Computer Aided Design

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Lecture No. # 23

Quadratic Shape Functions

Earlier we are talking of two noded that is linear elements and for two noded elements are elements, since we were talking of one dimensional element we had a uniform area of cross section. And there are 2 nodes are the two extremes, they are the nodes 1 and 2. And then we defined a local coordinate system given by zeta, we say zeta will be minus 1 here and zeta will be plus 1 over here.

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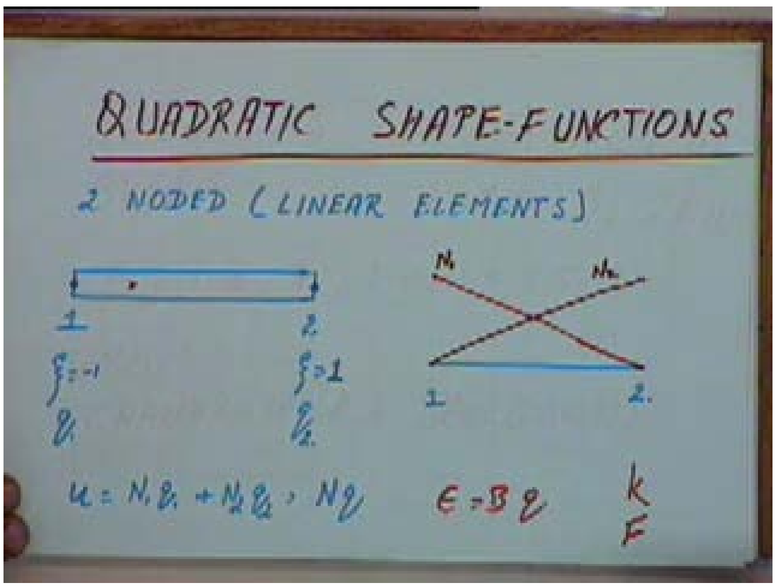

Similarly we decided that if the deformation here is q_1, the deformation here is q_2 then u which is the deformation in any arbitrary point here will be given by N_1 q_1 plus N_2 q_2 or it will be given by N time's q where N and q are matrices. This is what you had for the two noded elements. When we talk of a quadratic shape quadratic shape functions between two points it is not possible to define a unique quadratic curve, so we will need a three noded element.

If you remember the shape functions in this case were linear shape functions which looked like this, this is point 1 and this is point 2. And if I draw N_1 that looked something like this and let's say N_2 will be like this. This is N_2 and this is N_1. We said N_1 will be 1 at node 1 and 0 at node 2 and N_2 will be 1 at node 2 and 0 at node 1 where N_1 and N_2 are the shape functions which are basically the weightages we are assigning to q_1 and q_2 to find out the deformation at any arbitrary point within the element. Now what we will do? Here instead of a since we are talking of quadratic shape functions, we cannot define a quadratic curve two between two points.

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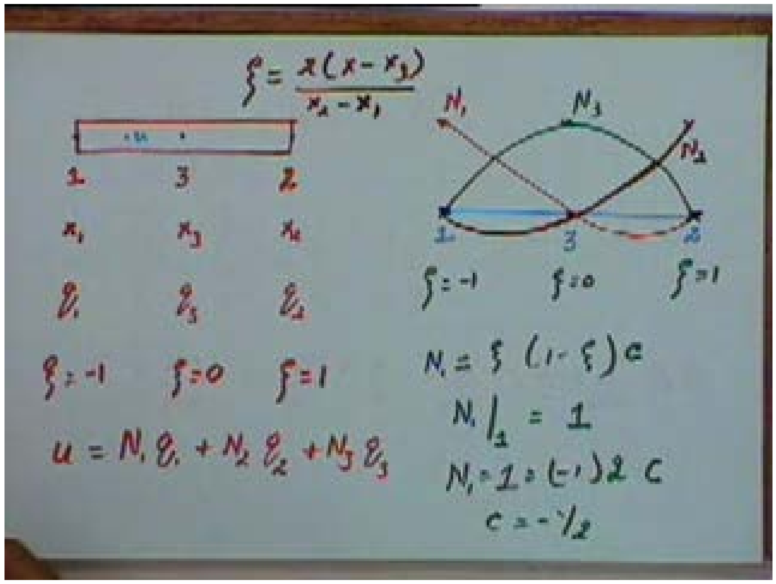

So now we will have a three noded element which looks something this. One node is at one end, the second node is as usual at this end and the third node will take somewhere in the center. So now we have 3 nodes. The positions are x_1, x_2 and x_3 and the deformations will be q_1, q_2 and q_3. And again if I want to find out u at any arbitrary point, that u will be a function of q_1, q_2 and q_3 but now we will do a quadratic interpolation instead of doing a linear interpolation. And we will again define the local coordinate system which will be let's say zeta and zeta at this point is minus 1, zeta here will be 0 and zeta here will be equal to 1. Again, it is something similar to that. So we will define a local coordinate system given by this description and we will define u to be equal to N_1 q_1 plus N_2 q_2 plus N_3 q_3 where N_1 N_2 and N_3 are the shape function associated with the three points 1 2 and 3.

And the way we will take the shape functions, this is my point node 1, node 2 and I say this is my node 3. N_1 is the weightage associated with the point 1. So a criteria that we will choose will be that N_1 will be 1 at this point and will be 0 at both the other points only then this will be a valid statement. If N_1 is 1 and N_2 and N_3 are 0, u at point 1 will be equal to q_1 and that is what we want. So the curve in this case would look something like this, that is 1 here, 0 here and 0 here. And it's a quadratic curve, so it would look something like this.

Similarly if I talk of N_2 this is N_1, if I talk of N_2 it has to be 1 here and 0 at both these points. It will be symmetrical to this curve, it will look something like this and this is N_2 and N_3 it has to be 1 here and 0 at both these locations. So N_3 would look something like this. And since at this point we have zeta equal to minus 1, zeta equal to 0 and zeta equal to 1. We will define N_1, N_2 and N_3 in terms of these zeta. And again N_1 is 1 at this point and 0 at both the other points and it is a quadratic function. So N_1 would have a term of zeta and 1 minus zeta in it. If it is zeta into 1 minus zeta multiplied by let's say some constant C then add zeta equal to 0, this whole term will be 0, add zeta equal to 1 also this whole term will be 0. So we will say N_1 will be equal to this, C is a constant and this gives us a quadratic function for N_1.

And then we know that N_1 at node 1 is equal to 1, at node 1 zeta is equal to minus 1 so we can put that value over here, we will get N_1 which is equal to 1 will be 1 into sorry, this will be minus 1 into 1 minus minus 1 which is 2 into C. So we will get C to be equal to minus half. So this way we will get N_1.

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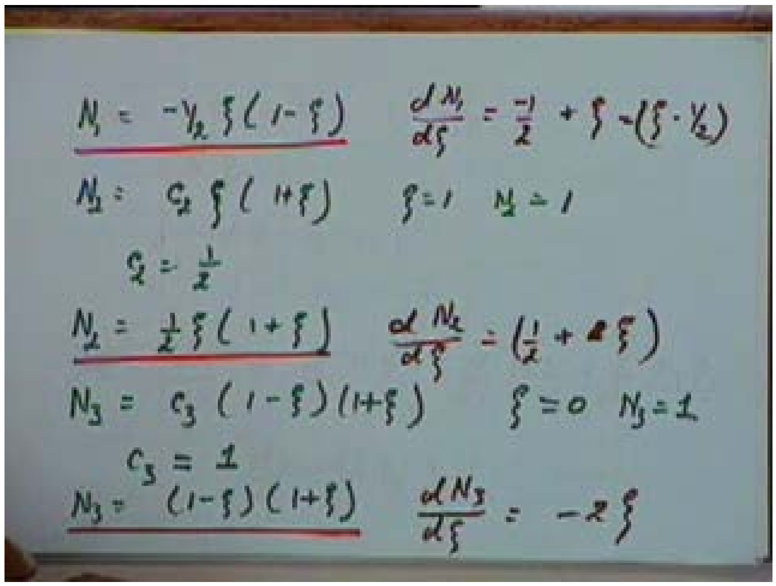

If you look at this expression, you will get N_1 to be equal to minus half into zeta into 1 minus zeta. This will ensure that this N_1 is a quadratic function of this shape is equal to 1 at this point, 0 at this point and 0 at this point. Similarly if we look at N_2, we will say N_2 will have some constant C. Let's say C_2 multiplied by multiplied by zeta and multiplied by 1 plus zeta. That is because at zeta equal to minus 1, it has to be 0 and at zeta equal to 0, it has to be 0. So we will have these two terms, zeta and 1 plus zeta in it. And again we will have at zeta equal to 1, N_2 is equal to 1. If I put this back over here, I will get C_2 to be equal to half or we will get N_2 to be equal to half of zeta into 1 plus zeta. Student: sir, excuse me. Yeah. Is there any purpose of naming the nodes in the linear element as 1 3 2, should because in while assembling so it will be in better when 3 or 4 will get completed of.

No, that doesn't happen that way because what will happen is this is 1, this is 2. From the next element you will get, this will be the first node of that, this will be a first node of that, the second node will be at that end. So the second location as it is we will see that the stiffness matrix of this will be larger, it won't be a 2 by 2 matrix, it will be a 3 by 3 matrix. So when you assemble a 3 by 3 matrix, you will see that this number will be more convenient. We will see that in the end because if I give 1, 2 and 3 I mean strictly speaking you can always do that. But this numbering will be more convenient as far as the assembling the matrix is concerned. Now we have got N_1 and N_2, similarly if you look at N_3, N_3 will be some constant C_3 multiplied by 1 minus zeta into 1 plus zeta and at zeta equal to 0, N_3 is equal to 1.

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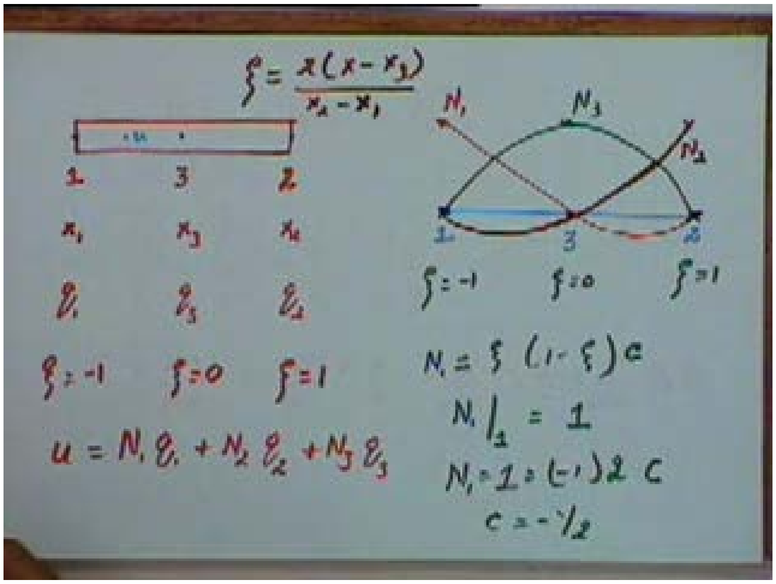

At zeta equal to 0 that is 1 and both these end points, it has to be 0. So we will have a term of 1 plus zeta and 1 minus zeta in it. If we put this, we will get C_3 will be equal to 1. Therefore N_3 is equal to 1 minus zeta into 1 plus zeta. So the three shape functions you will have, they will be this then this and this.

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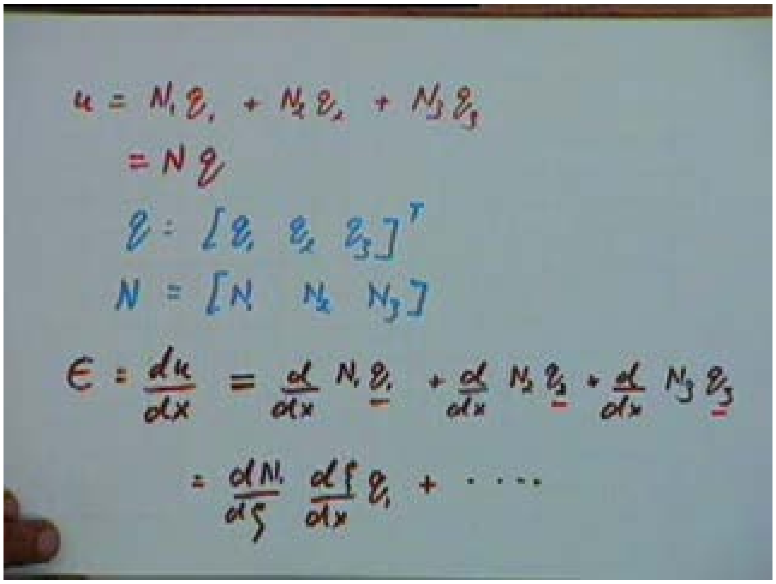

And we have mentioned u is equal to N_1 q_1 plus N_2 q_2 plus N_3 q_3 which we will write that as N times q where q is equal to q_1, q_2, q_3 transpose which is the element displacement vector. N is N_1, N_2, N_3 which is a set of shape functions. Any question up to this point? The next thing is in one dimensional element, the strain epsilon this is given by du by dx. So du by dx, if we take up this expression, it will be d by dx of N_1 q_1 plus d by dx of N_2 q_2 plus d by dx of N_3 q_3. Of these q_1, q_2 and q_3 are constants and N_1, N_2 and N_3 these are function of zeta and zeta in turn depends on x. So if you look up this formulation, we found out the expression for N_1, N_2 and N_3, this is N_3 in terms of zeta and zeta in terms of x. Zeta in terms of x is given by zeta will be equal to 2 times x minus x_3 divided by x_2 minus x_1. So if you use this, we can evaluate this derivative. If you notice the steps that we are going to follow will be the same as the steps we had for the two noded elements.

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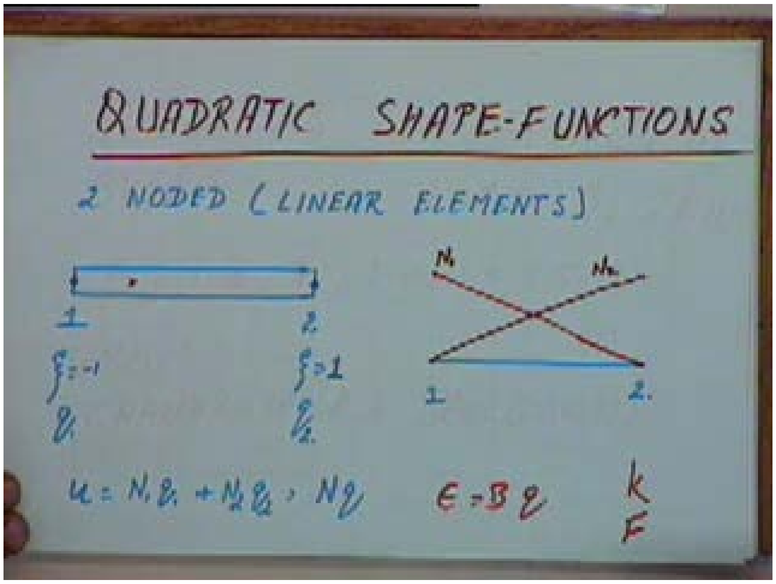

In the first two noded elements that we had taken, linear elements there also we defined u equal to N_q from that we found out epsilon. Once we found out epsilon that we got epsilon to be equal to B times q, here B is nothing but the derivative of N. Once you got epsilon then we put that in the potential energy expression and got expressions for the stiffness matrix and the force matrices and we are going to follow exactly the same sequence in this case also. So if we carry out this differentiation, the derivative of N_1 with respect to x will be nothing but derivative of N_1 with respect to zeta multiplied by derivative of zeta with respect to x multiplied by q_1 and so on. So again derivative of zeta with respect to x will be there in all the three terms. So what we need is the derivative of N_1, N_2 and N_3 with respect to zeta.

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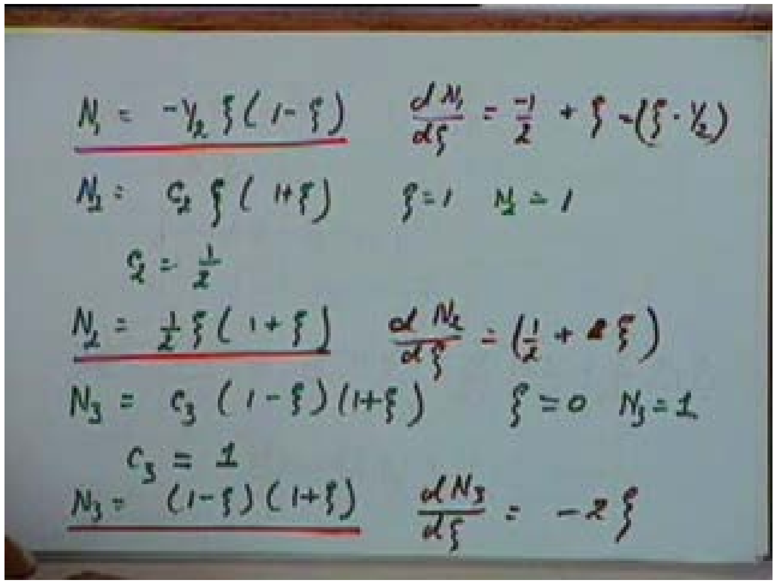

That we can get from our definitions of N_1, N_2 and N_3 from these terms. So here, so derivative of N_1 with respect to zeta that will give us, this will be minus half plus this zeta so it will be plus zeta which is zeta minus half. Is that right? 2 zeta minus 1 by 2. Yeah. N_2 this will give us half plus 2 half plus zeta and this df N_3 with respect to zeta, this will give us minus 2 zeta. So these are the 3 derivatives which we will have and we put them in this expression.

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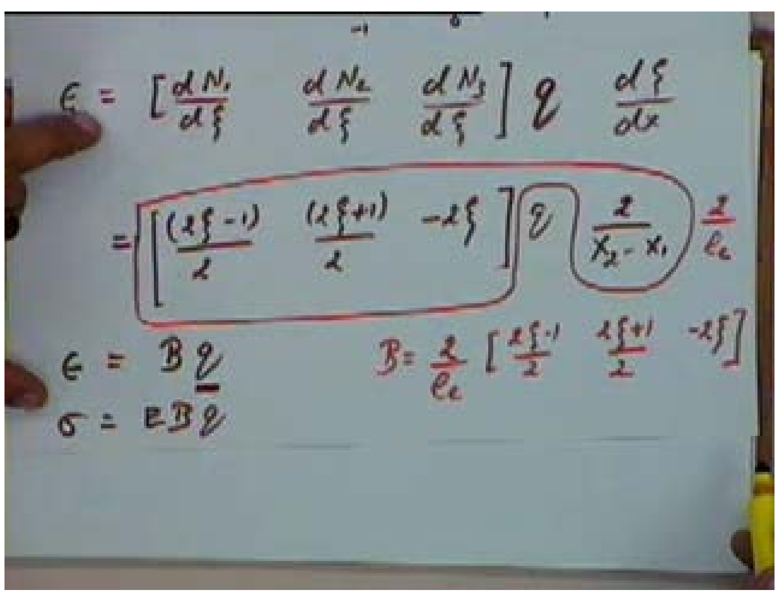

So what we will get will be, from this we will get epsilon to be and we write it again in a matrix notation multiplied by q and of course multiplied by d zeta by dx. d N_1 by d zeta, we will take from there will be 2 zeta minus 1 by 2, d into by d zeta is 2 zeta plus 1 by 2 and d N_3 will be minus 2 zeta. This multiplied by q and d zeta by dx will come out to be 2 divided by x_2 minus x_1. So this again we will write that this will be equal to B times q because B is this matrix multiplied by this constant. The x_2 minus x_1 is nothing but the length of the element. So this will be nothing but 2 divided by l_e. So we will get B will be equal to 2 divided by l_e multiplied by 2 zeta minus 1 by 2, 2 zeta plus 1 by 2 and minus 2 zeta. Any questions up to this point? If you notice, in this case the strain epsilon will be varying within the element because it depends on zeta. Zeta is basically, if this is my element zeta is changing along the length of the element. At this point zeta is minus 1, here it is 0 and here it is 1. So this strain is going to vary within this element and will vary as a linear combination of q_1, q_2 and q_3, the linear combination of the three parameters here. While in the previous case when we were taking the linear shape functions, the strain within the element was constant, so those were the constant strain elements.

In this case the strain is not constant but is going to vary linearly as a linear combination of these three. Then once you have an expression for epsilon, we can immediately get the expression for sigma which is nothing but E times B times q where sigma is a stress which is Young's modulus multiplied by the strain. Now let's look at the expression for the potential energy.

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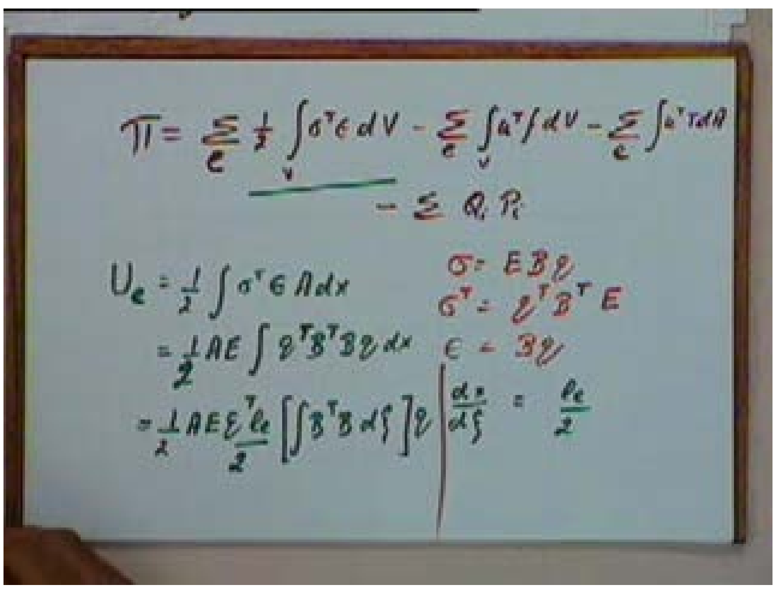

We will carry out the same steps again. Potential energy is given by summation over all the elements of half integral of sigma transpose epsilon dv minus summation over all the elements of integral u transpose f dv, again volume integrals minus summation over all the elements of integral u transpose TdA minus summation of Q_i P_i. So let's look at the first term, this term which is the strain energy term. So because strain energy of the element is equal to half integral of sigma transpose epsilon dv, again we are talking of one dimensional element. So we replace this by Adx, A into dx because within the element area is going be constant. So this will be equal to half of A sigma. Sigma is equal to E times B times q. So sigma transpose will be equal to q transpose B transpose multiplied by E, E is of course a constant it's not a matrix, so E can come out.

So this will, I will take E out integral of q transpose B transpose multiplied by epsilon where epsilon is equal to B times q. So for instead of epsilon, I will put B times q multiplied by dx and earlier also we have used the fact that dx by d zeta is going to be equal to l_e by 2 because my B matrix is in terms of zeta. So I prefer to do the integration with respect to zeta rather than with respect to x and that I can convert by using this relationship. So dx is equal to d zeta into l_e by 2. I can put that over here; q transpose is of course a constant. So I will get this to be equal to half of AE q transpose l_e by 2 is again a constant, so I will get l_e by 2 into integral of B transpose into B into d zeta and this whole thing is multiplied by q. Can we just draw a line.

Now B is the matrix which is given over here, this is my matrix B. If I do a B transpose into B, B is a rho matrix of size 3. So if I do a B transpose, it will be a column matrix of size 3 and that is being multiplied by B. So the product of the two will be a 3 by 3 matrix, a 3 by 3 matrix consisting of terms in zeta and zeta is changing from minus 1 to 1. So I can carry out this integral because it only has terms in zeta nothing beyond that. I can expand this integral and evaluate it.

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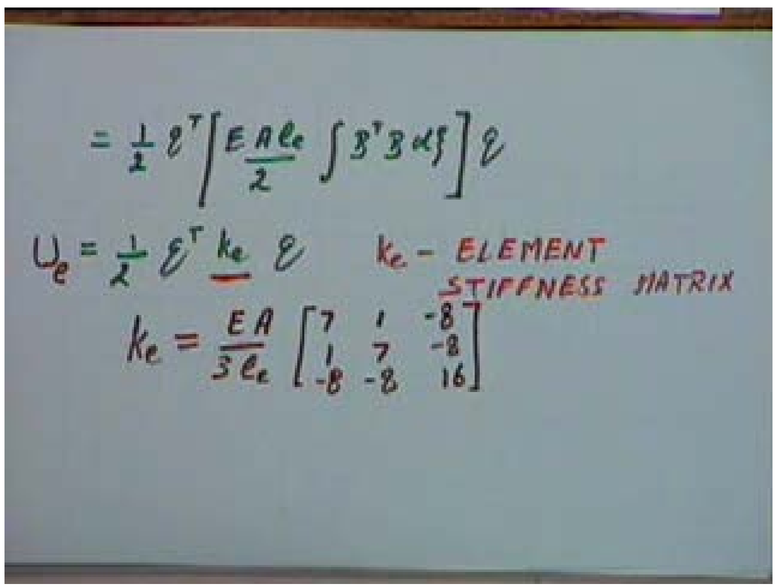

What we will get or rather what we will do now? This step let's say is equal to half of q transpose into we will have terms inside which will contain E into A into l_e by 2 into integral of B transpose B d zeta and this whole is multiplied by q. And this we will say is equal to half of q transpose k_e into q where k_e is the element stiffness matrix, k_e is called the element stiffness matrix. And if you carry out this integral we will get k_e to be given by, so k_e will be given by this matrix and the strain energy in the element is given by this expression. Is that okay? Any question up to this point? Then we will now take up the next term in the potential energy expression that is the term of these body forces.

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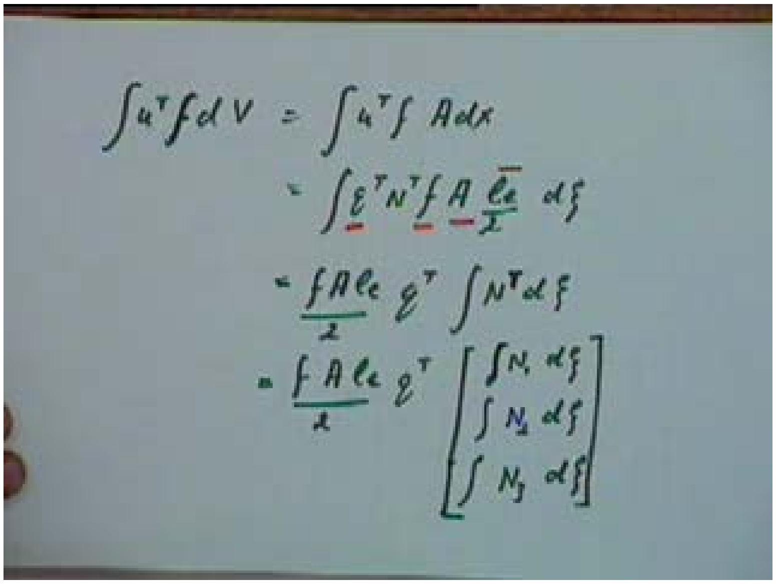

If you want to evaluate this integral, this is integral of u transpose fdv. This is nothing but integral of u transpose f Adx. This will again be u transpose is N times q, so this will be integral of q transpose N transpose fA dx. Again I will convert that into terms of d zeta so I will get this to be l_e by 2 times d zeta. Of these f is a constant, A is constant, length is a constant all these terms I will take them out and of course q transpose is also a constant. So if I take these terms out, I will get this will be equal to fA l_e by 2 into q transpose into integral of N transpose d zeta. This I will write that as fA l_e by 2 into q transpose into this will be integral of N_1 d zeta integral of N_2 d zeta and integral of N_3 d zeta.

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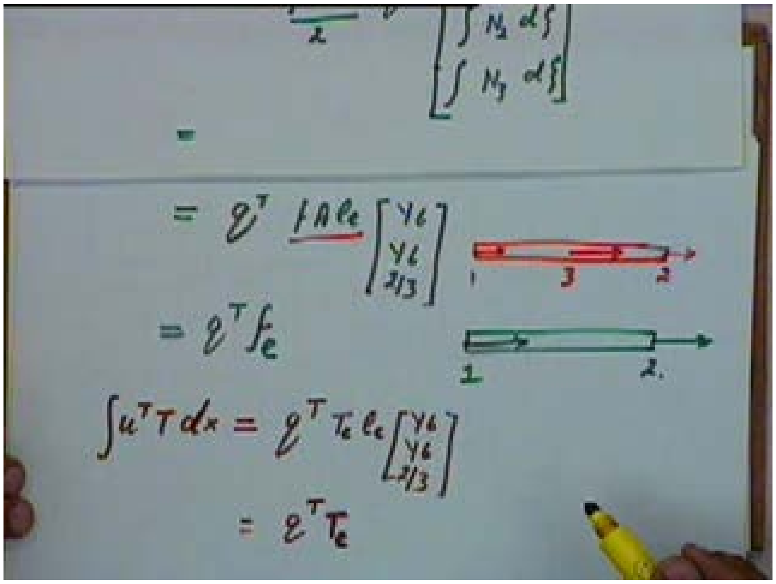

And again we can carry out the same integrals and this whole expression, we will find that this will be equal to, this whole expression will be equal to... Now f into A into l_e, f is the unit body force acting on the element force per unit volume, A into l_e is the volume of the element. So this thing will come out to be equal to q transpose multiplied by this term that is f into A into l_e and then inside we get three terms which will be 1 by 6, 1 by 6 and 2 by 3. Now this is the total body force acting on the element and this is this gives a proportion in which this force is effectively divided into the three nodes. We have three nodes this is 1, this is 2 and this is 3. So what we can say effectively one sixth of the body force is acting here, one sixth is acting here and two third is acting over here. The total body force acting is this which is divided in this ratio at the three nodes. And this we will say, this will be your q transpose multiplied by let's say the body force vector for the element.

So the total body force is acting on the element are divided into three nodes in this proportion. In earlier case when you are taking a linear element, we had two nodes one here and one at the other end and the total body force was acted was divided equally in the two nodes. Now it is divided in the ratio of 1:1:2 or rather 1:1: 4. One sixth is acting here, one sixth is acting here, two third is acting here. This is as far as the body forces are concerned.

Similarly if you consider tractive forces, the term for the tractive forces that we have is integral of u transpose into T into da, in this case it will be dx if we tractive force to be per unit length. And this again I will leave it up to you, you can carryout the integral. This will be equal to q transpose into the tractive into force on the element multiplied by 1 by 6, 1 by 6, 2 by 3. In this again we will write that as q transpose multiplied by let's say the tractive force vector for the element, this T and this T are different. This integral is just the same as the previous integral, you can try it out yourself.

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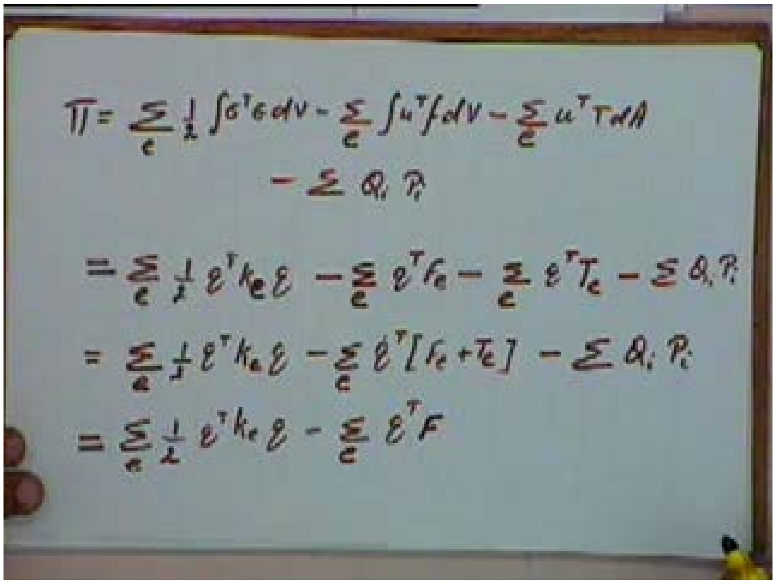

So now if you look at the potential energy expression expression, I will rewrite the expression here which is potential energy pi is equal to summation over all the elements of half of integral sigma transpose epsilon dv minus summation over all the elements of integral u transpose fdv minus summation over all the elements u transpose TdA minus sigma Q_i P_i. This first term will now become summation over all the elements for half of q transpose k_e into q minus the summation over all the elements of q transpose multiplied by F_e minus summation over all the elements of q transpose multiplied by T_e minus sigma Q_i P_i. Now this expression is the same as the expression we had earlier. So this is equal to summation over all the elements of half q transpose k_eq minus summation over all the elements of q transpose multiplied by F_e plus T_e, F_e plus T_e will give us the element force vector and minus summation of Q_i P_i. So this again I will write that as summation over all the elements half of q transpose k_eq minus summation over all the elements of q transpose multiplied by the element body force vectors. The point loads can be taken incorporated into this and then again the next step is also the same as what we had earlier.

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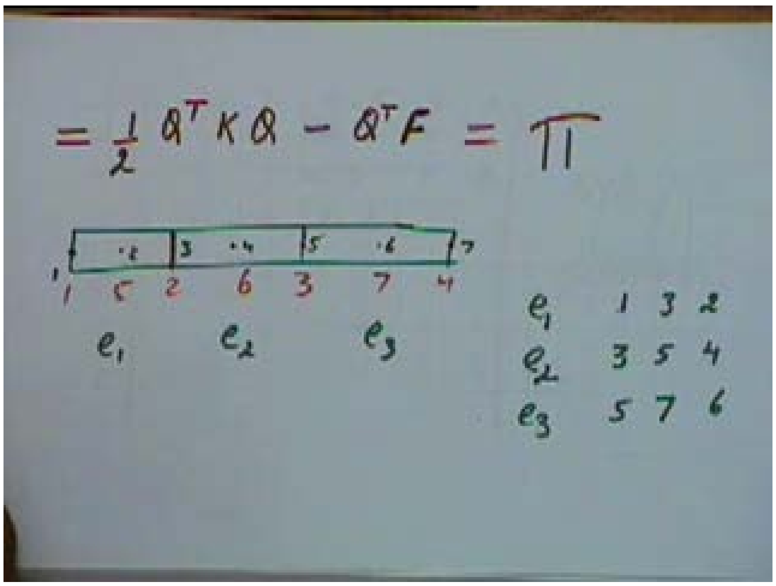

This summation of these matrix multiplications, we replace them by global matrices which will be of the type Q transpose kQ minus will become Q transpose F where this F is a global force vector force vector. So this is again the expression for the total potential energy in the system. The method for assembling the stiffness matrix, the basic method will still remain the same. If we have let's say elements of this type where I will say this is and so on. These numbers are the global node numbers. For each element let's say this is my element number 1, this is element number 2 and this is my element number 3. We will say that element 1 is defined between nodes 1 3 and 2. This is the first node, this is the second node, this is the third node. Element 2 is defined between 3 5 and 4. So this will be 3, this will be 5 and this will be 4. Element 3 is defined between 5 7 and 6. So my global stiffness matrix is going to have 7 locations in this case where there are 7 nodes.

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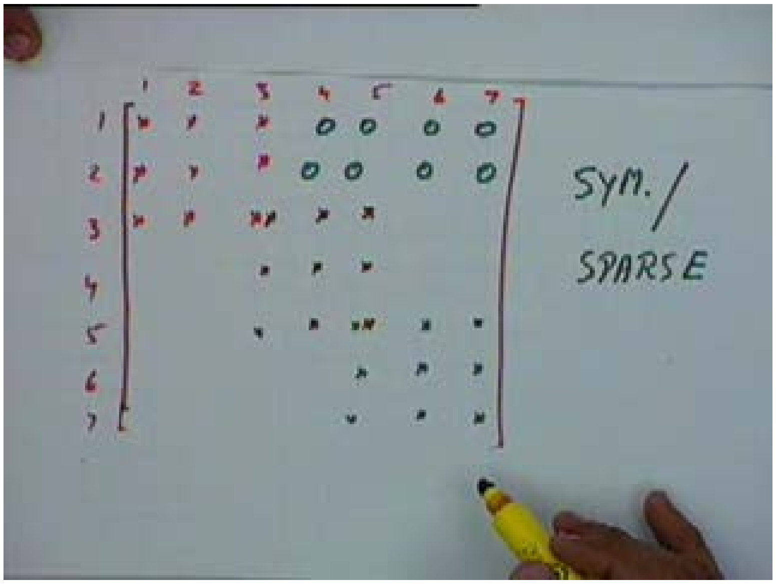

So if I take all those seven nodes, my first element is defined between nodes 1 3 and 2. So, first element will be defined between 1 3 and 2 at these locations. My second element is defined between 3 5 and 4. So, second element will be defined between 3 5 and 4. Mind you the sequence is important. I am writing this as 1 3 2 and not as 1 2 3 because the second node for this element e_1 is node number 3. I can't interchange these numbers here. If I interchange them, all my forces will straight away go wrong. So, element 2 is between 3 5 and 4, so 3 5 and 4. Stiffness matrix for the first element will take these 9 locations, second elements will take these 9 locations and third is 5 7 and 6. So this is, I will take 5 7 and 6 so 5 7 and 6 and so on and the properties of the stiffness matrix are still retained.

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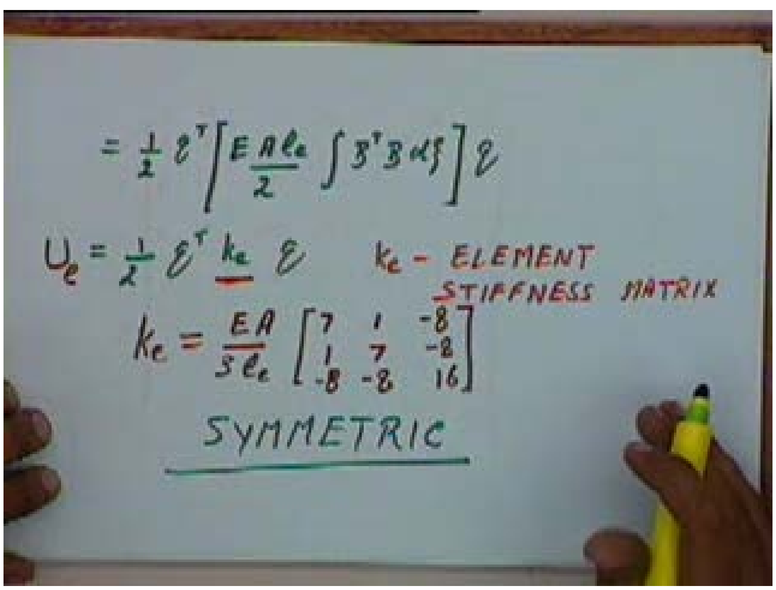

That means earlier we had said that a stiffness matrix for the two noded elements was a symmetric matrix. This matrix is also symmetric and if we do the numbering properly, we will still get a banded matrix. That means we have only diagonal elements and elements at a fixed distance from the diagonal, all other elements will be 0 and so on. But if my numbering is not correct that means in this case let's say if I do my numbering like 1 2 3 4 5 6 and 7 then my matrix will not remain a banded matrix because the first element will be defined between 1 2 and 5. So it will become 1 2 and 5, this element will become non-zero. So it will not remain a banded matrix but it will be a sparse matrix. So this global matrix should be either symmetric or it will be sparse. Student: sir what is that local numbering 1 2 3, make a difference here. This numbering?

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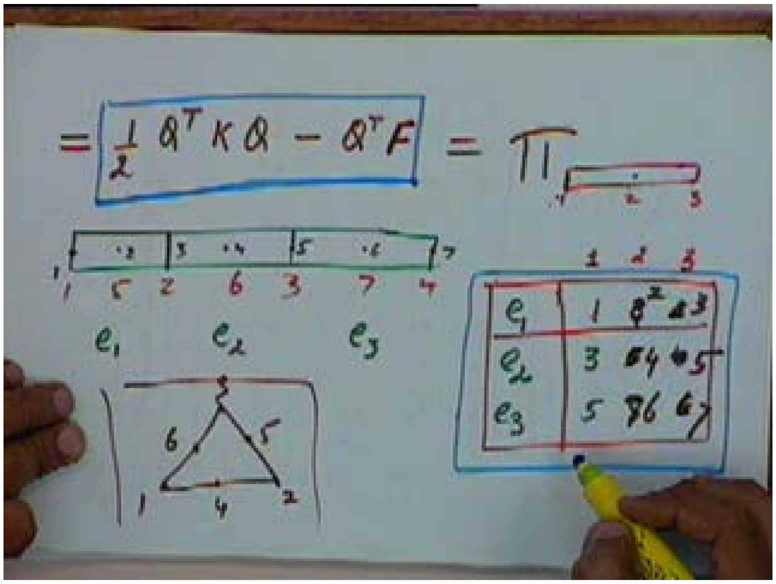

See here in this table what I am basically saying is that for each element, the local numbers 1 2 and 3 correspond to global numbers 1 3 and 2. The local number 1 for this node is global number 1, global node number 1. Student: I said what difference will it make had the local node numbering been 1 2 3. Has a local numbering been in a sequence? That means if I had taken my local numbering like this 1 2 and 3. What is eventually more important is this global numbering but then my global numbering will not be straight like this because if I am doing the numbering like this, my node numbers will become this is my local number 1, this is my local number 2, this is my local number 3, this will become 2, this will become 3.

My element 2 is defined between 3 4 and 5, 3 4 and 5. Element 3 we defined between 5 6 and 7. So in this case you will still get the same patterns, your matrices will remain the same but what happens is if you have a if your see this quadratic element that we have taken, right now it is only for a one dimensional element. Later on we will see if we have a triangular element, even there we will take a 6 noded triangular element. There also we will do numbering like this. Student: I am asking, I don't see any advantage. There is no specific advantage as far as, there is no specific advantage that you will gain from this numbering because eventually if you can always change your global numbering to suit this particular pattern.

But typically when an element is defined, you will find that eventually it basically comes on to the fact that it is by tradition, that you always define. Student: See the other way round if we number 1 2 3 4 5 6. Again? Student: I think the number is a sequence 1 2 3 4 5 6 this element. In this, anyway this will come to when we come to two dimensional elements but in this if you, even if you number the elements 1 2 3 directly. As far as your formulation is concerned, you won't lose anything but typically you will always find the numbering be 1 2 and then 3. Possibly because you are extending it from linear elements, possibly because of that reason but as far as the technical numbering is concerned, it will probably not make a difference. Any other question on this particular thing? So the potential energy expression is given by this formula.

Now if you notice the potential energy expression is the same as what we had earlier. Our boundary condition is also going to be same as what we had earlier. So our method of solving is also going to remain the same. Eventually any type of element or any type of finite element formulation we choose, we will try to get the same formulation and then use the same method that we have discussed already, whether it is two dimensional elements or three dimensional elements, our potential energy expression will be the same, the method of solving it will remain the same. The only thing that will change will be this matrix. That means the elements will have different number of nodes and a different numbering, different kind of local numbering and a global numbering. So the potential energy expression will always remain the same. What we will do from now onwards is that for different types of problems for two dimensional as well as three dimensional problems, we will just see how to define elements, how to define shape functions and how to get this potential energy expression. Any questions on this?