Computer Aided Design

Dr. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi

Lecture No. # 24

2D-FE Problems

Today we will be starting our discussion on 2 D finite element problems but before we go into that couple of things about one dimensional problems.

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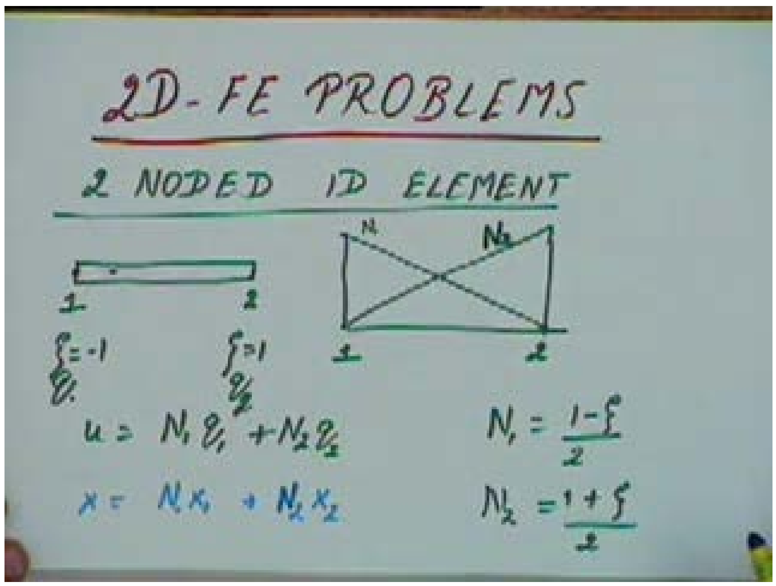

If we consider the two noded elements, in a 2 noded 1 dimensional element which is defined like this, the two nodes and at this end we have said zeta is equal to minus 1 and here zeta is equal to 1. And the shape functions we have taken linear shape functions which had a shape something like this, this is N_1 and N_2 is something like this. This is node 1 and this is node 2 and we said that u is equal to N_1 q_1 plus N_2 q_2 where u is the deformation at any arbitrary point and q_1 and q_2 are the deformations at the two ends. Now for N_1 we said it will be a linear relationship, N_1 is 1 here and 0 at the other end. So this expression was 1 minus zeta by 2 and N_2 was 1 plus zeta by 2. If you notice in this case I can also write X to be equal to N_1 X_1 plus N_2 X_2.

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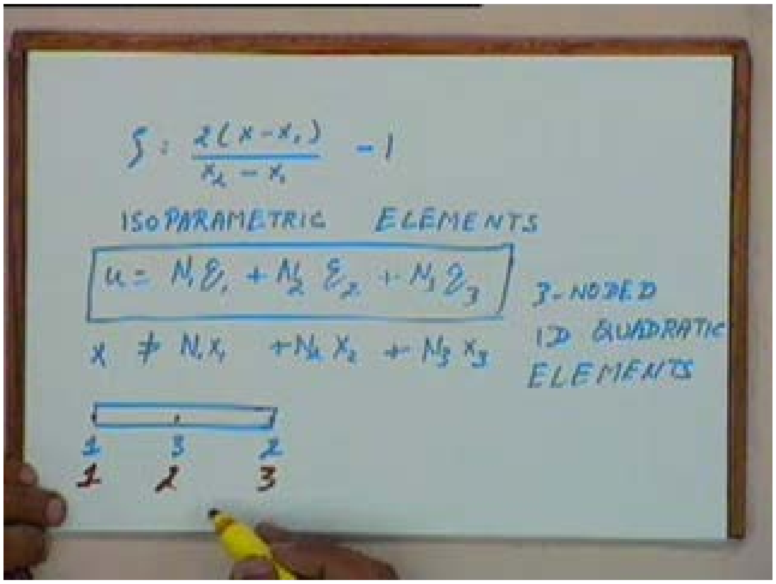

Just to verify this, we have said zeta is equal to 2 times X minus X_1 divided by X_2 minus X_1 minus 1. This is how we have defined zeta and N_1 and N_2 have been defined like this, N_1 and this is N_2. If we put the value of N_1, N_2 and zeta back into this equation, you can verify that X will be equal to N_1 X_1 plus N_2 X_2 and that should be clear otherwise also because X is a linear combination of X_1 and X_2. We have defined zeta to vary linearly between node 1 and node 2. So X will be equal to N_1 X_1 plus N_2 X_2 and whenever we have a situation of the type that u is given by an expression like this and X is also given by a similar expression. Such elements are referred to as isoparametric elements, isoparametric formulation of the problem.

These are isoparametric elements for an isoparametric formulation that means the location X and the deformation u are given by similar expressions. Such elements are called isoparametric elements. In fact we will notice that when you go to 2 D problems, 2 D or higher dimension problems we will prefer to use isoparametric elements in a number of situations that simplifies the formulation to a certain extent. We will be seeing in two dimensional problems how that happens. If you notice the quadratic elements where zeta is also defined in a similar manner and where we said u is equal to N_1 q_1 plus N_2 q_2 plus N_3 q_3. This is not an isoparametric formulation because in this case we cannot say, x will not be equal to N_1 X_1 plus N_2 X_2 plus N_3 X_3 that should be quite obvious where N_1, N_2 and N_3 are going to be quadratic and zeta will be defined in a linear manner, zeta will be varying linearly along the length of the element.

So this relationship will not be true for 3 noded 1 D elements, 1 D quadratic elements. So these elements are not isoparametric and the only other thing is that yesterday we are talking of three noded quadratic element, we had done numbering like 1 2 and 3 and in this case even if we do the numbering in any other way, if we do the numbering like this or the other way around eventually it will not make a difference. The only thing that might, that will change is how the global matrix is to be formed, one has to be consistent from the beginning to the end. The numbering, the local numbering the way it is done, the numbering will not make a difference.

One just has to be consistent from the beginning to the end. And with this we will now go onto two dimensional problems.

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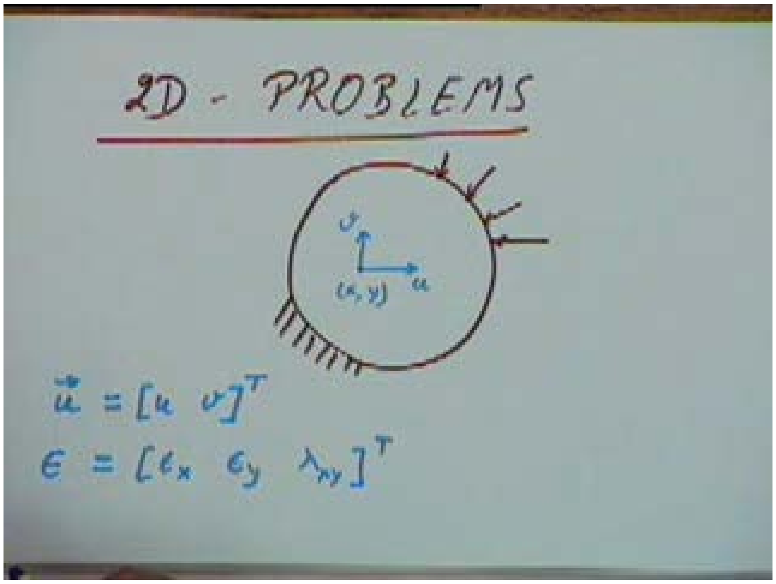

And in two dimensional problems, if you have any object like this which is having some boundary conditions and some external forces acting on it. What we will do is we will take any arbitrary point here, the location of this point is let's say xy and we will say that the deformation would be u in the x direction and v in the y direction and then we will say that the vector u will be the vector u v transpose, this is the vector of the two deformations. This u and this u are different, this is a vector u and this is a scalar quantity which is a deformation in the x direction. We might not put this arrow every time but by context it should be clear. Then the strain epsilon will be given by epsilon x epsilon y and lambda xy, there will be three terms and this will be the strain vector.

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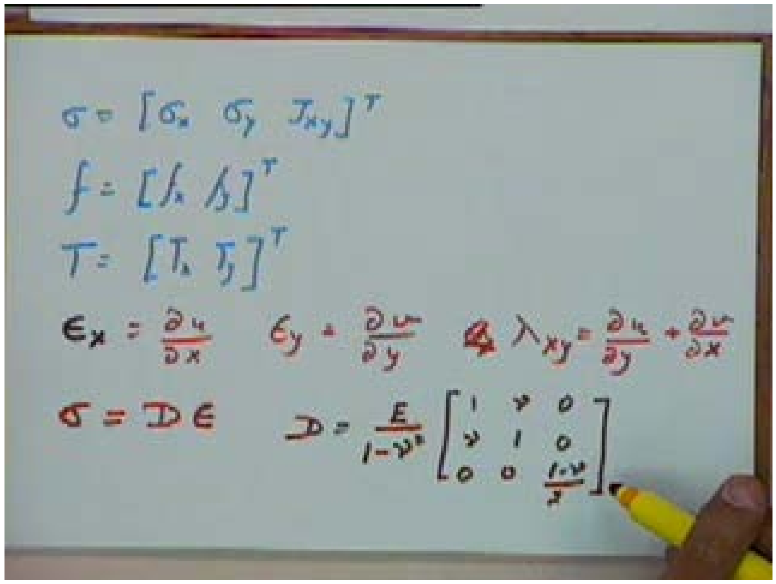

Similarly the stress, the stresses would be given by sigma will be sigma $_x$ sigma $_y$ and tau xy. As far the forces, the body force is acting on the body that is volumetric forces will be fx fy transpose that means there can be a body force may be gravity or may be magnetic force or some such force acting throughout the body or opposite in volume basis. We will use this notation for that and then for tractive forces they will have components of T_x and T_y transpose. Then the next thing will be epsilon $_x$.

We will say epsilon $_x$ will be equal to del u by del x, epsilon $_y$ will be equal to del v by del y and epsilon $_z$ sorry here we have two dimensional case that won't be there, lambda xy will be equal to del u by del y plus del v by del x and sigma that is the stress vector will be given by D time's epsilon where epsilon is the strain vector and D would be given by where nu is the Poisson's ratio. So if we know epsilon $_x$, epsilon $_y$ and lambda $_{xy}$, we can compute sigma by this relationship and this relationship has been derived directly from basics of solid mechanics. We have Poisson's ratio which will contribute to the stresses.

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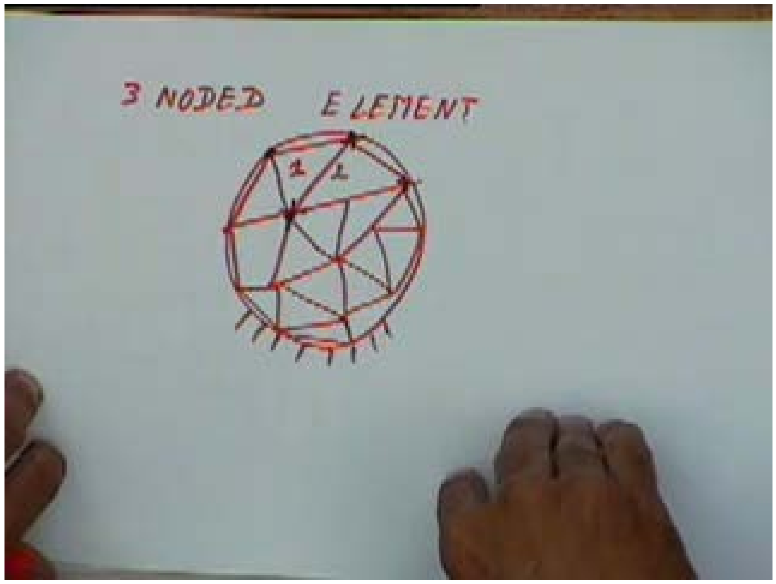

Now with this background or with this notation, we will be solving the two dimensional problems. The first type of element that we will take is what we call as a 3 noded element. If we have a body like this with some boundary conditions, we divide this body into a set of triangles so those triangles might look something like this. In each of these elements, let’s say this is the element number 1, this is 2, each of these elements have 3 nodes that is one node here, here and here. This is element number 1, element 2 will have three nodes that is this, this and this and so on. So we just consider one element.

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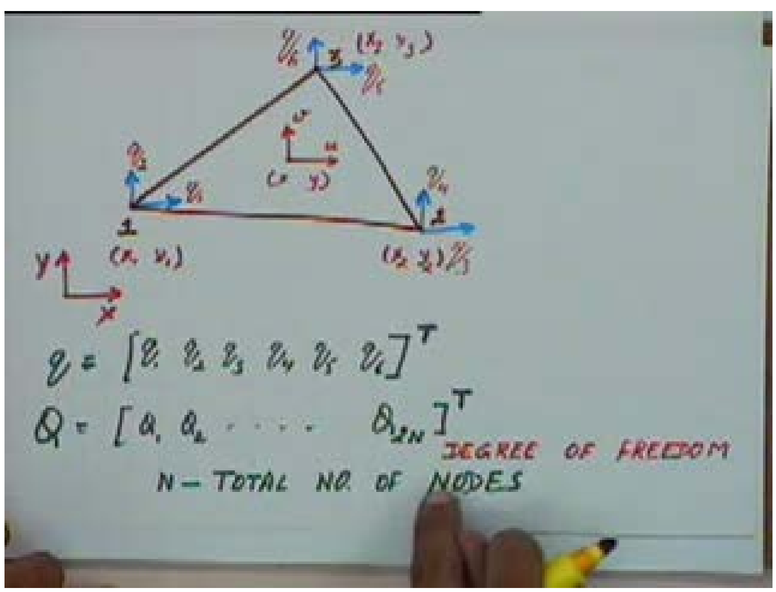

There are three nodes of the element and for these three nodes; we will define deformation in the two directions as these two vectors. Let's say we have some xy system as the coordinate system the origin, this is x direction and this I am saying the y direction. Now this is the deformation in the x direction maybe we will call that q_1, this maybe we will call that q_2. Similarly this may be it will be q_3, this will be q_4, this is q_5 and this q_6. So we have three nodes and each node we have deformation both in the x and the y direction. The x coordinates will be given by x_1 y_1, x_2 y_2 and x_3 y_3. If we consider any arbitrary point here with coordinates of xy that will have a deformation in the two directions given by u and v. So now we have got three nodes on this element 1, 2 and 3.

The deformations, if we write down the deformation vector, the deformation vector would look like q_1, q_2, q_4, q_5 and q_6. This is the deformation vector, the q_1 to q_6. Since we have three nodes, this vector we want to have 6 elements in it. Earlier when we are talking of the one dimensional case and we have two nodes, the element deformation vector had only two elements. In a 3 noded element, we had only 3 elements in q. Now in a three noded element, we will have 6 elements. It's a column vector. And later on when we talk of the global vector, we will define a global vector Q which will be like this which will consist of let's say Q_1, Q_2 till Q_{2N} where N the total number of nodes where N is the total number of nodes and the global deformation vector will be given by this vector. It will have 2 N elements, if N is the number of nodes.

Therefore individual location of that won't correspond to a particular node, that will only correspond to deformation in one direction of that node. And each individual location will normally be referring to that by the term called degree of freedom. So we say that this vector has got 2 N degrees of freedom. The term degrees of freedom is basically used because if I consider complete formulation that means a figure like this.

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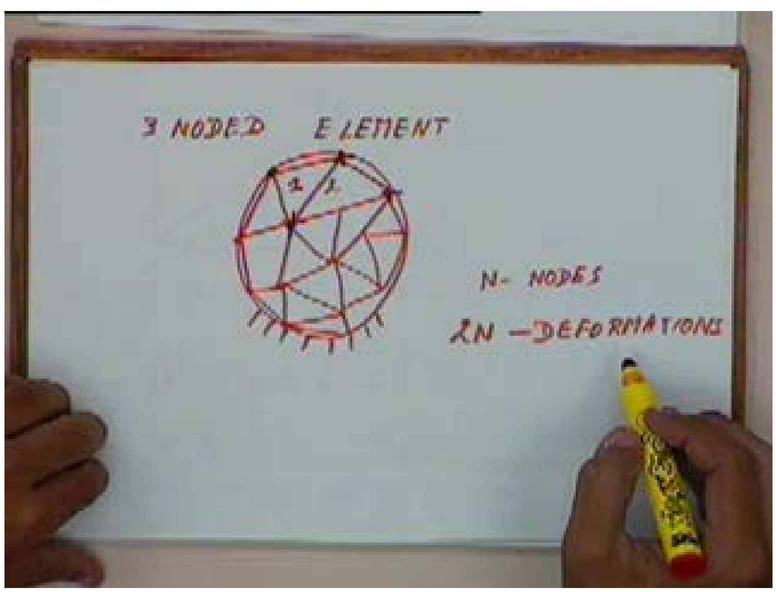

And it has N nodes then the two N deformations which have to be specified, the 2 N deformations. So we will say that this has got 2 N degrees of freedom. So this is the element deformation vector. This is the global deformation vector.

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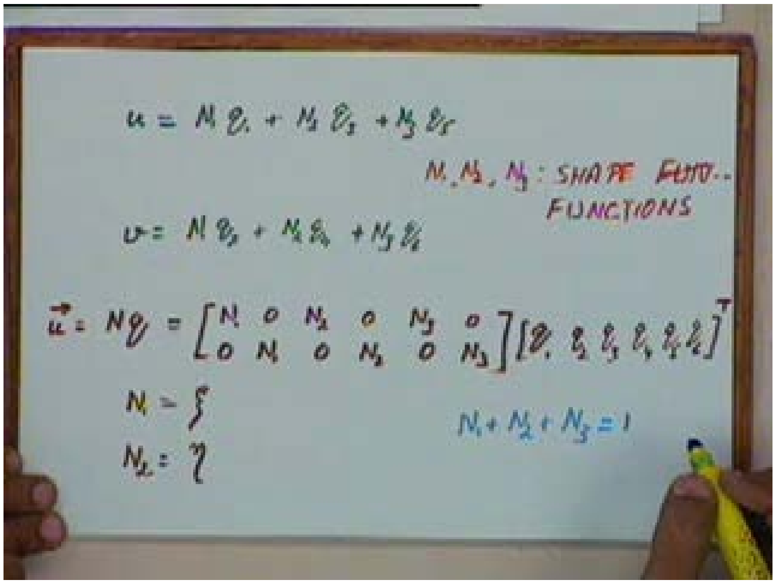

Now if we consider u which is a deformation at any arbitrary point, the x direction deformation at any arbitrary point xy. Now this deformation u will depend on q_1, q_3 and q_5, we will say it will be a combination of q_1, q_3 and q_5. This q_2, q_4 and q_6 are deformation in the y direction and q_1, q_3 and q_5 are the deformation in the x direction. So the deformation u will depend on q_1, q_3 and q_5 and we will take it to be a combination of the 3. So we will say that u will be equal to N_1 q_1 plus N_2 q_3 plus N_3 q_5 where N_1 N_2 N_3 are the shape functions.

Similarly deformation in the y direction will be given by N_1 q_2 plus N_2 q_4 plus N_3 q_6 because q_2, q_4 and q_6 are the deformations of the 3 nodes in the y direction. And if we write this down in a vector notation, we will say that the vector u will be equal to N times q which is equal to N_1 0 N_2 0 N_3 0 0 N_1 0 N_2 and 0 N_3. This multiplied by the vector q. The vector q is a column vector of q_1 to q_6. I will just write it as q_1. So the first row multiplied by this column will give us this equation and the second row multiplied by this column will give us this equation and let's say if we put N_1 equal to zeta and we put N_2 to be equal to eta.

We can put these expressions over here, the third constraint is that N_1 plus N_2 plus N_3 has to be equal to 1 at all locations that is because if I consider this element it's a it's a it's a combination of the three. It need not be linear, even if it is not linear it's a combination of the three. That means the deformation at any point is a combination of deformation here, deformation here and a deformation here. So total weightage of the total weightage we are giving to any of the three has to be 1 at any point, has to be 1 at all the points. That means the weightage given to this point plus the weightage given to this point plus the weightage given to this point has to be equal to 1 throughout. So N_1 plus N_2 plus N_3 will be equal to 1 and therefore we will get N_3, we will say N_3 will be equal to 1 minus zeta minus eta.

So we will get u to be given by this expression where N_1 is zeta, N_2 is eta and N_3 is 1 minus zeta minus eta. Now right now what we have done is that we have given an expression for u as a function of N_1, N_2 and N_3 but we haven't yet specified as to which point we are talking of. We haven't yet related N_1, N_2 and N_3 with the x values. That means if I am taking any particular point over here, I am trying to find out the deformation at this point. This relationship gives me the relation between u and q as a function of N_1, N_2 and N_3. But I am yet to relate N_1, N_2 and N_3 with the xy values. The xy values are the location of this point x and y, I have to specify N_1, N_2 and N_3 as a function of x and y.

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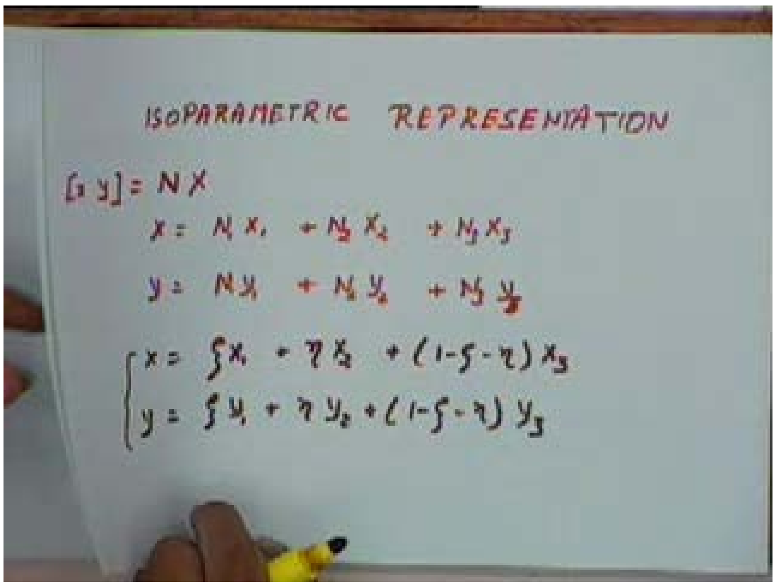

So relating these two, what we will do is that we will assume or we will take what I have what I just explained earlier isoparametric elements or an isoparametric representation. And if you assume an isoparametric representation, we have said u is given by this expression, so x also has to be given by the same expression. So xy will also be equal to N times let's say x or we can say x will be equal to N_1 X_1 plus N_2 X_2 plus N_3 X_3 and y will be equal to N_1 y_1 plus N_2 y_2 plus N_3 y_3. Yeah. We are assuming the representation given, that's the way we are defining the element right now. So x is given by this relationship, y is given this relationship and if you put these values N_1 is zeta, N_2 is eta and so on, I can simplify this further. We will get x to be equal to zeta times x_1 plus eta times x_2 plus 1 minus zeta minus eta times x_3 and y will be equal to zeta times y_1 plus eta times y_2 plus 1 minus zeta minus eta times y_3. For this system or this set of equations, I will simplify them further and I will get x to be equal to zeta times x_1 minus x_3 plus eta times x_2 minus x_3 plus x_3.

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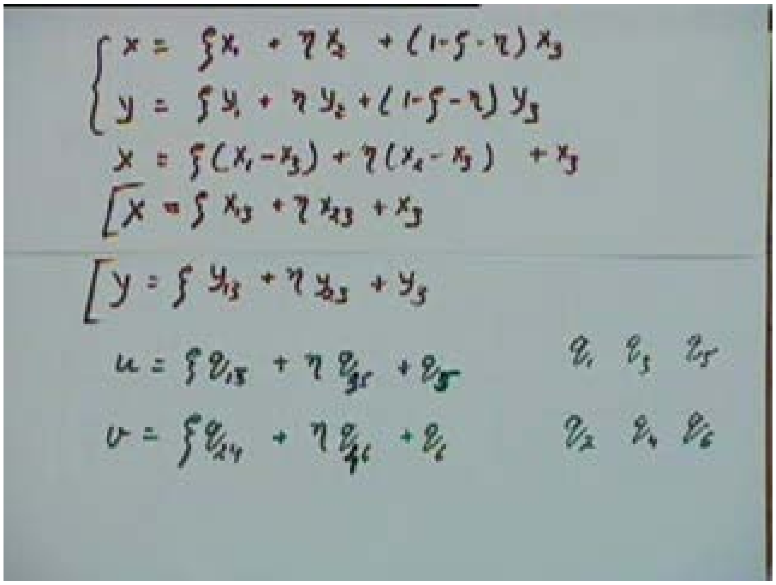

And this will, this I will write as zeta times x_{13} plus eta times x_{23} plus x_3 where x_{13} x_1 minus x_3 x_{23} x_2 minus x_3. So this will be equal to x. And similarly I will get y to be equal to zeta time's y_{13} plus eta times y_{23} plus y_3. So x is given by this, y is given by this solution. Similarly if I write down u and v, I will be getting u to be equal to zeta time's q_{15} sorry q_{13} plus eta times q_{15} plus q_3 sorry q_5. And v will be equal to zeta times q_{24} plus eta times q_{26} plus q_6, this should be quite straight forward.

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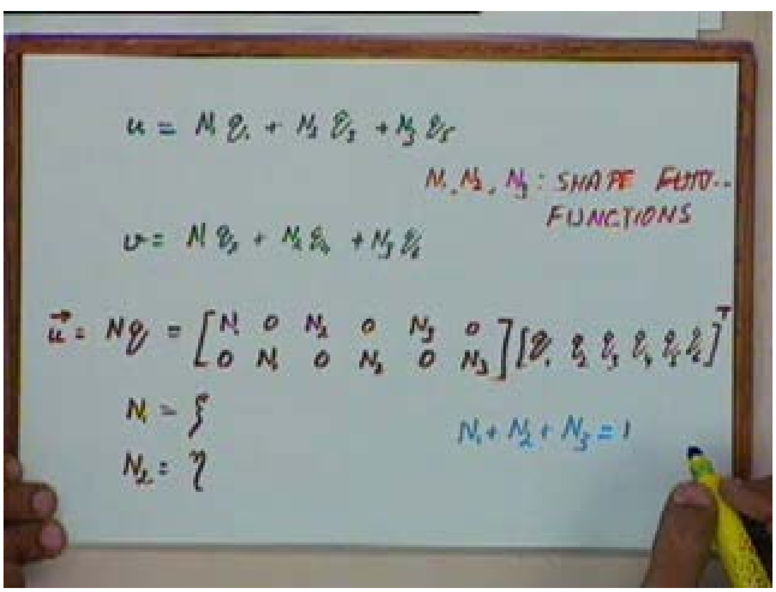

All that I have done is I have put N_1 equal to zeta and N_2 equal to eta and N_3 equal to 1 minus zeta minus eta in these equations. Student: When it be q q_{35} in first equation of q u. Yeah. Here

q_{35}. Yeah, because let me just check x_{13} x_{23}. Is that okay? Yeah thanks. This q_{35} and this will be q_{46}. The basic thing is that the deformations in the x direction are q_1, q_3 and q_5 and in the y direction the deformations are q_2, q_4 and q_6. So this way I have got expressions for x and y and expressions for u and v. Essentially what I am saying with saying now is that if I take any particular point, let's say if I take some value of zeta and some value of eta, zeta and eta are both varying between 0 and 1. If I take some value of zeta and some value of eta, I can put it in these equations. I will get some value of x and some value of y.

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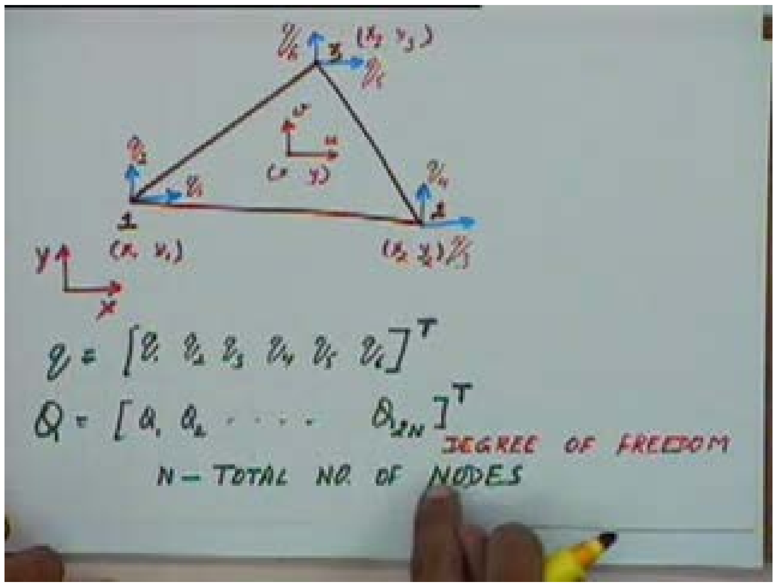

If this is that point xy then the deformation here are given by these expressions. So if I take some specific point here, some specific point with some xy values I will first have to find out the zeta and the eta values over here. In finding the zeta and eta values, I will have to make use of these equations. Once I find zeta and eta, I put them back over here and get the values of u and v. Maybe what we can do is we can take a small example just to see how xy and u and v are related.

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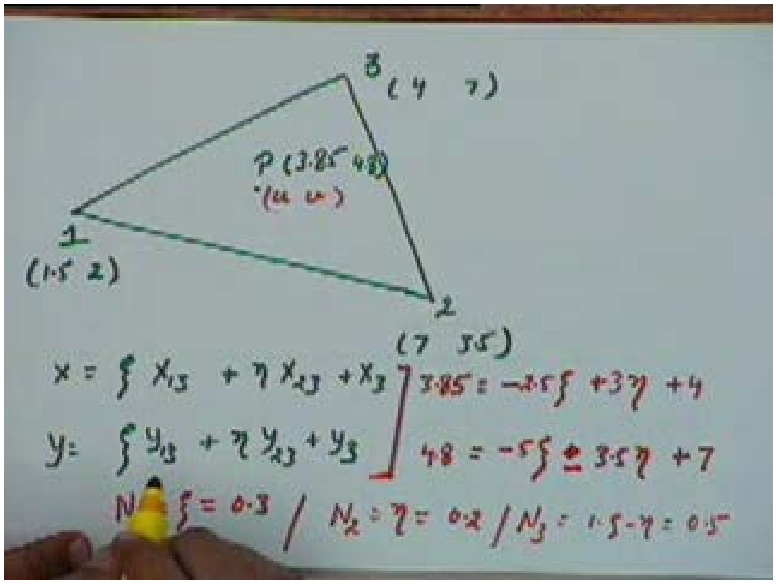

I say coordinates of this point are (1.5 2), my coordinates of this point are (7 3.5), the coordinates of this point are 4 and 7 and I have some point here P whose xy values are given by (3.85 4.8). So I know x is equal to zeta times x_{13} plus eta times x_{23} plus x_3 and y is equal to zeta times y_{13} plus eta times y_{23} plus y_3. So these two equations will give me x as 3.85, so 3.85 is equal to zeta times x_{13}, x_{13} is x_1 minus x_3 that will give me minus 2.5 zeta plus x_{23} is x_2 minus x_3 that will give me 3 times eta plus x_3 which is 4. Again similarly from here, the y value is 4.8. So I will get 4.8 is equal to minus 5 zeta which is y_{13}, y_1 minus y_3 2 minus 7 is minus 5 plus y_{23} 3.5 and 7 that will give me minus 3.5 eta and plus y_3 which is 7. So these two equations I can solve them simultaneously. I will get zeta which is the same as the N_1 to be equal to 0.3. I will get N_2 or eta to be equal to 0.2 and of course N_3, I will get will be 1 minus zeta minus eta which is equal to 0.5. Once I get zeta and eta, if I want to find out the u v values here that I can find out by the equation.

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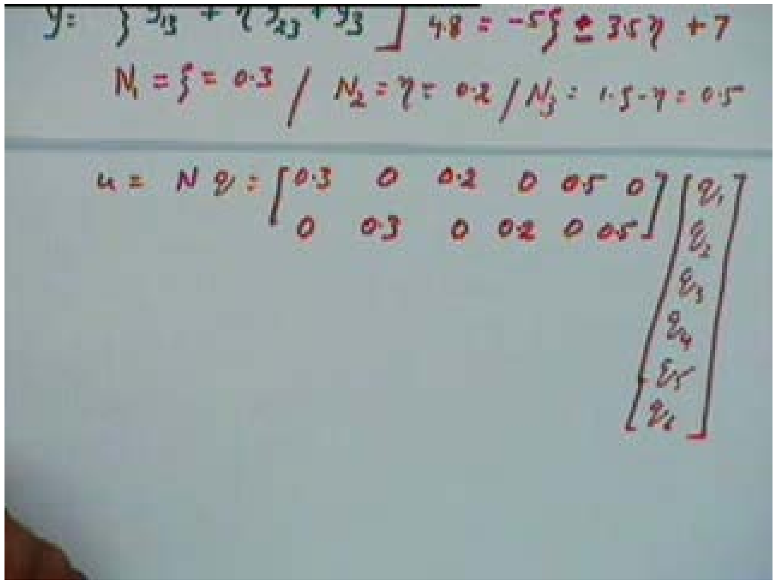

I know N_1 N_2 and N_3, so u is equal to N times q that will be equal to N_1 which is 0.3, N_2 is sorry N_2 is 0.2 0 and 0.5 0. The second row will be 0 0.3 0 0.2 0 0.5 and this is multiplied by the q column vector, this q_1 q_2 q_3 q_4 q_5 and q_6. This is how we can find the deformations at any arbitrary point inside this element. So the u to be equal to N times q where N is given by this matrix at this particular point. Is that okay? Any questions up to this point? What I will do, I will stop here now. In the next class, we will see how we can relate these deformations or how we can relate this these expressions with the stresses and strains and then go onto solve the potential energy expression. We will take that up in the next class.