Computer Aided Design
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Lecture No. # 26
2D - FE Problems (Contd.)

Today will continue with the 2 D finite element problems and in the last class we have derived this relationship.

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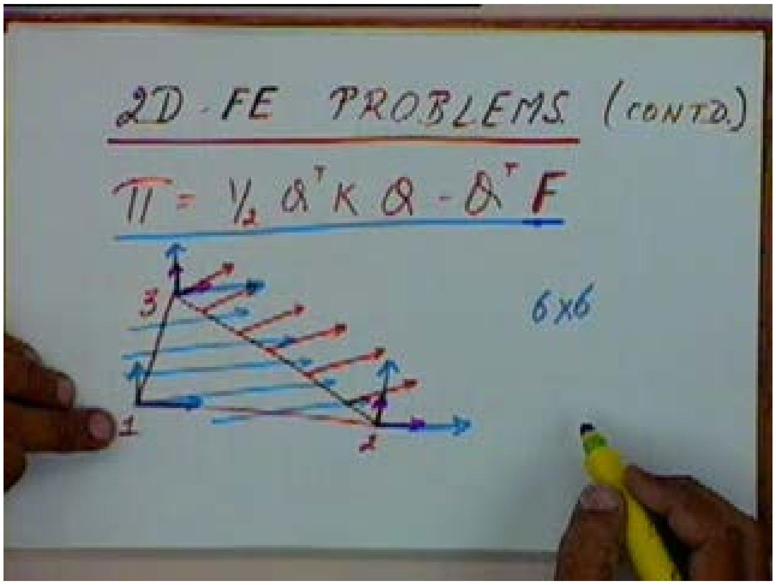

That is the total potential energy is given by half of Q transpose KQ minus Q transpose F, when we are using 3 noded constant strain triangular elements. And for deriving this we had said that if we have an element 1 2 3 and there are some tractive forces acting at the edge 2 3 then the total tractive force acting on this edge can be split equally between the nodes 2 and 3. We get the total tractive force acting in the x direction will be split equally between these two that is between the nodes 2 and 3.

And similarly the total tractive force in the y direction will also be split equally between the nodes 2 and 3 and similarly that body forces that are there, they will also be split equally between the 3 nodes. So, we will have some body forces acting effectively at these 3 nodes. That is the total body force acting will be split equally between the 3 nodes. And this way we can compute the global force matrix by including the forces on each of these 3 nodes plus all the point nodes at each node that way we can compute the total or the global force matrix.

And the stiffness matrix that can be computed by finding the local stiffness matrices for each of the elements which will be 6 cross 6 matrices and then we can compute a global stiffness matrix. Let's take a small example of how we can compute the matrices in this case. The process is going to be absolutely similar to the process for one dimensional element.

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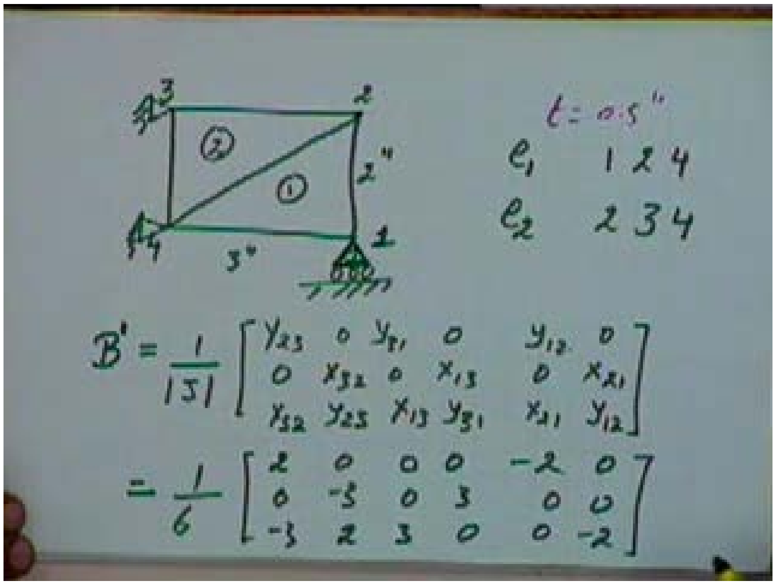

So if you take two elements like this that means this is node 1 2 let's say I call this 3 and this is 4 and let's say I call this as element number 1 and this as element number 2. I just have a plate with 2 triangular elements, it's a very simple situation and let's say I consider that and here and there is some load acting at let's say at node 2. So this is my node number 1. So my element 1 is defined between nodes 1 2 and 4 and element 2 is defined between nodes 2 3 and 4 or I can take any sequence 2 3 4 or 3 4 2 that doesn't make any difference. If I consider first element and I write the B matrix for the first element that is B for the first element that will be equal to 1 by the determinant of the Jacobian multiplied by y_{23} 0 y_{31} 0 y_{12} 0 0 x_{32} 0 x_{13} 0 x_{21} and x_{32} y_{23} x_{13} y_{31} x_{21} and y_{12}, this is the B matrix for the first element. And I mentioned earlier that the determinant of a Jacobian will be 2 times the area of the element.

So let's say if I take the sides, let's say this is 2 inches and this is 3 inches and thickness of, if I say the thickness t is equal to 0.5 inches then the determinant of the Jacobian will be 6 and these values y_{23}, I can get these values like this y_{23}, the y_2 minus y_3. For this node y_2 node number 2 is node 2, node number 3 is node 4. So y_2 minus y_3 will be 2 and similarly y_1 minus y_2 that is y_1 minus y_2 will be minus 2 and here I will get 0 minus 3 0 3 0 0 and then here I will get minus 3 2 3 0 0 minus 2, this will be the B matrix for the first element.

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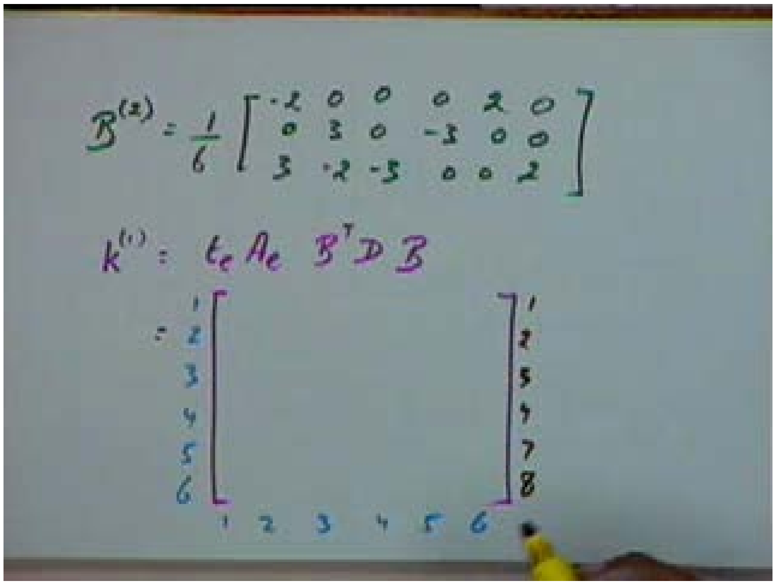

And similarly for the second element would come out to be B for the second element would be 1 by 6 multiplied by minus 2 0 0 0 2 0 0 3 0 minus 3 0 0 and 3 minus 2 minus 3 0 0 2. And once I have the B matrices I can combine, I can get the stiffness matrices by saying that let's say the stiffness matrix for the first element that should be given by thickness multiplied by area multiplied by B transpose DB where B transpose will be a transpose of this matrix that will give me a 6 by 3 matrix, D will be a 3 by 3 matrix and D is again a 3 by 6 matrix. So this whole thing is going to give me a 6 by 6 matrix. And so this should be a 6 by 6 matrix and these are the local node numbers.

And if I look up my element 1 that is defined between nodes 1 2 and 4, so the nodes are, global node numbers will be 1 2 and 4. So this way in the global matrix these locations will occupy the locations 1 2 3 4 and 7 8. That is because corresponding to node number 1, the global locations will be 1 and 2, corresponding to node number 2 the global locations would be 3 and 4 and corresponding to node number 4 the global locations will be 7 and 8. So when I make my global matrix, my global locations will be 1 2 3 4 7 and 8 that is because I have numbered them as 1 2 and 4.

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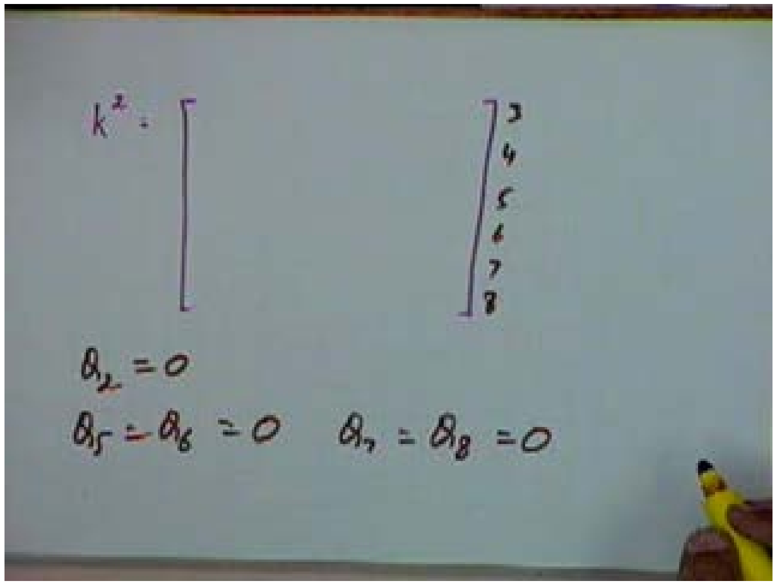

And similarly when I consider my second element that is the stiffness matrix for second element that will again be a 6 by 6 matrix. And that is the second element is defined between locations 2 3 and 4, so the global locations occupied will be 3 4 5 6 and 7 8. So this will go to the global locations 3 4 5 6 7 and 8. And if I look at the boundary conditions in this case I have taken nodes 4 and 3 to be fixed and node 1 is fixed in the y direction. Now corresponding to that my boundary conditions will be Q_2 will be equal to 0 because this is constant in the y direction, it can move freely in the x direction. Similarly if I take node 3 that will give me Q_5 will be equal to Q_6 will be equal to 0 and corresponding to the node 4, we will get Q_7 equal to Q_8 equal to 0.

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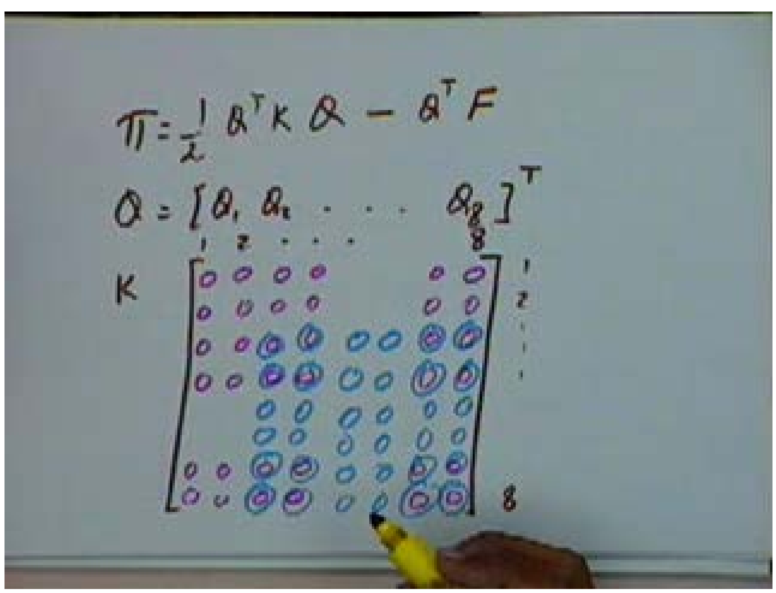

And in my potential energy expression pi, this term Q is going to contain 8 terms and K will be an 8 by 8 matrix. So this term K be an 8 by 8 matrix and in this 8 by 8 matrix, my first element, the first element will occupy these locations 1 2 3 4 and 7 and 8. So 1 2 3 4 and then 7 and 8. Similarly in this direction 1 2 3 4 and 7 and 8. So these locations will be occupied by elements of the first stiffness matrix and the second stiffness matrix as 3 4 5 6 7 8 locations will be occupied by the second stiffness matrix. So that will take a locations so 3 4 5 6 7 8 4 5 6 7 8 and all these.

So these are the locations of the second stiffness matrix, these are the locations of the first stiffness matrix and this way we will get the global stiffness matrix. And as I mentioned these are the boundary conditions we have. So to complete this potentially expression, this minus Q transpose F, F will again contain 6 8 terms. And if I use the method of elimination, my Q_2 Q_5 Q_6 Q_7 and Q_8 all 5 are constraint. So by the method of elimination the equation arise that I finally get, they are of the form K prime Q prime equal to F prime.

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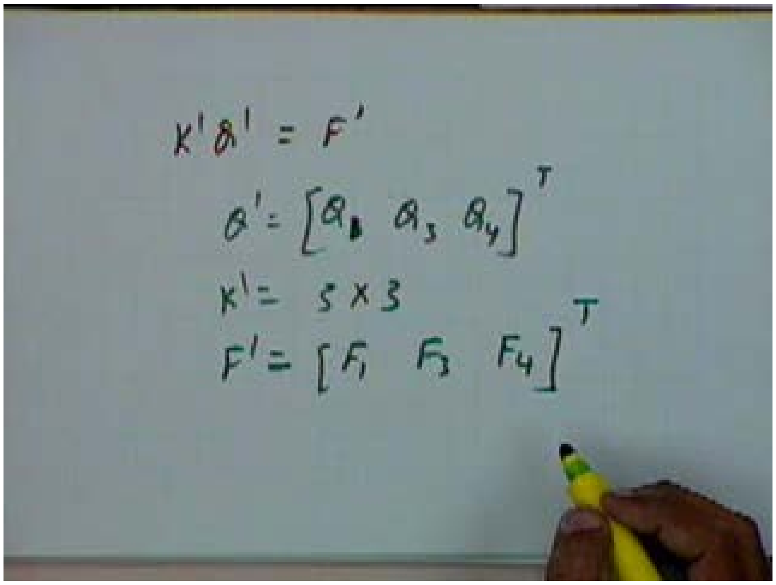

And K prime is obtained from K by deleting locations of the constraint nodes. So in this case 2 5 6 7 8 are constraint. So that means 2 5 6 7 and 8 all these locations, all these columns will get deleted and similarly 2 5 6 7 and 8 all these columns will also get deleted, all these rows will get deleted, so 2 5 6 7 8 6 7 and 8. All these will get deleted and eventually we will get this 3 by 3 by matrix as the K prime matrix and Q prime will have the other 3 nodes which will be Q prime will be equal to Q_3 sorry Q_1 Q_3 and Q_4 transpose that will be Q prime and K prime will be a 3 by 3 matrix.

And similarly F prime will also contain only 3 elements which will be F_1 F_3 and F_4 and this set of equations can be solved to find out the values of the Q's. This will give us 3 equations 3 variables, we can solve them to get Q_1 Q_3 and Q_4.

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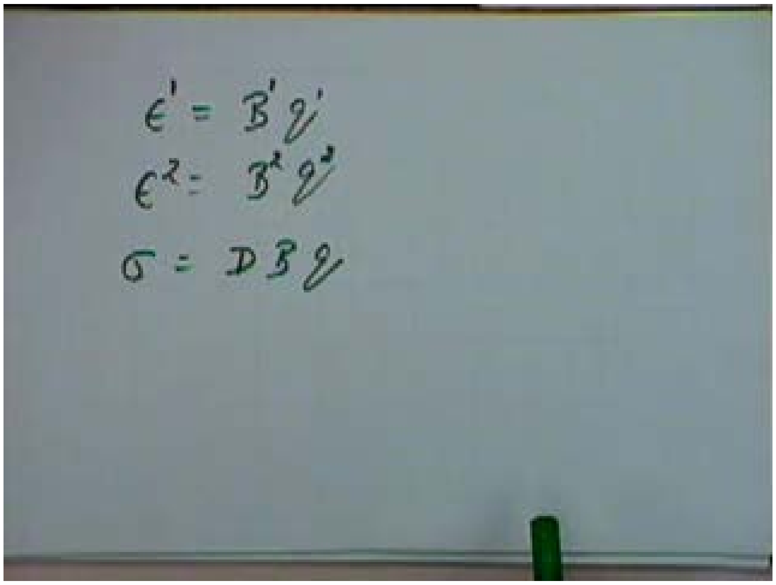

And once we have that then we in order to compute the strains, we say epsilon which is the strain is equal to B times Q for each element. So for the first element it will look like this and similarly for the second element we will get where Q_1 is the displacement matrix for the first element, Q_2 is second element and so on. And for getting the stresses sigma we will get it, we will say it will be equal to D times Bq and we can get stress, the strain and stresses in both the elements. So the only thing that changes in the cases of 2 D elements is how this global matrix is combined because now we will have twice the number of twice as many locations in the stiffness matrix as there are the number of nodes.

If you have 4 nodes, there will be 8 locations that is the only difference. Now this is as far as 3 noded triangular elements are concerned and as far as the potential energy approach is concerned. If you quickly look at how to you use the Galerkin's approach for the same element that means if you have an element like, a triangular element like this and we have said that its displacement matrix is given by Q_1 Q_2 Q_3 till Q_6 that is for this element.

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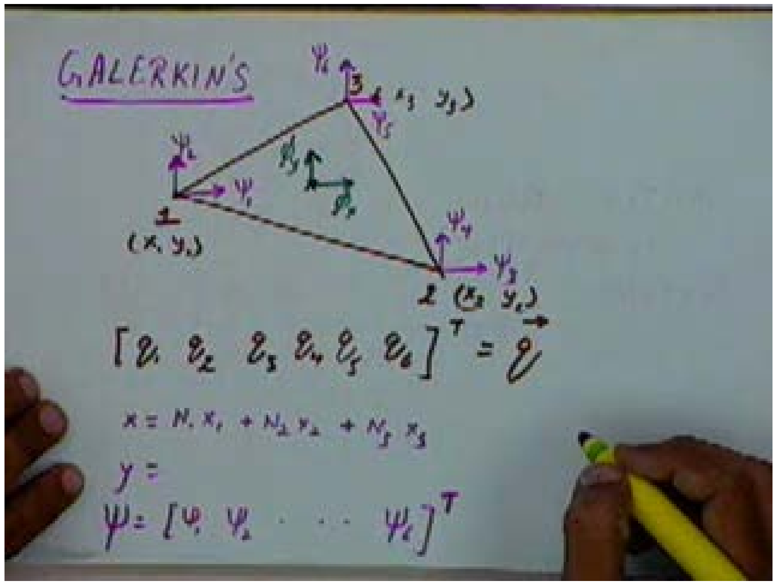

And I said the locations are given by x_1 y_1, x_2 y_2 and x_3 y_3. We have already mentioned that we will say x will be equal to N_1 x_1 plus N_2 x_2 plus N_3 x_3 and so on. And similarly y will be N_1 y_1 plus N_2 y_2 and so on that we have already seen. Now when you are using the Galerkin's approach, we have to give a set of virtual displacement to this element. And let's say the virtual displacements are given by psi_1, psi_2, psi_3, psi_4, psi_5 and psi_6 and we will define a virtual displacement vector which will be given by let's say psi which is psi_1, psi_2 till psi_6 transpose. This is a virtual displacement vector for this element that gives a virtual displacement of the 6 degrees of freedom.

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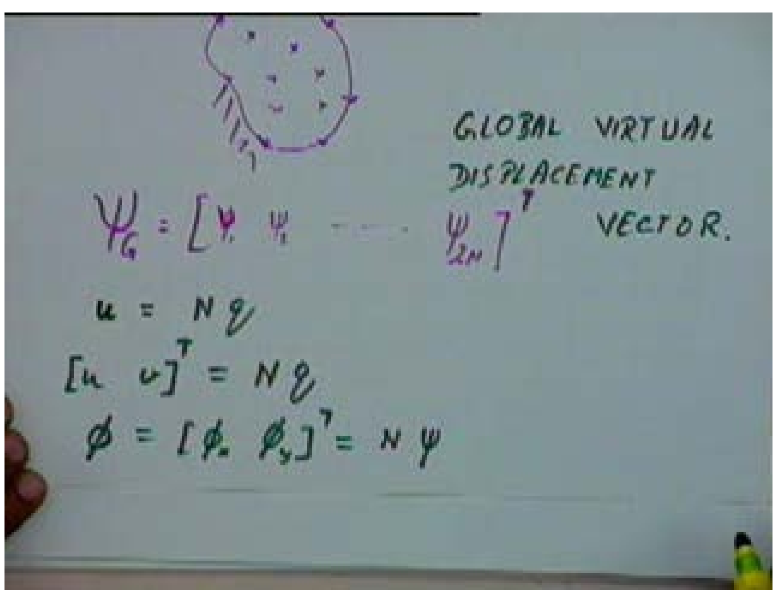

And if we have a general 2 D element like this consisting of a number of nodes, we will define a global vector which will be let's say psi global which will consist of psi_1 psi_2 till psi_{2N} where N is the number of nodes. This will be the global; this will be the global virtual displacement vector. Now coming back to this element, if you will take any arbitrary point here and we want to find out the virtual displacements at this point let's say the virtual displacement at this point is given by phi, phi $_x$ and phi $_y$. So the virtual, if you look at the strain, the strain at that point or let's say if you look at the u vector u first, we say u is equal to N times q, u is nothing but uv transpose and we said this is equal to N times q. Similarly if we look at the virtual displacement that is phi which is nothing but phi $_x$ phi $_y$ transpose and we will say that this will be equal to N times psi.

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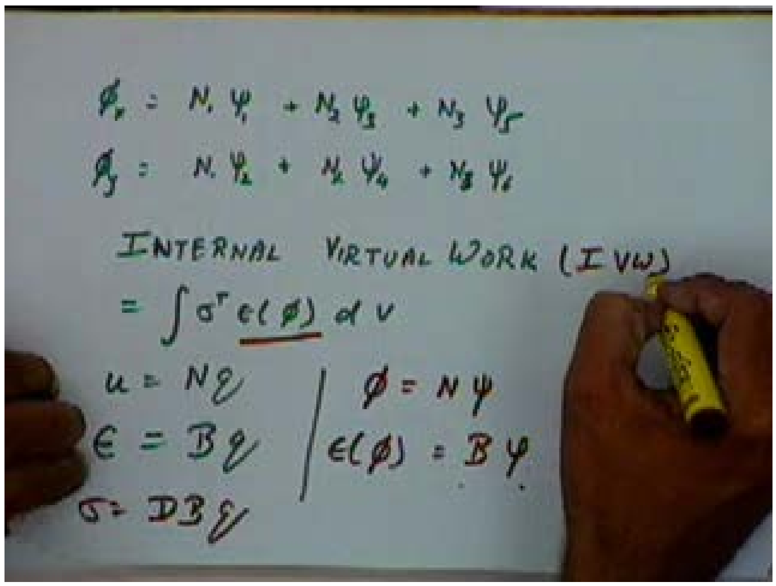

What that means is we will say phi $_x$ is equal to N_1 psi_1 plus N_2 psi_3 plus N_3 psi_5 and phi $_y$ will be equal to N_1 psi_2 plus N_2 psi_4 plus N_3 psi_6. This is similar to expression that we had for that this expression, u equal to Nq the expression we had for deformations. These are virtual deformations, so the expression will be going to be the same. And if you will look at the expression for the internal internal virtual work done, internal virtual work that is equal to the integral of sigma transpose multiplied by epsilon of phi multiplied by dv. Here epsilon of phi is the strain that would be caused due to the virtual displacement. So we have said if u is equal to Nq then we can differentiate u and get an expression for epsilon which is given by epsilon is equal to Bq.

Similarly if I contrast this with my relationship of phi, phi is equal to N times psi. So epsilon of phi will be equal to B times psi. The process, the steps involved in getting this matrix B would be the same as what we did yesterday for getting this matrix B. So epsilon of phi will be equal to B times psi. So if you look at this expression now epsilon of phi will be B times psi and sigma. If I say sigma will be equal to DBq and the sigma transpose will be q transpose B transpose into D. So I can put those expressions over here and the internal virtual work done, the internal virtual work I can get an expression for that which will be which is integral of sigma transpose epsilon phi dv is nothing but thickness of the element multiplied by dA.

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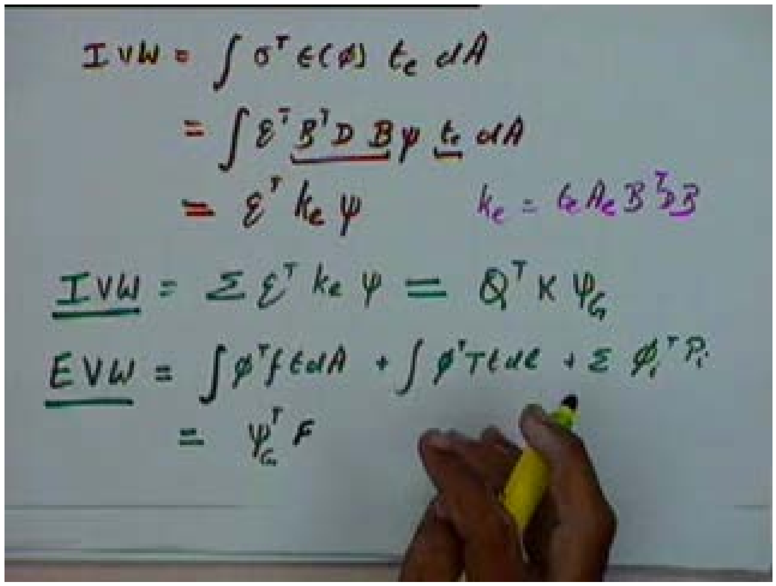

So this will be equal to the integral of q transpose B transpose D B psi into thickness into dA and again qBD B and psi and t_e all these are constants. So this will be equal to integral of dA will given me the area of the element. So finally I will get this to be equal to q transpose multiplied by k_e multiplied by psi where k_e will be these three terms multiplied by t_e into the area of the element. This is same as what we had when we are using the potential energy approach, k_e will be equal to t_e into A_e multiplied by B transpose DB. So this is the expression for the internal virtual work done for an element. If we consider the internal virtual work done by the complete body that would be equal to sigma q transpose k_e psi and that will again assemble as global matrices and get Q transpose k psi global.

Again the process of assembling the global matrices is the same as what we had earlier. And similarly if we look at the external virtual work, external virtual work that we said earlier or we have shown earlier has integral of phi transpose ftdA plus integral of phi transpose t into the thickness dl plus sigma phi_i transpose P_i and again we had shown that this will be equal to psi transpose global multiplied by F. The process will again be the same, we will put expression for phi we will put expression for f carry out the integral and will assemble these force matrices. So the Galerkin's approach will gives us that internal virtual work done is equal to the external virtual work done.

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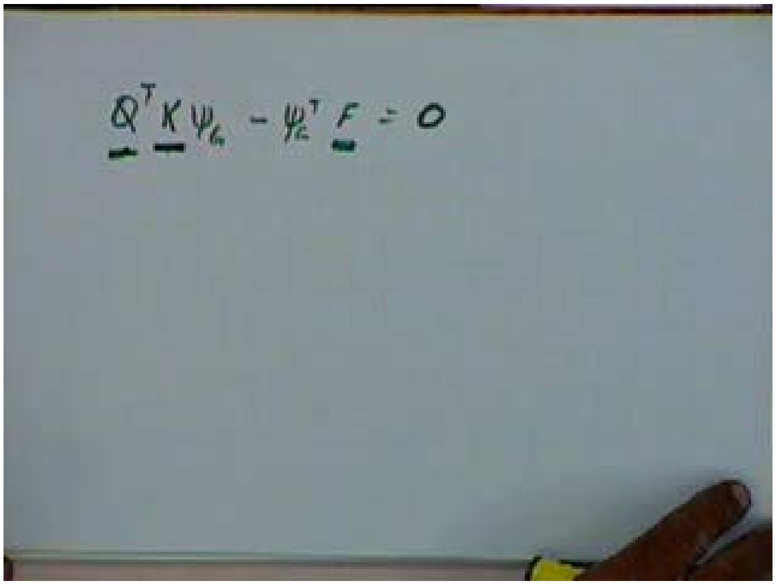

So this will effectively give us an equation of the form Q transpose K psi global minus psi global transpose times F is equal to 0. This is the equation we will get by taking this and this together. The expressions for the stiffness matrix and for the force matrix and for Q will remain the same. And again when we look at this, we will say that this should be valid for all values of virtual, virtual work given virtual displacement. Since it has to valid for all values for the virtual displacements, we will finally get the equation will be similar Q transpose K minus F, Q transpose K will be equal to F or KQ will be equal to F. And again we will use the same boundary conditions and solve these systems of equations.

So eventually whether we use the Galerkin's approach or the potential energy approach we will finally comedown to the same expression for the stiffness matrix and force matrices and the displacement matrices. Again the final equations that we will get they will be the same and the method of solving will also remain the same. Any questions up to this point? And the next thing that we will take up in two dimensional problems is the case of 4 noded elements.

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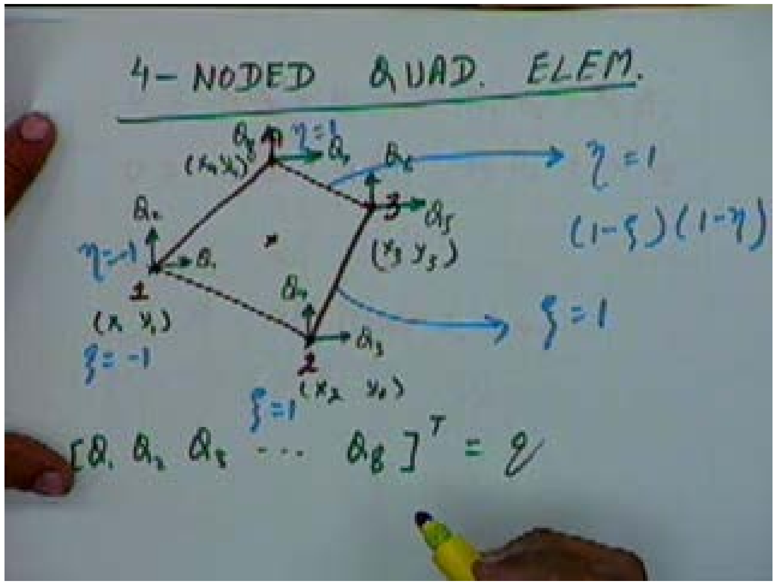

If you have 4 noded quadrilateral elements that means our element looks like this. So we have 1 2 3 and 4, so we have 4 nodes and deformations of these 4 nodes. So then the Q_1 to Q_8 will be the 8 deformations at these nodes and the deformation vector for this element will be given by this 8 tuple. And look x_1 y_1, the coordinates are x_1 y_1, x_2 y_2, x_3 y_3 and x_4 y_4. Now again just like the elements earlier, we have to give 4 straight functions in order to get the deformations at any arbitrary point.

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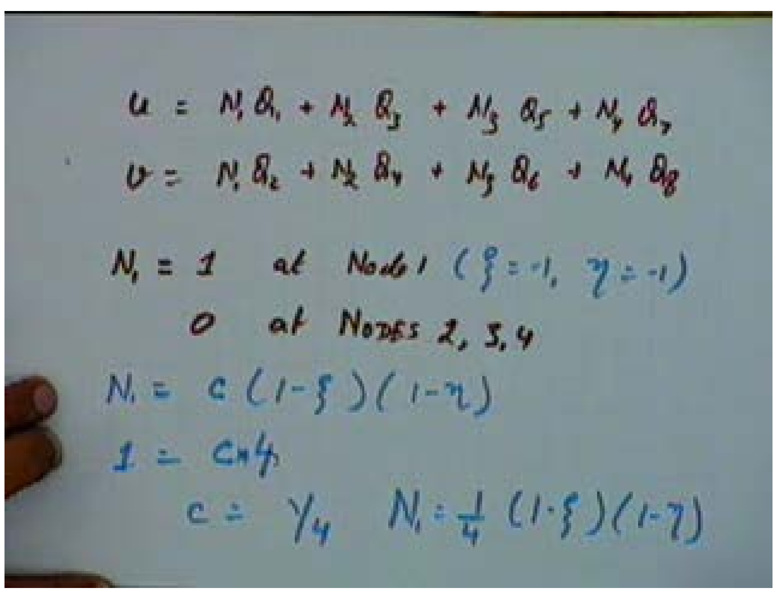

So if we want to get the deformation in the x direction, we will say that the deformation in the x direction u will be equal to N_1 Q_1 plus N_2 Q_3 plus N_3 Q_5 plus N_4 Q_7. And v will be equal to N_1 Q_2 plus N_2 Q_4 plus N_3 Q_6 plus N_4 Q_8 where N_1 N_2 N_3 N_4 are the 4 shape functions. So, if we consider N_1, we want N_1 to be equal to, we want N_1 to be equal to 1 at node 1 and it should be 0 at all the other 3 nodes. Similarly N_2 should be 1 at node 2, it should be 0 at all the other 3 nodes and so on. And so we will say N_1 is equal to 1 at node 1 and 0 at nodes 2 3 and 4, similarly for the other nodes. In order to formulate these shape functions, what we will do is we will define 2 parameters zeta and eta. And we will say that, let's say along this direction from 1 to 2 zeta let's say at this point zeta will be equal to minus 1 and here zeta will be equal to plus 1.

Similarly in this direction, we will say eta will be equal to minus 1 and here will say eta will be equal to plus 1. So along this edge eta is minus 1 and along this edge eta is plus 1 and similarly along this edge zeta is minus 1 and along this edge zeta is plus 1. Now this is in some sense a local coordinate system that we have defined. And if we look at this local coordinate system then N_1 has to be 1 at node 1 that means at zeta and eta values of minus 1 and minus 1, N_1 should be 1 and it should be 0 at all the other 3 nodes. If I consider this line and this line, this line has an equation given by zeta equal to 1 and this line has an equation given by eta equal to 1.

So a term of the form 1 minus zeta into 1 minus eta will be 0 along this edge as well as along this edge that means it will be 0 here, here as well as here. So if we say N_1 will be some constant multiplied by 1 minus zeta into 1 minus eta. Then N_1 has to be 0 at all these 3 nodes, it has to be 0 along this edge as well as along this edge. And to get this value of C we will again use the fact that at node 1, N_1 has to be equal to 1. If N_1 has to be 1 at this node that means at this node zeta is equal to minus 1 and eta is equal to minus 1. So we will put these values over here, we will get 1 will be equal to C multiplied by 2 into 2 which is 4. So we will get C to be equal to 1 by 4 or we will get N_1 to be equal to one fourth of 1 minus zeta multiplied by 1 minus eta. This is the shape function we will choose for N_1. Is that okay?

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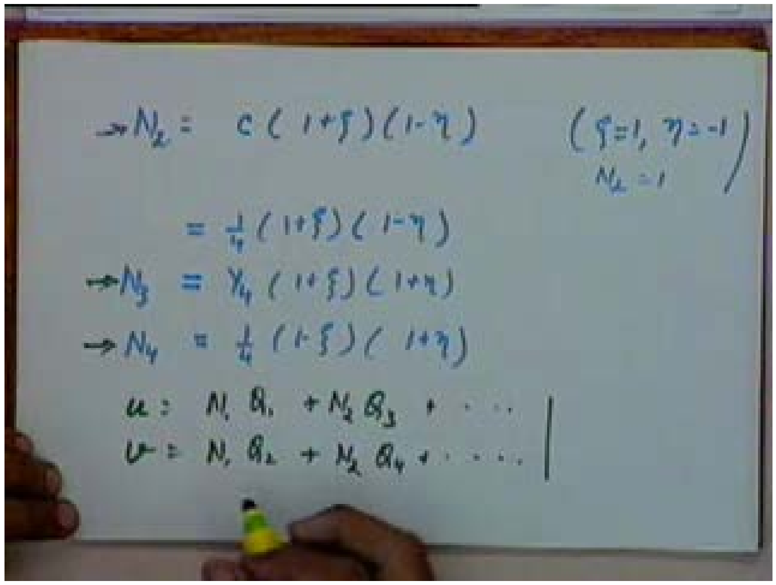

Similarly if we take N_2 and N_2 has to be 1 at node 1 and has to be 0 at all these 3 nodes. So along this edge as well as along this edge, it has to be 0 and the equation of this edge is zeta equal to minus 1. So for N_2 we will say it will be of the form some constant multiplied by 1 plus zeta into 1 minus eta and at zeta equal to 1 and eta equal to minus 1, N_2 is equal to 1 we will use this condition now. And we will get N_2 to be equal to one fourth of 1 plus zeta multiplied by 1 minus eta and similarly we can get N_3 and N_4. N_3 will be one fourth of 1 plus zeta into 1 plus eta and N_4 will be one fourth of 1 minus zeta into 1 plus eta.

We will get the 4 shape functions like this. And once we have the 4 shape functions, we can then write expression for u as N_1 Q_1 plus N_2 Q_3 and so on and similarly v will be equal to N_1 Q_2 plus N_2 Q_4 plus and so on, so where N_1 N_2 N_3 and N_4 are given by these expression and similarly for N_1 or we can write this in a matrix form as u equal to Nq where N will be N_1 0 N_2 0 N_3 0 N_4 0 and 0 N_1 0 N_2 and q is equal to Q_1 Q_2 till Q_8.

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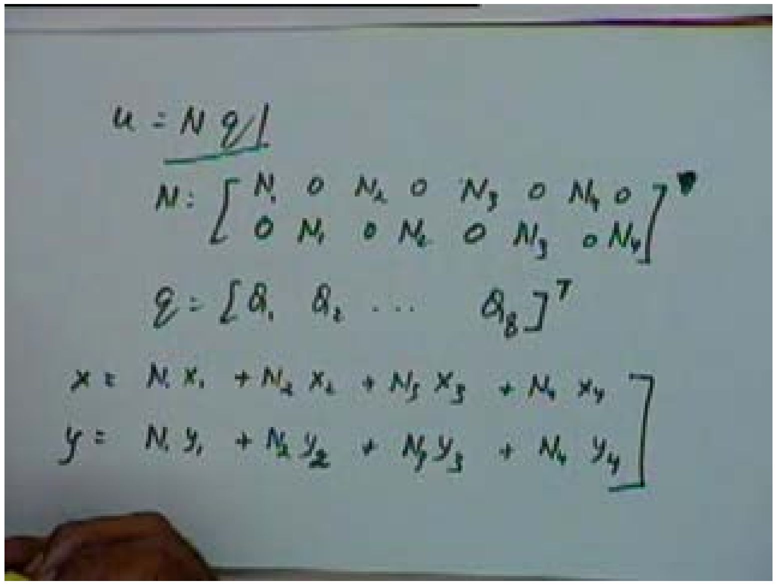

And again we will choose a isoparametric representation, we will say x will be equal to N_1 x_1 plus N_2 x_2 plus N_3 x_3 plus N_4 x_4 and y will be equal to N_1 y_1 plus N_2 y_2 plus N_3 y_3 plus N_4 y where N_1 N_2 N_3 N_4 are the 4 shape functions in terms of zeta and eta and x and y are given by these relations, u and v are given by this relation. So this way we have to formulate the basic equations for relating the deformations and the position vectors.

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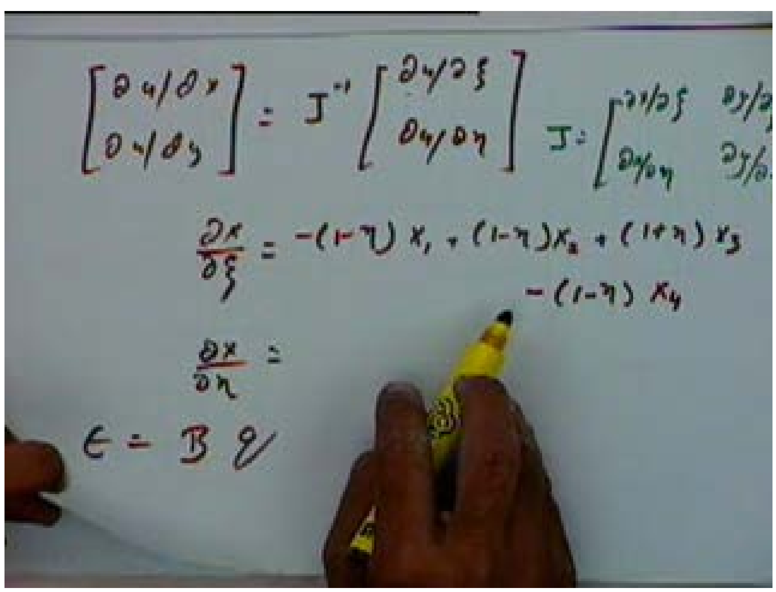

And now again to get the strains that is the epsilons, we need the expressions for del u by del x and del u by del y and this we derived last time that this is equal to Jacobian inverse multiplied by del u by del zeta and del u by del x I am sorry del u by del eta where the Jacobian is J, the Jacobian is given by del x by del zeta, del y by del zeta, del x by del eta and del y by del eta. And in order to find expressions for del u by del x, I now need to find out del u by del zeta, del u by del eta and these four terms.

If I try to get del x by del zeta, that is this first term I will have to take this equation and differentiate it with respect to zeta and all N_1 N_2 N_3 N_4 we have terms of zeta in it. So del x by del zeta will come out to be minus of 1 minus eta times x_1 plus 1 minus eta times x_2 plus 1 plus eta times x_3 minus 1 minus eta times x_4. This will be the expression for del x by del zeta.

Similarly del x by del eta will also be an expression like this and so on. And after getting all these expressions here, I can finally come up with an expression for epsilon which is equal to B times q but now B, this matrix B will contain expressions of zeta and eta. B is not going to be a constant matrix, so B will contain expressions of zeta as well as eta.

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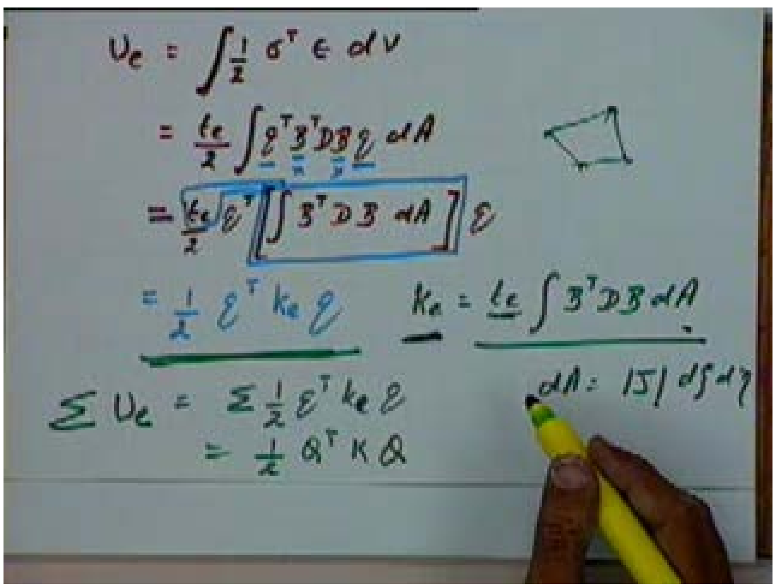

And if I use this expression and try to get the expression for the strain energy that is u_e which is given by integral of half sigma transpose epsilon dv. This will be equal to t_e by 2 t_e into dA will be dv and sigma transpose will give me q transpose B transpose D, epsilon is B times q so B times q. And again in this expression, q transpose is a constant, q is a constant but B transpose and D they are not constants, they are functions of zeta and eta.

So in order to evaluate this integral, this will become t_e by 2 into q transpose into integral of B transpose D B dA, this whole thing multiplied by q. So let's say I will take t_e inside and what we will do is we call this matrix, this thing multiplied by t_e as a stiffness matrix.

So this will be equal to half of q transpose k_e multiplied by q where k_e will be equal to t_e sorry t_e into the integral of B transpose D B dA. And for finding out this stiffness matrix, one has to carry out this integral because B consists of terms of zeta and eta and dA, a differential area which will also consists of terms of zeta and eta. In fact we can show that dA will be equal to mod J multiplied by d zeta d eta, this determinant of J will again have terms of zeta and eta. So this integral will contain terms of zeta and eta and this stiffness matrix can only be computed by carrying out this integral using numerical techniques.

So for finding out the stiffness matrix in the case of a 4 noded element, one has to carryout this integral using numerical techniques and only then we can get the values of the stiffness matrix but the final equation that we get will remain the same. We have got this stiffness matrix for the element and then when we consider the total strain energy in the body that will be equal to sigma of half q transpose k_e q which will again be equal to half of q transpose kQ. So final equation will remain the same but for finding out the stiffness matrix, one will have to use a process of integration. Without using this, without carrying out this integral, one cannot find out the stiffness matrix.

So unlike a 3 noded elements and all the one dimensional elements, in this case for finding out the stiffness matrix we have to carry out this integration. This integral is going to be so quite complex because we will have the term of Jacobian which also contains zeta and eta and B and B transpose they also contain terms of zeta and eta. So, whole thing going to be should be quite complex. Any questions up to this point? The programmer is supposed to do it, whichever finite element program you are using, the movement we use a 4 noded quadrilateral element it will solve this integral to get the stiffness matrices, it cannot be solved by hand.

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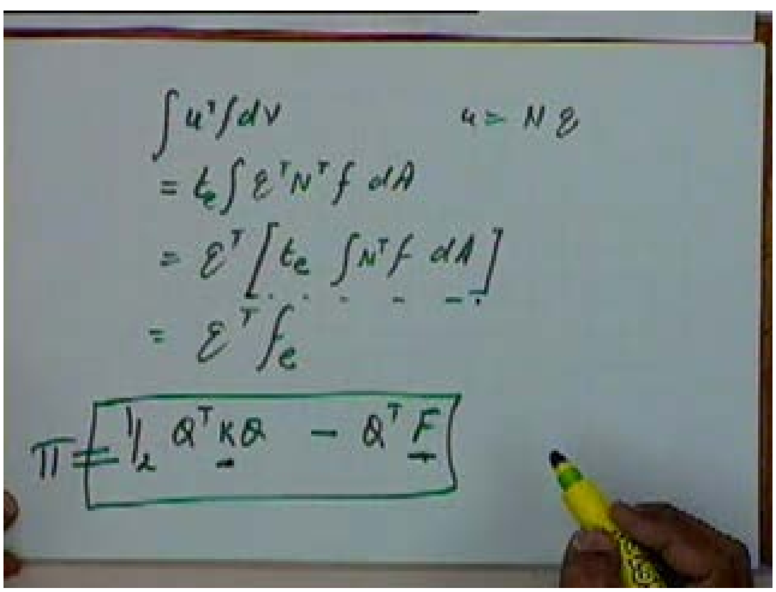

And similarly when we combine the force terms, the body force terms that is the potential energy contribution due to the body forces, u transpose is N cube. So we have u equal to

N cube, the u transpose will be q transpose N and q transpose into N transpose. So this will be equal to integral of q transpose N transpose f into the thickness into dA, q transpose is of course a constant but N transpose is not a constant that has terms of zeta and eta in it. So this is equal to q transpose into t_e into integral of N transpose fdA. So this is equal to we will say q transpose into f_e where f_e is the body force vector for the element, element body force vector but again fe is computed after computing this, carrying out this integral. Unless you can carry unless we carry out this integral we can compute f_e.

So in the case of a four noded element we have to compute this integral to find out the body forces, the effective body force acting at the four nodes. Similarly, the tractive force that will also have to be split into 4 nodes by carrying out an integral. Eventually we will combine all these to get a equation of the form of half of Q transpose KQ minus Q transpose F. The process of finding of this F and this K will involve numerical techniques. We will have to solve complex integrals. Any questions up to this point?

Essentially this equation remains the same. The process of solving this, this is the expression for pi. This equation is, this expression remain the same and the process of solving it will also remain the same and what changes is your straight functions and the process of computing k and F. We will see as the elements become more and more complex, finding out these k and F matrices, the process of finding that out also becomes more and more complex. Any questions up to this point? And that way we will stop here and the next time we will see some more elements and then we will go onto three dimensional problems.