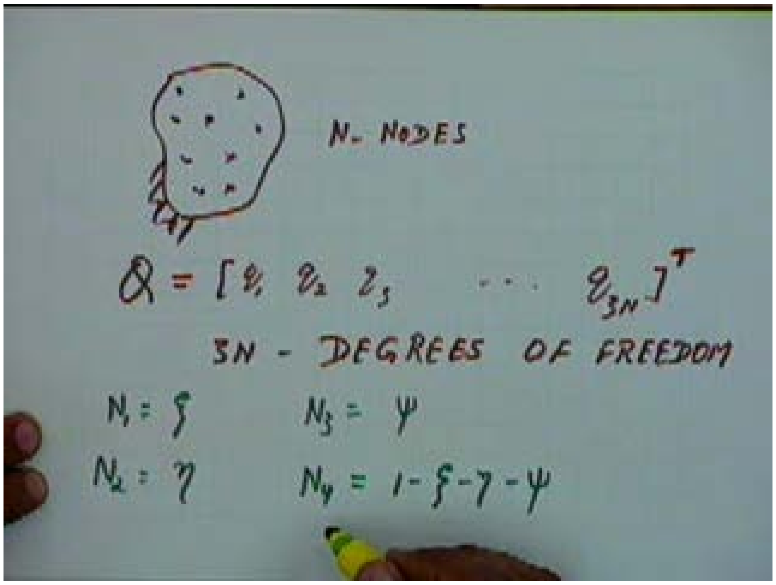
f_x, f_y and f_z are the body forces in the x y and z direction respectively and the tractive forces will be again T_x T_y and T_z which are 3 forces in the x y and z direction respectively. These are basically a pressure which will be acting on the surface of the body and the simplest element that will take will be a 4 noded element, a 4 noded tetrahedral element which will look something like this.

(Refer Slide Time: 00:03:09 min)



So it's a 4 noded element 1 2 3 and 4, the deformation in the 3 directions at each node let's say node 1 are q_1 q_2 and q_3 where these are the x y and z directions. Similarly at each node we have 3 deformations, node 2 will have q_4 q_5 q_6, 3 will have q_7 q_8 q_9 and 4 will have q_{10} q_{11} q_{12}. So the deformation vector q that will be given by 12 elements which will be q_1 q_2 q_3 so on till q_{12}, this will be a deformation vector. And if we have a body with some boundary conditions which have the total of let's say n nodes then the deformation vector, global deformation vector will have 3N elements which will be q_1 q_2 q_3 so on till q_{3N}.

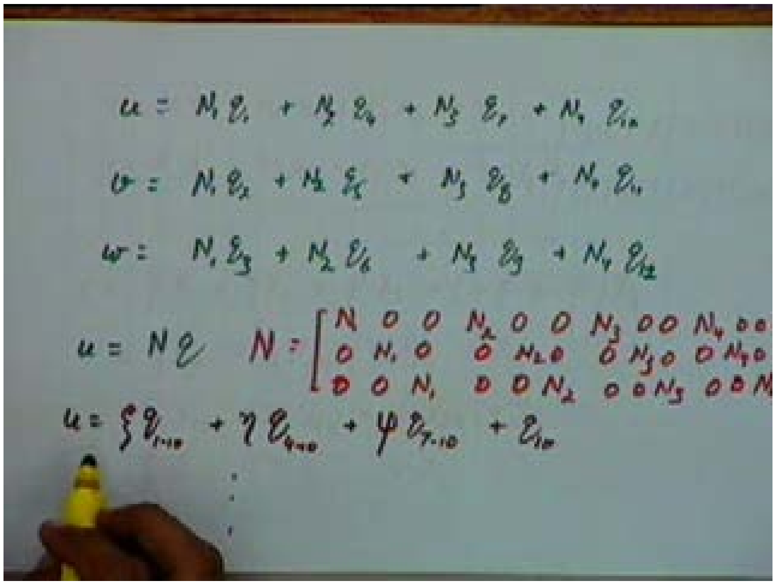
(Refer Slide Time: 00:04:02 min)



So there will be 3N elements, so the total amount degrees of freedom will be 3N which are 3N degrees of freedom and if we module the deformations using same functions, for this element corresponding to the 4 nodes will take 4 different shape functions, corresponding to 1 will take a shape function N_1, for node 2 will take N_2, for node 3 N_3 and so on.

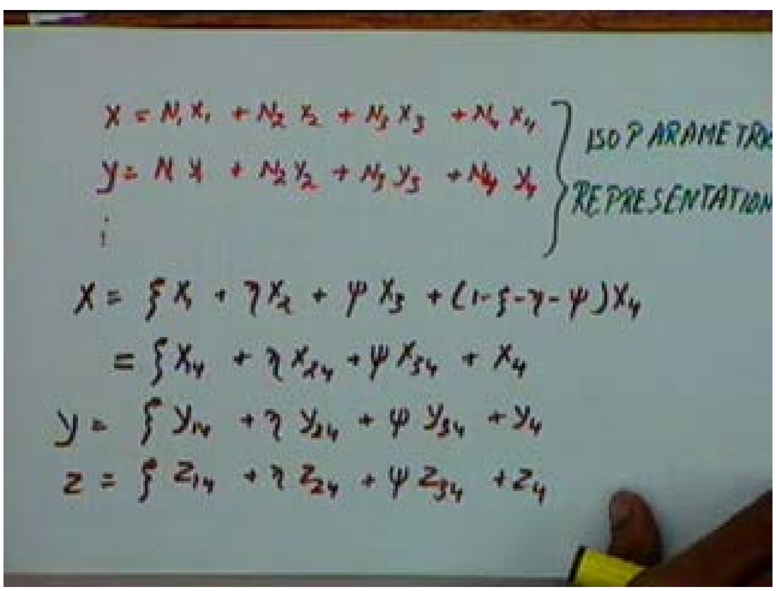
So we will have 4 shape function N_1 N_2 N_3 and N_4 and to start with we will assume let's say N_1 is a parameter let's say zeta, N_2 is a parameter eta, N_3 is psi and N_4 will be 1 minus each of these, so 1 minus zeta minus eta minus psi. This is because N_1 plus N_2 plus N_3 plus N_4 has to be equal to 1 at all points and N_1 will be 1 at node 1, will be 0 at the other 3 nodes, similarly the other 3. There is just an extension of the two dimensional elements that we had, I just extending that to 3 dimensions. And then we will say u will be equal to N_1 q_1 plus N_2 q_4 plus N_3 q_7 plus N_4 q_{10}.

(Refer Slide Time: 00:06:10 min)



The u which is the deformation in the x direction, the deformation in the x direction for node 1 is q_1, for node 2 is q_4, for node 3 will be q_7 and for node 4 will be q_{10}. So if u will be given by this equation and v will be given by N_1 q_2 plus N_2 q_5 plus N_3 q_8 plus N_4 q_{11} and w will be this. And if you write this in a matrix form, we will say u is equal to N times q where the matrix N will be N_1 0 0 N_2 0 0 N_3 0 0 N_4 0 0 0 N_1 0 0 N_2 0 0 N_3 0 and 0 N_4 0 and 0 0 N_1 0 0 N_2 0 0 N_3 0 0 N_4.

(Refer Slide Time: 00:08:12 min)

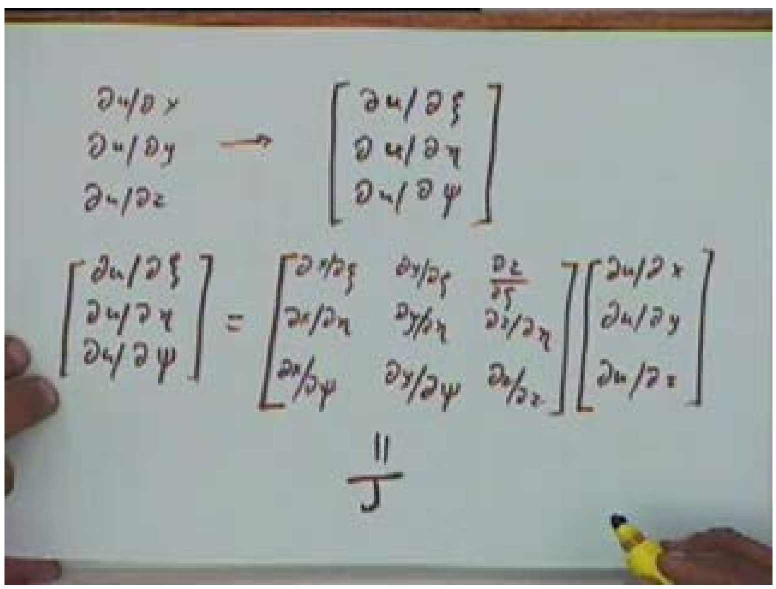


This matrix will be a matrix N and u will be equal to N times q. And again if we assume an iso parametric representation will be saying x will be equal to N_1 x_1 plus N_2 x_2 plus

N_3 x_3 plus N_4 x_4 and y will be equal to N_1 y_1 plus N_2 y_2 plus N_3 y_3 plus N_4 y_4 and so on. Similarly for z and this is assuming an iso parametric representation and in these relationships, if we substitute our expressions for N_1 N_2 N_3 and N_4 in terms of zeta and eta these expressions, this set of equations would become x will be equal to zeta x_1 plus eta x_2 plus psi x_3 plus 1 minus zeta minus eta minus psi into x_4 and this I can say will be zeta into x_{14} plus eta into x_{24} plus psi into x_{34} plus x_4. This is again very similar to what we did in the two dimensional case.

Similarly y will be equal to zeta into y_{14} plus eta into y_{24} plus psi into y_{34} plus y_4 and z will be zeta into z_{14} plus eta into z_{24} plus psi into z_{34} plus z_4. So these will be the relationships for x y and z and if we do the same thing here for u v and w, we will get u will be equal to, N_1 will be replaced by zeta so, we will get zeta q_1 minus q_{10} will get, so it will be $q_{1\ 10}$ plus eta into $q_{4\ 10}$ plus psi into $q_{7\ 10}$ plus q_{10} and so on. There will be an expression in terms of zeta, eta and psi. And now in order to get, we need to get del u by del x, del u by del y and del u by del z.

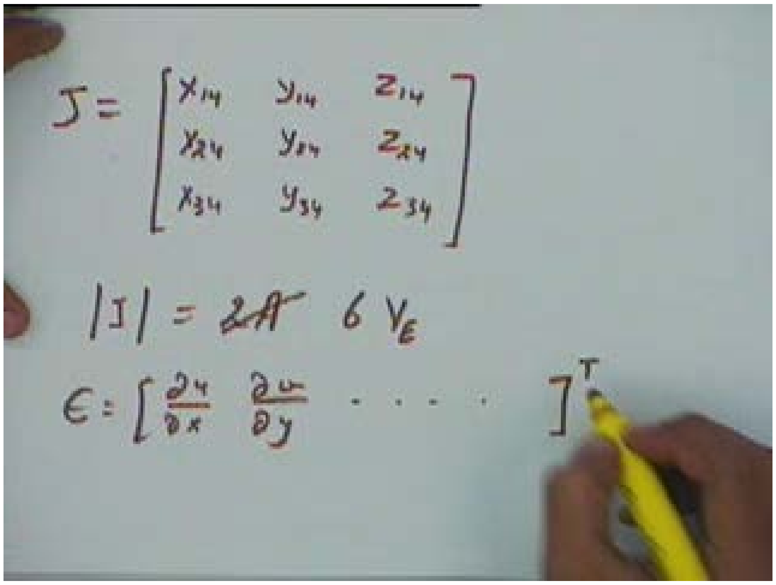
(Refer Slide Time: 00:11:29 min)



We will relate these to del u by del zeta del v by del zeta sorry del u by del eta and del u by del psi and the relationship would be that this vector we write it again del u by del zeta, del u by del eta and del u by del psi. This will be equal to del x by del zeta del y by del zeta del z by del zeta. So it will be del x by del eta and del x by del psi del y by del psi and del z by del psi multiplied by del u by del x del u by del y. This first term will be equal to this row multiplied by this column, del u by del zeta will be del x by del zeta or del x by del zeta just a sec. So del u by del zeta will be del x by del zeta into del u by del x, del y by del zeta into del u by del y plus del z by del zeta into del u by del z that will be del u by del zeta. And similarly del u by del eta and del u by del psi.

In this expression is again there is Jacobian and del x by del zeta, each of these terms we can evaluate from these expressions. del x by del zeta is nothing but x_{14} and del x by del eta is nothing but x_{24} and so on. So this Jacobian, this Jacobian we will get that this would be equal to del x by del zeta is x_{14}, del y by del zeta will be y_{14}, z_{14} and here we will get x_{24} we get y_{24} and z_{24}, this will be x_{34} y_{34} and z_{34}.

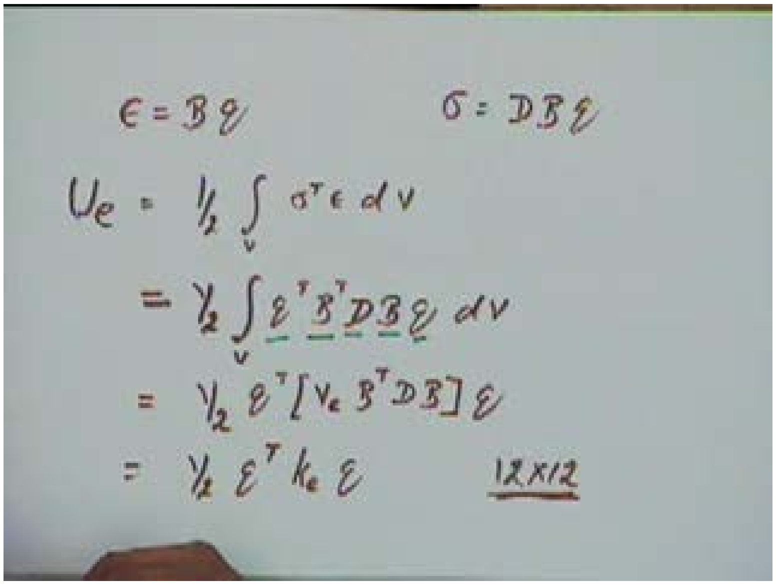
(Refer Slide Time: 00:14:11 min)



So the Jacobian will be this and you should remember in the two dimensional case, we had said that the determinant of the Jacobian there was it will be two times the area. In this case it is the two times area we will get 6 times the volume of the element and again then we can prove it mathematically. The determinant of this Jacobian is 6 times the volume of the element. So again we will use this relationship for finding out the relationship between epsilon and q. So now in this we know the Jabobian, we will know these terms so we can find each of these terms.

Once we can each of these terms we will get similarly, we can get expressions for del v by del x del v by del y and del v by del z and so on. So you can get all these terms and we can compile the terms for epsilon, epsilon is nothing but del u by del x for epsilon x, del v by del y for epsilon y and so on.

(Refer Slide Time: 00:16:02 min)

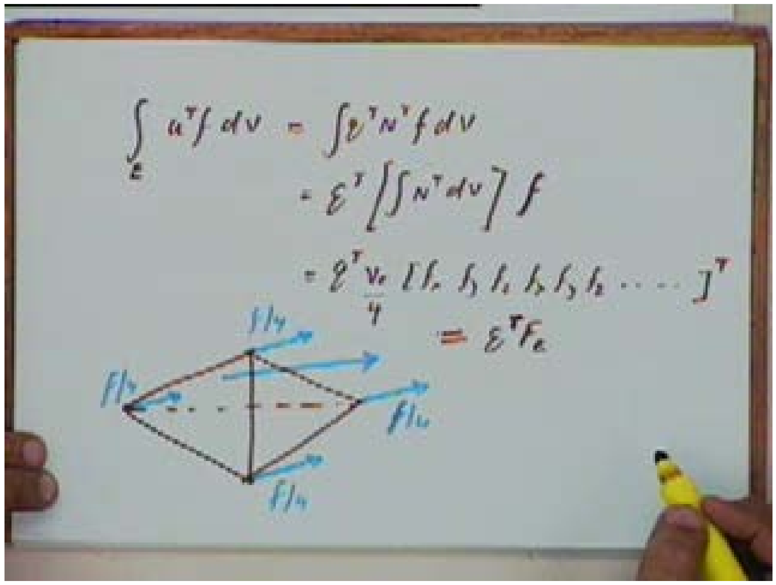


We can compile it just like we did in the two dimensional case and get the relationship for epsilon which is equal to B times q where B will contain elements which have been obtained from this matrix and this matrix. We will take the inverse of the Jacobian put it on this side and so on. So, the B matrix will be slightly a complicate expression but it will have expression only of x_{14} y_{14} z_{14} x_{24} y_{24} z_{24} and so on. And from this side we get expression of q_{12} q_{17} and for the Jacobian these terms will contain term of x_{14} y_{14} z_{14}. Here will contain terms of x_{24} y_{24} and z_{24}, this is x_{34} y_{34} and z_{34}.

The terms here del u by del zeta should be $q_{1\ 10}$, this is $q_{7\ 10}$ and this is sorry this is $q_{4\ 10}$ and this is $q_{7\ 10}$. Basically expressions of this will have a term only of x y and q, it won't have any expression to zeta, zeta eta or eta. So when we find just B matrix, this will be independent of zeta, eta and q, zeta eta and psi. So again the strain within an element will be a constant in this case. And quickly if you see how do find out the strain energy, we know the strain energy is half of the volume integral of sigma transpose epsilon dv. This will be half of the volume integral sigma, sigma is equal to DBq. The sigma transpose will give us q transpose B transpose D D and D as symmetric matrix. So D and D transpose are the same, epsilon as Bq will say B times q multiplied by dv. And again if you look at this q transpose is a constant, B is a constant, all these terms are constants.

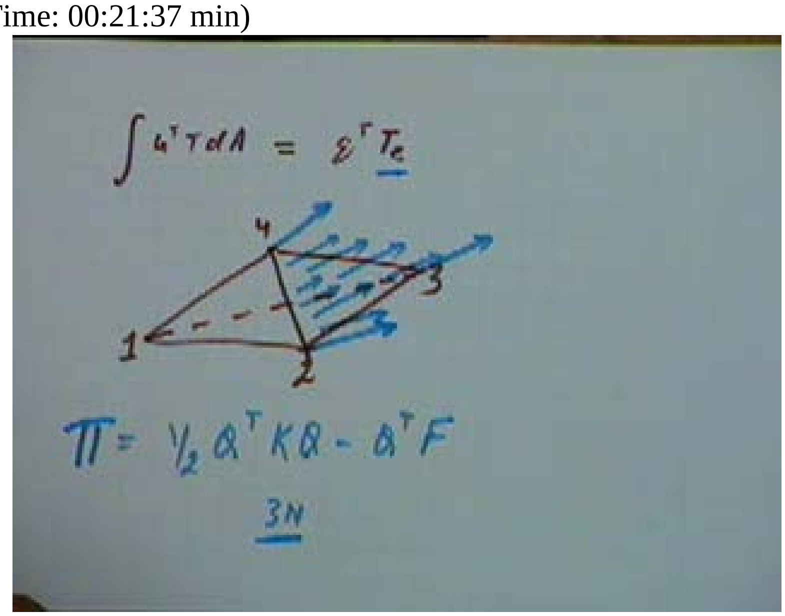
So the integral of this will be equal to half of q transpose into the volume of the element multiplied by B transpose DB times q. And again this will be equal to half of q transpose $k_e q$ where k_e is the element stiffness matrix and k_e is a 12 by 12 matrix because each element has got 12 degrees of freedom. And similarly if we consider the force terms and potentially the contribution due to the body forces, we get that to be integral over the element of u transpose fdv, u is equal to nq so this will be equal to integral of q transpose N transpose into fdv.

(Refer Slide Time: 00:19:18 min)



q transpose and f are going to be constants but N is going to be where is zeta and eta. So we will get this to be q transpose into integral of N transpose dv multiplied by f and if we carry out this integral, we can finally show that this will be equal to q transpose into volume of the element by 4 multiplied by fx fy fz fx fy fz and so on. There are 12 terms of this type, transpose of that. Effectively what that means is if we have a 4 noded element like this which has some body force acting in any arbitrary direction. The contribution of this can be split equally into the 4 nodes, where let's say this is f by 4 f by 4 f by 4, f by 4 where this f is a total body force acting on this element, that we can see from here that v_e into fx is a total body force acting, divided by 4 is what is the acting on the first degree of freedom.

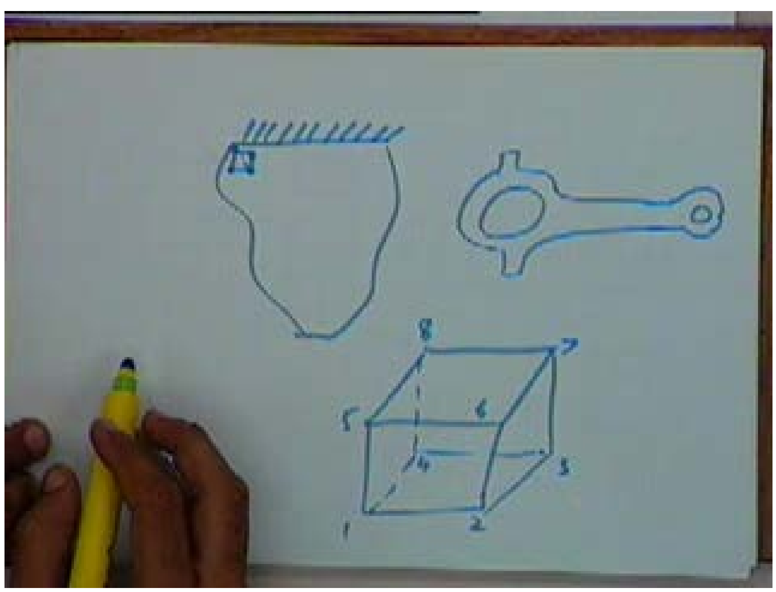
(Refer Slide Time: 00:21:37 min)



v_e into fy is the total body force acting in the y direction, divided by 4 is what is acting on the fourth degree of second degree of freedom and so on. And this expression again we will write that as q transpose into the element body force vector. Similarly for tractive forces we can show that integral of u transpose TdA will be equal to q transpose multiplied by T_e. The T_e will be the contribution on each of the nodes, if we have a 4 noded element like this and we have tractive forces acting on one of the phases. Let's say on this phase, phase 2 3 4, the some pressure acting. Then the total tractive force will split that equally into three components acting on these 3 nodes and we will get this T_e expression. And then again if you come combine the potential energy for the complete body which will be, we will get global equation of the type Q transpose KQ minus Q transpose F. Because these are the element vectors, element tractive force vector, this is the element body force vector and this is the element stiffness matrix, this is the element displacement vector.

If you find the total U in, the total strain energy in the body that will be sigma u_e which will be sigma of half of q transpose k_eq which is equal half of Q transpose kq. And again the process of finding out the global stiffness matrix will remain the same. The only difference will be that now we will have the total number of degrees of freedom will be 3 N where N is the number of nodes, so total number of degree of freedom will be 3 times that. And this vector Q will also have 3 N elements, F will also have 3 N elements. We will get this equation, again a method of solving it will also remain the same. Whatever the boundary conditions be, we will use either the penalty approach or the method of elimination. So this is how we can solve for 3 dimensional problems using the simple 4 noded tetrahedron element. Any question on this? And today there will be one more thing that when we are talking of a three dimensional body, let's take any arbitrary any three dimensional body.

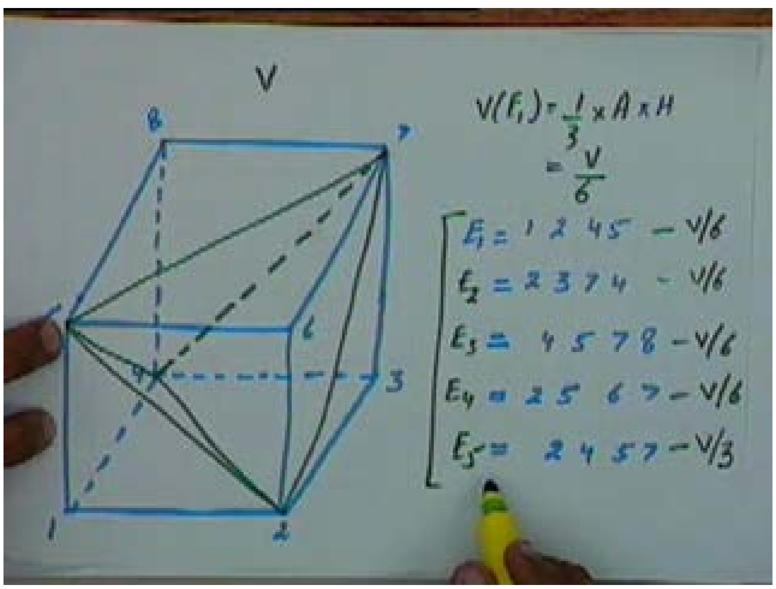
(Refer Slide Time: 00:24:35 min)



That is take any arbitrary shape and lets say it's got some boundary conditions and we want to split this body into a set of 4 noded elements. Then for each of the nodes we will have to give the coordinates that are there. We have to find out the coordinates of each of the node and for general three dimensional body this can become a very complex task. For instance if we talk of let's say a simple rod or if you talk of a connecting rod and we want to split this into a set of 3 dimensional elements and we want to get the coordinates of each of the nodes that is not a very easy task.

So typically what we do is we can split this into a set of cubes and if you have a cube, we will split this cube into a set of tetrahedrons. So first, this body will be split into a set of small cubes. The size of the cube will depend on how fine a mesh we wanted to we want to have. So if we take a set of small cubes in this, for those cubes lets say if we have a cube like this 1 2 3 4 5 6 7 8, this is a 8 noded cube and if you want to use tetrahedron elements, we will split this 8 noded cube into tetrahedrons where you can split a cube into tetrahedrons. Consider this cube, it has got 8 nodes 1 2 3 4 5 6 7 8.

(Refer Slide Time: 00:26:46 min)



We can split this into 4 into 5 tetrahedrons. The first one is 1 2 4 5. So 1 2 4 and 5, so this in this corner we will get one tetrahedron. Similarly 2 3 7 4, 2 3 7 in this is 7, so this is 1 edge and 4 and 7 are connected like this. So 2 3 7 4, that is with 3 as the corner we will get one slice we will cut the corner of, that will give us another tetrahedron, 4 5 7 8 consider 4 5 7 and 8 so if I join 5 and 7, so 4 5 7 and 8 that is with 8 as the corner, we will get the third tetrahedron. And 2 5 6 7 that means with 6 as the corner, we will get another tetrahedron which will be 2 5 7 and 6. So with 4 corners that is 1 3 and 6 and 8 we will get one tetrahedron each. So there are 4 tetrahedrons and the volume that is enclosed inside that is 2 4 5 and 7.

Just imagine in this cube 4 of its corner have been cut off, the remaining volume will consist of nodes 2 4 5 and 7 that will be the fourth tetrahedron. So this way we can split this cube into a set of 5 tetrahedrons and if we consider this cube having lets say total volume of v and if we consider first element 1 2 4 5 that is 1 2 4 and 5, its base triangle is half the base of the cube, height is the same. So what will be the volume of this tetrahedron? The volume of this tetrahedron, volume of element one base area into height into one third. So, 1 by 3 into area of the base into height. Now this will give us the total volume by 6. So volume of this tetrahedron will be one sixth of the total volume of the cube.

Similarly, the second the second element that is 2 3 7 4, 2 3 7 and 4 that will also have a volume of v by 6. So this has the volume of v by 6, this will have the volume of v by 6, 4 5 7 8 that is also symmetrical because it is about the node 8. So this will also have a volume of v by 6 and this will also have a volume of v by 6. So the tetrahedron which is left in the center will be have will have a volume of v by 3, there is a total volume v minus each of these volumes. So we will get the centered tetrahedron with the volume of v by 3 that means that tetra tetrahedron would be twice the size of any other tetrahedrons. So in this case we are getting 5 tetrahedrons, 4 of which have identical volume and the fifth one is having twice the volume. Is that okay? So this is the another way of splitting this cube into 2 tetrahedrons where it's splitted into 6 tetrahedron of equal volume.