**Computer Aided Design**
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**Lecture No. # 22**

# Penalty Approach and Multi Point Boundary Conditions

Today we will be continuing with the discussion on penalty approach and then we will see what are called as multi point boundary conditions. So far we have only seen single point boundary conditions that is constraints of the type Q $Q_i$ equal to $a_i$ or let's say $Q_1$ is equal to $a_1$.

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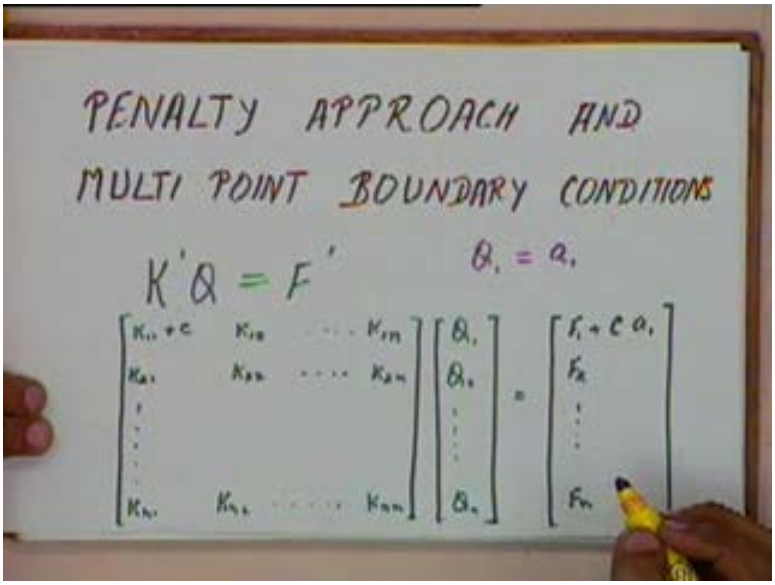


And we have seen that if you have a constraint of the type $Q_1$ equal to $a_1$ then using the penalty approach, we get this as the system of equations where K prime is the stiffness matrix obtained by adding a term c to the diagonal element of location 1 and this is a K prime matrix. The Q matrix remains the same and the F matrix, that is F prime is obtained by taking the global force matrix and adding to the first location c times $a_1$ where c is some large number. Now in this system of equations, the first question is how do we decide value of c. Just saying that c is some large number is not sufficient. When you have to solve it algorithmically, we need some way of estimating the value of c. If you notice this parameter c is appearing only in the first equations, first equation that means this row is multiplied by this column is equal to this term. That is the only the where only place where this value c is coming, c is not appearing anywhere else in the equations. So if I just write down this equation, the expression would be $k_{11}$ plus c times $Q_1$.

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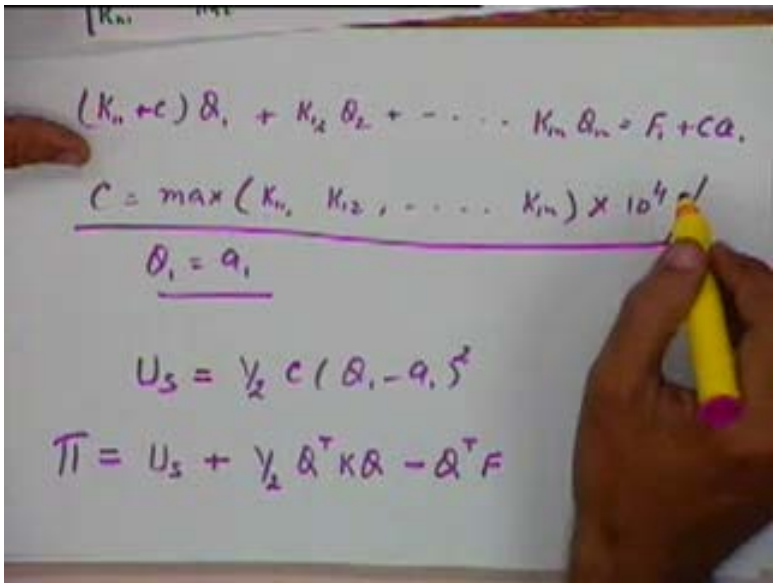


And in this system of equation or in the given system of equation, this is the only equation which has the parameter c. So if I say that the c should be larger than $k_1$ $k_1$ $k_{11}$ $k_{12}$ $k_{1n}$, all these coefficients and larger than all these coefficients by an order of magnitude. So we can say that c, we will take that to be equal to maximum of $k_{11}$ $k_{12}$ so on till $k_{1n}$ and we will take the maximum and multiply that by some order of magnitude, let's say 10 to the power 4 or 10 to the power 5 something like that. We will take the highest of all these coefficients and multiply that by 10 to the power 4 or 10 to the power 5, 10 to the power 6 some large number like that and that will give us the value of C.

We know that the larger we take the value of c, the closer will be our, more correct our result will be because we finally want that $Q_1$ should be equal to a one. So if we take a very large value of c here, we finally get this $Q_1$ to be equal to $a_1$. And how do we show that? Already I have mentioned that $U_s$ that is the strain energy in the spring is equal to half of c times $Q_1$ minus $a_1$ squared. And the total strain, the total potential energy system in the system was pi which is equal to $U_s$ plus the other terms that is half of Q transpose k Q minus Q transpose F.

Now in this potential energy expression, we are trying to minimize the potential energy. And when you are trying to minimize the potential energy, this will be a minimum only when this term becomes very small where c is large and $Q_1$ minus $a_1$ is not equal to 0, this term will become very large. So only when $Q_1$ tends to be equal to 0, we will be able to minimize the potential energy. So if we take a very large value of c, in this system of equation we will finally approach towards the condition of $Q_1$ equal to $a_1$. And for taking a large value of c, we can normally take a condition of this type. This we can change, this factor instead of 10 to the power 4 you can change may take it any factor but c should be larger than all these coefficients and larger by an order of magnitude. Let's take a small example for this penalty approach and then maybe it will become clearer, the same example that we had earlier.

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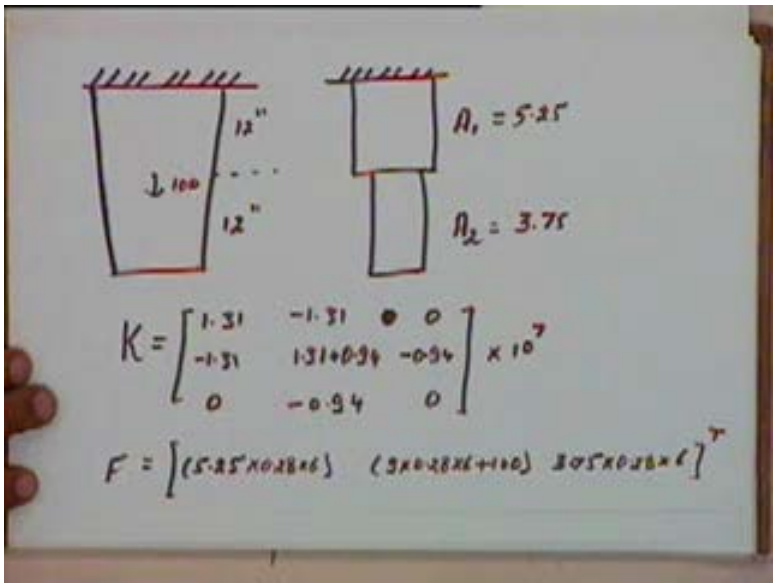


Instead of force of 100 pounds, this is since is 12 inches, the load is since it's also 12 inches and this we had approximated by 2 elements of this type. In the area of cross section for this was if I remember correctly 5.25 square inches, the area of cross section for this was 3.75 square inches and the global matrices that we had they looked something like this. The global stiffness matrix was 1.31 minus 1.31 0 minus 1.31, this is 1.31 plus 0.94 this term is minus 0.94, this is 0 minus 0.94, this is 0. This into I think 10 to the power 7, these are the global stiffness matrix. And the global force matrix that was 5.25 into 0.28 into 6. This is 9 into 0.28 into 6 plus 100 and the third term was 3.75 into 0.28 into 6, the transpose of whole. This was the global force matrix. Now if I use the penalty approach then I have to modify both the K matrix as well as the F matrix by adding the c terms.

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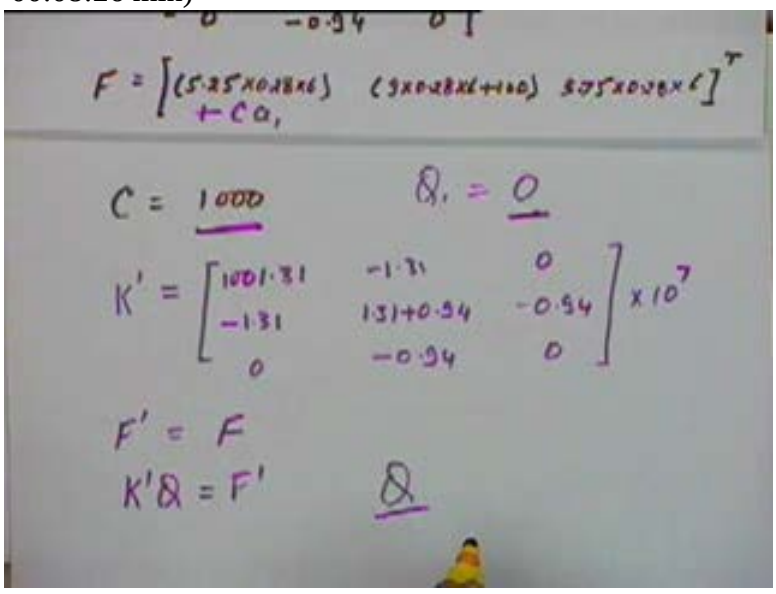

So let's say I will take c to be equal to some value, let's say 1000 and my boundary and my boundary condition in this case is $Q_1$ is equal to 0. So if I take c to be 1000, I have to modify my K matrix by adding 1000 to the location 1 1. So here I will add, this will become plus 1000 and in the F matrix in the first location, I have to add this plus C times $a_1$. Again $a_1$ is equal to 0, $a_1$ is this term and c is this. So I modify matrices, it would become something like this K prime would be I have added 1000 into 10 to the power 7. So this will become 1001.31 minus 1.31 0 1.31 and I have modified first matrix F prime that should be the same as F in this case because $a_1$ is equal to 0.

And my system of equation would be K prime Q equal to F prime. K prime is 3 by 3 matrix, Q is a is having 3 variables, F prime is having 3 locations. I can solve this system of equations to get the value of Q, in a simple set of linear equations I can easily solve it out. So this is how you can use the penalty approach for solving single point boundary conditions. Since these are the single point boundary conditions because these are giving constraints on only one node at a time. $Q_1$ is a deformation of one node, this constant is giving a deformation of only one node or controlling the deformation of only one node. So we call them single point boundary conditions. In contrast to this, we will now see what are referred to as multi point boundary conditions, multi-point boundary conditions.

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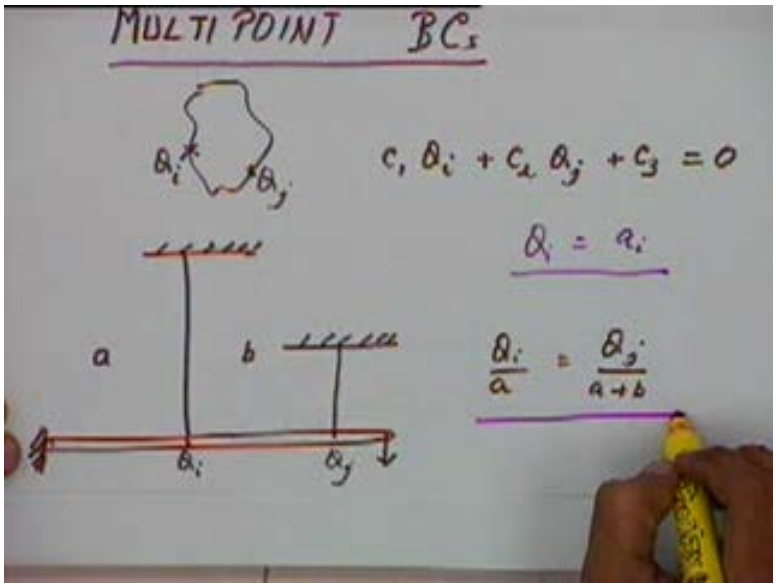


In multi-point boundary conditions we will basically take, let's say if we have any general element like this we take two nodes on this, let's say one node here and another node here. let's say this is $Q_i$ and this is $Q_j$ and we say that $Q_i$ and $Q_j$ will be related linearly that means for just say some constant $c_1$ times $Q_1$ $Q_i$ plus some constant $c_2$ times $Q_j$ plus there is another constant $c_3$ will be equal to 0. That means the deformation of these two nodes is constrained in a linear manner. Typically the situations where these kind of boundary conditions are used, let's say if you are talking of a situation like this.

If we have a system like this where this is a lever which is tied by two ropes and we assume that this lever is let's say that is compared to the ropes. The lever is hinged at one end and we are applying some load at the other end. So then $Q_i$ and $Q_j$ they have to be related in a linear manner that means let's say this distance is a and this distance is b, we will get a relationship of the type that $Q_i$ by a will be equal to $Q_j$ divided by a plus b. So this is a constraint which we are relating to displacements $Q_i$ and $Q_j$ and this is what we referred to as a multi-point boundary condition.
And so far we have handled boundary conditions of the type $Q_i$ is equal to $a_i$. We know, we can use the penalty approach or the elimination method to solve problems with these type of boundary conditions but if we have a boundary condition of this type, the same method will not be sufficient or we have to modify that method slightly to see how we can solve constraints of this type.

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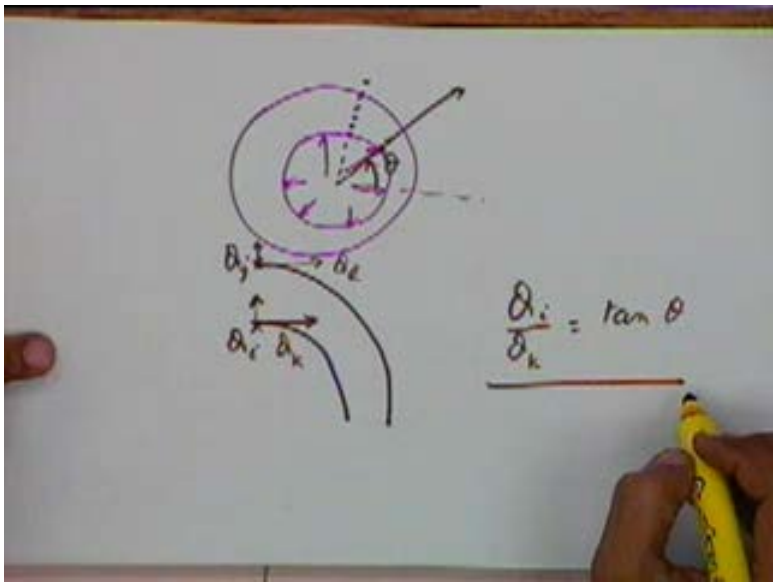


In other situation where multi point boundary conditions are needed or let's say if you have a circular cylinder of uniform thickness and that is having internal pressure. In this case since I am assuming a uniform thickness and the pressure is also uniform, if I have nodes here like this the nodes are all going to move radially outwards. So this mode will get deflected may be to this position, this position and this position respectively but all these nodes should be moving radially outwards.

Now if I take just a section of this cylinder and I model by body just let's say only the quarter of the body like this, I have one node here another node here. This is $Q_i$, this is $Q_j$ this is where let's say a deformation in the y direction. Let's say deformation in the x direction are $Q_k$ and here they are let's say $Q_l$. Now $Q_i$ and $Q_k$ they have to be related to one other because any point here will be moving in a direction at an angle theta. So the relationship between $Q_i$ and $Q_k$ will be of the form that $Q_i$ by $Q_k$ will be equal to tan theta or something like that. So again this is your multi point boundary condition because you are basically saying that a node here has to move only in the radial direction that means the x direction deformation and the y direction deformation they have to be related to one other.

This is a two dimensional case, it's not a one dimensional case. So this is also a multi-point boundary condition. Similarly you have number of situations where deformations of more than one node are constrained together by a by a linear relationship between them. So if we have multi point boundary condition let's say if we take a general multi point boundary condition $beta_1$ $Q_{p1}$ plus $beta_2$ $Q_{p2}$ is equal to $beta_0$ where $beta_1$, $beta_2$ and $beta_0$ they are constants and $Q_{p1}$ and $Q_{p2}$ are two, are the deformation of two nodes.

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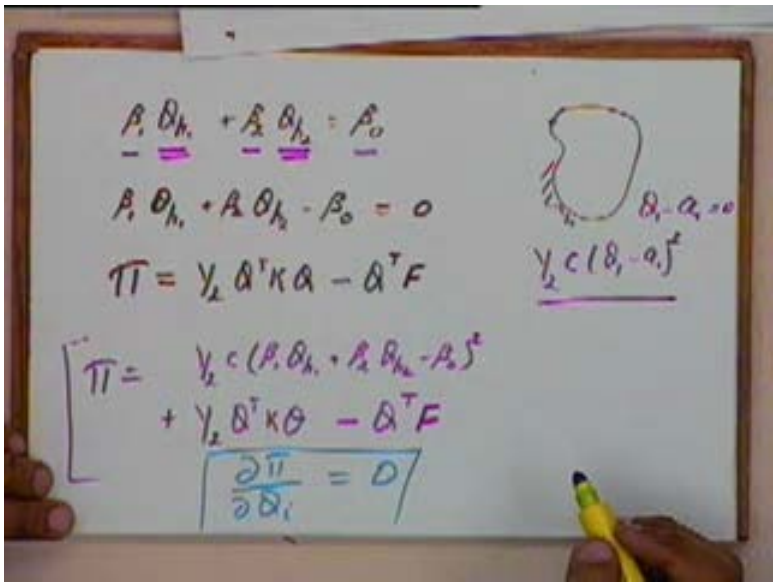


$Q_{p1}$ is the deformation of the node number $p_1$, $Q_{p2}$ is the deformation at node number $p_2$. If we have a boundary condition of this type, we will rewrite it like this $beta_1$ $Q_{p1}$ plus $beta_2$ $Q_{p2}$ minus $beta_0$ is equal to 0 and again we will use the penalty approach. Now our potential energy expression pi is given by half of Q transpose K Q minus Q transpose F that means if you have any general system, for this system this is the potential energy expression. Now we want to give it a boundary condition of this type. Let's first see simply by analogy, earlier we had a boundary condition of the type $Q_1$ minus $a_1$ equal to 0. So the penalty that we gave where penalty was of the type half of c times $Q_1$ minus $a_1$ whole squared, this is the penalty we had given to the potential energy.

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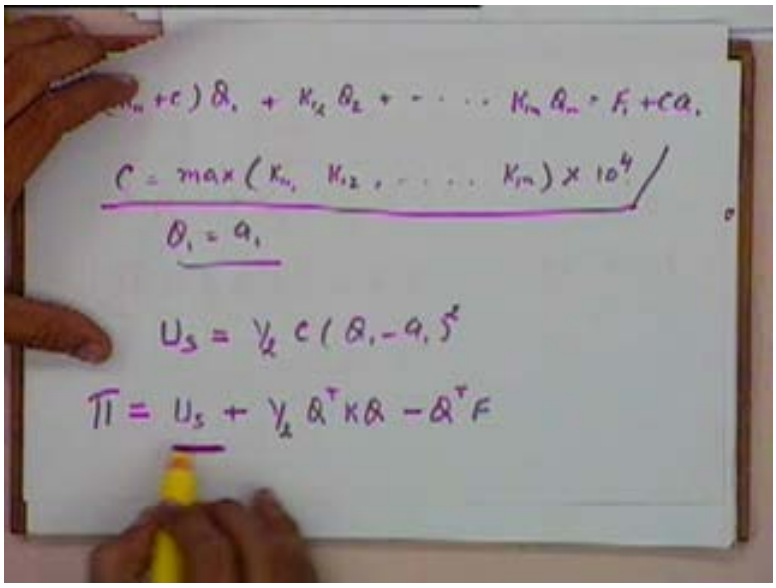


If you, this potential energy of the spring half of c into $Q_1$ minus $a_1$ whole squared. This is the penalty we had given to the potential energy expression. Now we will do a similar thing over here. We have this is equal to 0, so we will we will give a penalty which should be given by half of c into this whole thing $beta_1$ Q $p_1$ plus $beta_2$ Q $p_2$ minus $beta_0$ whole squared. This is the potential energy penalty that we will give to this potential energy expression. So we will get pi to be equal to this plus half of Q transpose K Q minus Q transpose F. So with this penalty, the total potential energy of the system would become this. And if this is the potential energy expression, we will again use this condition that we had earlier that del pi by del $Q_i$ will be equal to 0, for minimizing the potential energy.

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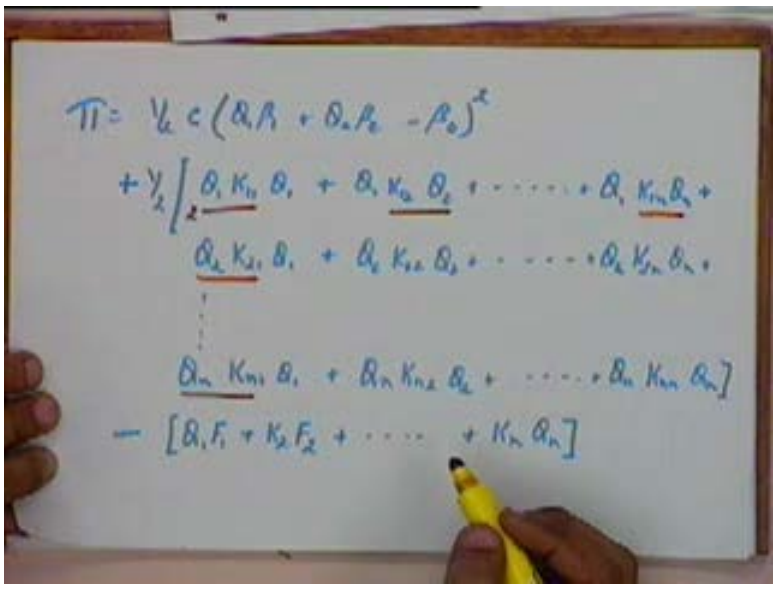

So we will use these constraints and get a system of equations from this potential expression. Let's quickly go through that exercise. Now this expression I will, potential energy pi is half of c times…

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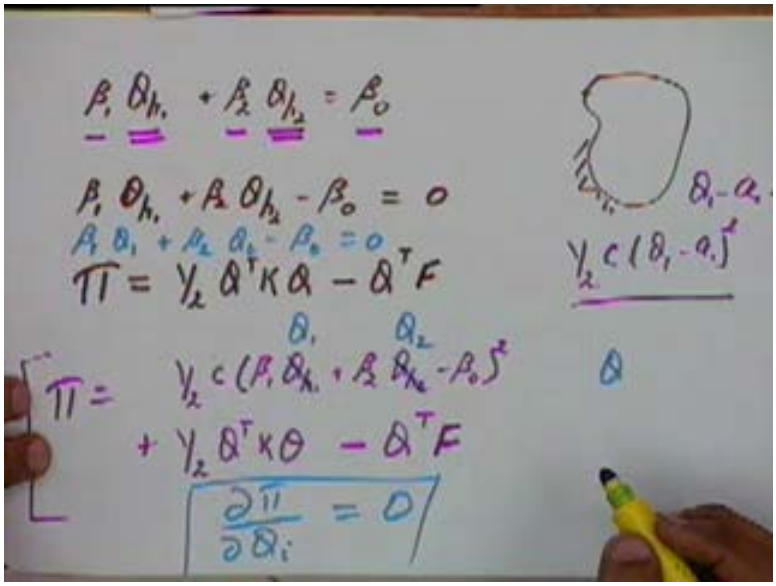


Right now let's take $Q_{p1}$, we will take this expression to be $beta_1$ $Q_1$ plus $beta_2$ $Q_2$ minus $beta_0$ equal to 0. So $p_1$ $p_2$ I am taking $Q_1$ and $Q_2$ directly, just for having simpler expression. So this $Q_{p1}$ I will replace that by $Q_1$ and $Q_{p2}$ I will replace that by $Q_2$. So this will be the half of c times $Q_1$ $beta_1$ plus $Q_2$ $beta_2$ minus $beta_0$ whole squared plus we have half of Q transpose k Q. Again we will write the complete expression that is half of we will get $Q_1$ $K_{11}$ $Q_1$ plus $Q_1$ $K_{12}$ $Q_2$. This expression, this part of the expression is the same as what we had earlier. The only thing we have now is we have a different term for the penalty. This is the penalty we have in this case.

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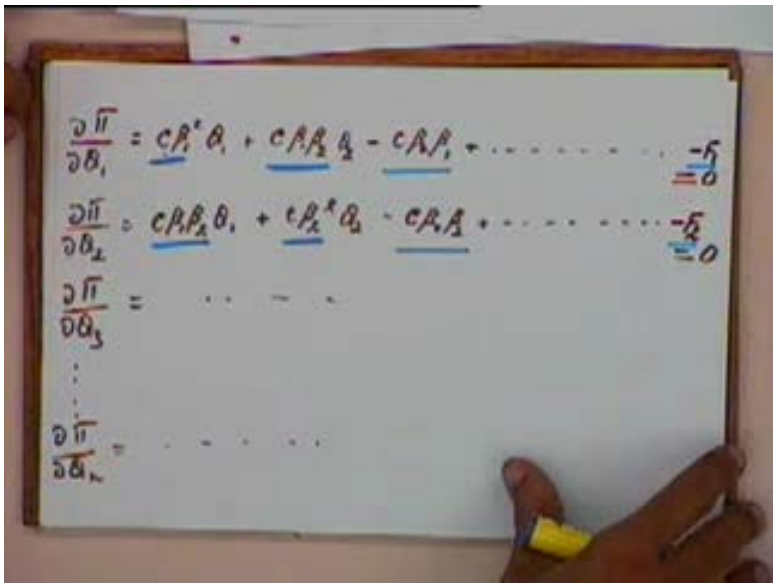

If I take this expression and try to evaluate del pi by del $Q_1$, I am differentiating my potential energy with respect to $Q_1$. So if I differentiate with respect to $Q_1$ I will get two times this thing, this term, this term and from a column I will get this term so on until this term. And if I differentiate this whole thing, I will get c into 2, this whole expression multiplied by $beta_1$. So if I write that down I will get c times $beta_1$ squared into $Q_1$ plus c times $beta_1$ $beta_2$ into $Q_2$ minus c times $beta_0$ $beta_1$ that's this is what I am getting from this, two times this whole thing multiplied by $beta_1$. So 2 into and half will cancel, c into $Q_1$ into $beta_1$ squared plus c into $beta_1$ into $beta_2$ into $Q_2$ that is this term and so on. This is what I will get from this expression plus the term that I will be getting from these expressions would be the same as what I had earlier. There will be no change in those terms.

There will just be a similar term that we had earlier, there will be absolutely no change. And from here also I will get a term of minus $F_1$. The last term here would be minus $F_1$ and this whole thing has to be equal to 0 because del pi by del $Q_1$ has to be 0. Similarly if I take del pi by del $Q_2$, if I differentiate this with respect to $Q_2$ I will get this whole thing multiplied by $beta_2$ that will give me c times $beta_1$ $beta_2$ $Q_1$ plus c times $beta_2$ squared $Q_2$ minus c times $beta_0$ $beta_2$. And the rest of the terms from this would be the same as the terms coming from this column and this row. And this thing finally I will get minus $F_2$ will be equal to 0. I differentiate with respect to $Q_3$, differentiating this with respect to $Q_3$ will not give me anything but when I differentiate the rest of it, I will get terms from the third column and the third row and all these would be the same as what we had earlier. So, all the other derivatives will remain the same.

The only change would be in a differential with respect to $Q_1$ and with respect to $Q_2$. And if you look up these terms, these two are constants and I can combine them with the force terms. This term will get added to the coefficient of $Q_1$, this term will get added to the coefficient of $Q_2$. Similarly in this equation, this term will get added to the coefficient of $Q_1$ and this will get added to the coefficient of $Q_2$.

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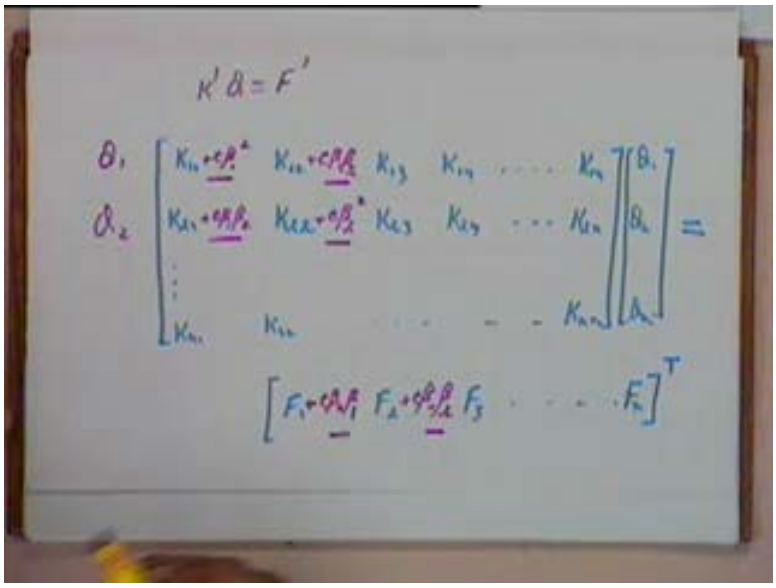

So if I write down this complete equation, complete set of equations in the matrix form, this whole thing will be equal to I will write the force terms here. The coefficient of K $Q_1$ in the first equation I have got this external term of c $beta_1$ square. So here I will add c times $beta_1$ squared. The coefficient of $Q_2$ I have got c $beta_1$ $beta_2$, so here I will add this plus c times $beta_1$ $beta_2$. Similarly if I take up the second equation, coefficient of $Q_1$ has got c times $beta_1$ $beta_2$. So I will add this plus c times $beta_1$ $beta_2$ and the coefficient of $Q_2$ I have got c $beta_2$ square, so this plus c times $beta_2$ squared. And the force terms $F_1$, to $F_1$ I will be adding c $beta_0$ $beta_1$ that will become $F_1$ plus c times $beta_0$ $beta_1$ and the second equation I will add c times $beta_0$ $beta_2$, it will become $F_2$ plus c times $beta_0$ $beta_2$.

So this complete set of equations would become equation of the type k prime Q equal to F prime but the difference is that now when I am evaluating k prime, I will have to evaluate k prime by adding these terms in the locations corresponding to $Q_1$ and $Q_2$. In the first row, in the first row diagonal element I will add c $beta_1$ square, in the location 1 2 I will add c $beta_1$ $beta_2$. In location 2 1 I will add c $beta_1$ $beta_2$ and in the diagonal element for second row I will add c $beta_2$ square and similarly I will modify the force terms by these expressions and then I can solve for the system of equations. Any questions up to this point? So now we know how to solve for single point as well as multi point boundary conditions but one thing that we have not seen so far. Here let's say if you have any finite element problem with some boundary conditions, in order to constrain this body at these boundary, at this boundary some forces have to be applied from the boundary.

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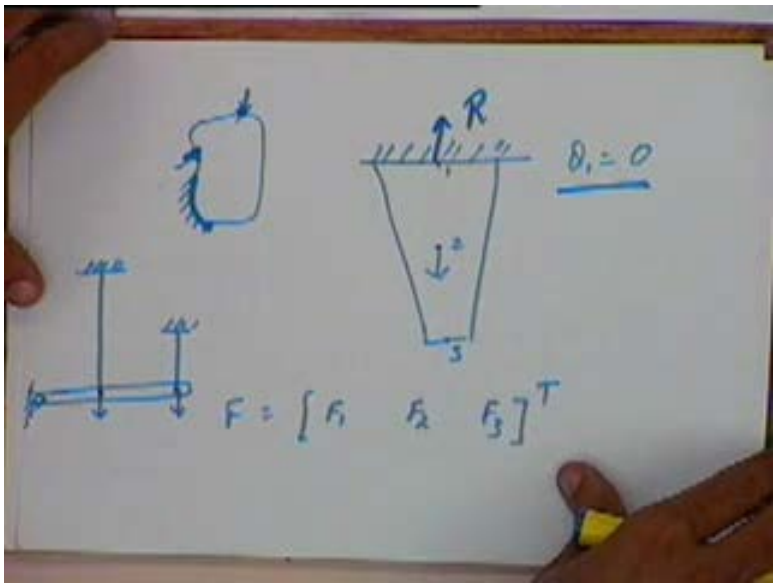


For instance when we were talking of in the simple case, in this case you mention that $Q_1$ is equal to 0 but in order to get $Q_1$ equal to 0, there will be some reaction from here. When we made the first matrix for this body, we had this is say node 1, this was node 2 and this was node 3. We have three terms $F_1$, $F_2$ and $F_3$. In none of these three terms we have taken into account this reaction, where this reaction is also an external force acting on this body. Similarly when we are talking of multi point boundary conditions, some node here and some node here they are constant together. That's possible that there would be some reactions at these two points as a result of which the deformation of these two nodes will be constant in some manner. We haven't yet seen anyway of finding out the value of these reactions.

So let's see how do we evaluate these reactions. The other one dimensional case that I had taken, if I consider this rope and this rope, I am considering this body, this rod to be infinitely stiff. If I consider this rope and this rope, there will be some reaction at this point. The rod will tend to pull the ropes down and we haven't yet derived at those reactions. So whenever you are talking of any kind of boundary conditions, there will be some reactions at each of the nodes. Let's see how do we get the value of these reactions. Let's first see the simple single point boundary condition.

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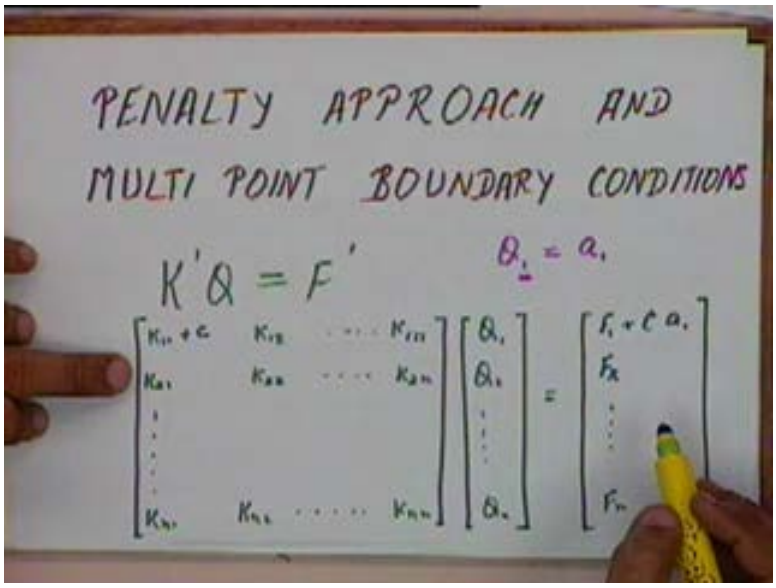


If I consider this single point boundary condition, I am sorry this system of equations the first equation is this row multiplied by this column giving us this term which I can write down as $Q_1$ multiplied by $K_{11}$ plus c plus $Q_2$ $K_{12}$ so on till $Q_n$ $K_{1n}$ and is equal to F plus c times $a_1$.

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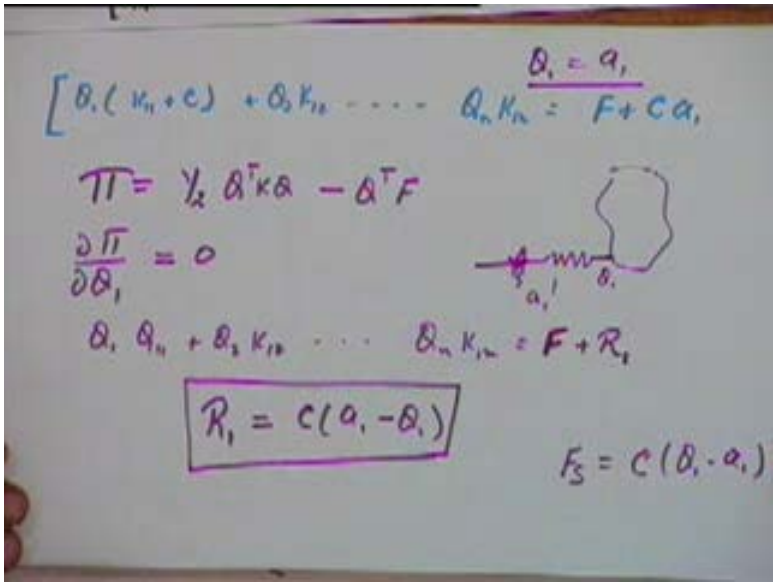


In contrast to this if you look at, if you just look at the potential energy expression that we had earlier half of Q transpose K Q minus Q transpose F and I differentiate this with respect to Q sorry with respect to $Q_1$, the first equation that I will get will be if I put this and the first equation would be $Q_1$ $Q_{11}$ plus $Q_2$ $K_{12}$ $Q_n$ $K_{1n}$ will be equal to F.

And on the node one, we are saying that there will be some reaction. So the actual force acting here will be other external forces plus the reaction force. So if I compare these two, what I will basically what I will get will be at this reaction force at the node one will be equal to c times $a_1$ minus $Q_1$. Is that okay? c times $a_1$ minus $Q_1$ and if you remember the formulation at this node $Q_1$, we had added a spring like this and we said that this end will be given a deflection of $a_1$. As a result of that this spring will have a deflection of $Q_1$ minus $a_1$. So this force, the force in the spring will be equal to c which is the stiffness of the spring multiplied by $Q_1$ minus $a_1$.

The force of the spring will be equal to the reaction and magnitude. If you consider this node at this point, whatever reaction you are having from here that will be equal to the force in the spring which is exactly what this relationship is telling us or if you just consider the magnitude, you can say c times $Q_1$ minus $a_1$. So the reaction would be c times $Q_1$ minus $a_1$ that is for single point boundary condition of the type $Q_1$ is equal to $a_1$ and when we are using the penalty approach.

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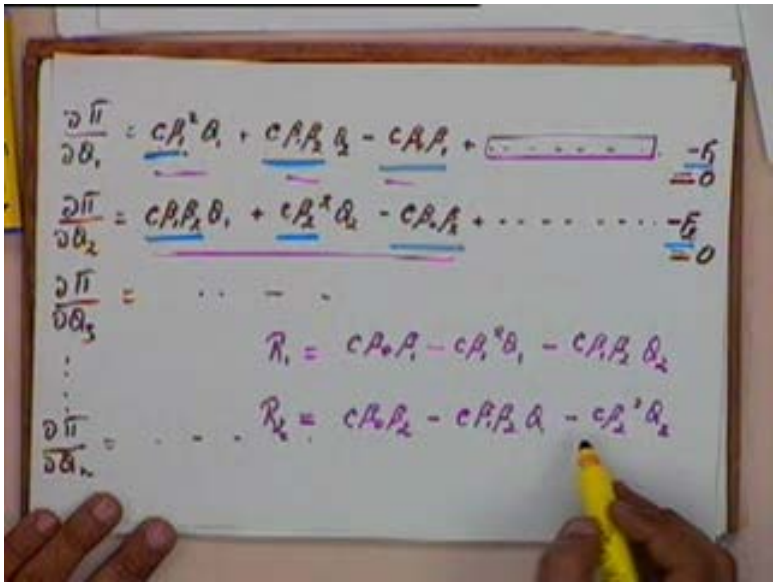

Similarly if you consider multi point boundary conditions, if you consider this system of equations, the reaction force, at the from this first equation that you will get that I will just write it here itself $R_1$ that should be equal to these three terms put together which will be c times, I have to take this to the right hand side. So c times $beta_0$ $beta_1$ minus c times $beta_1$ square times $Q_1$ minus c times $beta_1$ $beta_2$ times $Q_2$. And similarly the reaction at the node two would be c times $beta_0$ $beta_2$ minus c times $beta_1$ $beta_2$ $Q_1$ minus c times $beta_2$ squared $Q_2$. This would be the reaction at the second node, this would be the reaction at the first node because these term would remain unchanged. These three if I take them to the right hand side, I shall get my reaction terms.

Similarly if I take these three to the right hand side along with $F_2$, I will get my reaction terms. So reaction from the boundary would be given by these two expressions. Is that okay? Any questions up to this point?

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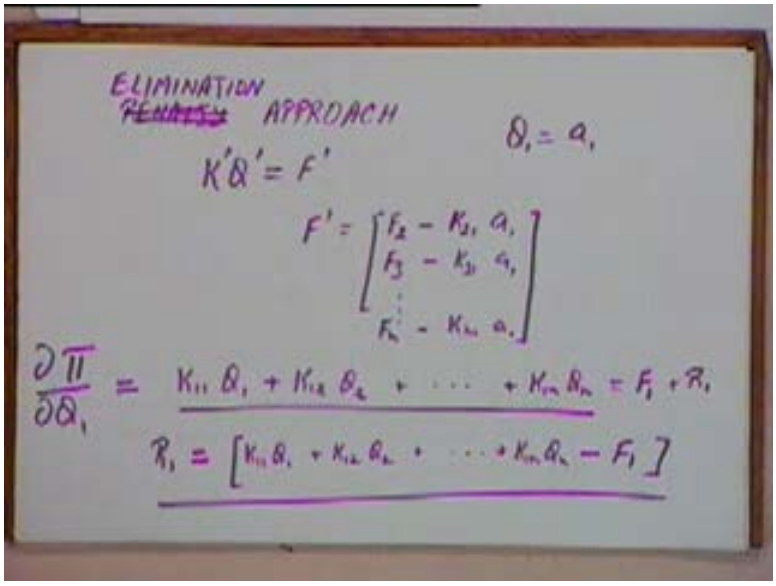


The only thing that now remains in this is when you are using the penalty approach, how do you find out the reactions. If you remember in the penalty approach we had equations given as K prime Q prime equal to F prime. If you have a constraint of the type $Q_1$ is equal to $a_1$, K prime was obtained by deleting the first row sorry not penalty elimination. If I use the method of elimination, if I delete the first row and the first column in the K matrix I get K prime. Q prime was obtained by deleting the first row and F prime was also obtained by deleting the first element from there. And essentially you are having F prime to be equal to $F_2$ minus $K_{21}$ $a_1$, this is $F_3$ minus $K_{31}$ $a_1$ and so on. $F_n$ minus $K_{n1}$ $a_1$, this was F prime. And K prime and Q prime are obtained just by deleting the elements.

How do we find the reaction in this case? So again if we look at the first equation, if I take my potential energy expression, differentiate that with respect to $Q_1$, I will get $K_{11}$ $Q_1$ plus $K_{12}$ $Q_2$ up to $K_{1n}$ $Q_n$ will be equal to the force at node one. And the force at this node one will be actually the external forces $F_1$ plus the reaction $R_1$. So from this equation I can directly say $R_1$ will be equal to this whole expression that is $K_{11}$ $Q_1$ plus $K_{12}$ $Q_2$ $K_{1n}$ $Q_n$ minus $F_1$. So this will be the expression for finding out the reaction at node one.

Once I have solved this system of equation, the node values of $Q_1$ $Q_2$ till $Q_n$ all the variables will be known and we can find out the reactions by this expression. This is in the elimination method. And for multi point boundary conditions, elimination method cannot be used. Multi point boundary conditions can be solved only by the penalty approach. Any questions with respect to either the elimination method or the penalty approach and single point and multi-point boundary conditions? Then no more conditions, no more questions I will stop now. What we will do is in the next class, we will go onto what are called as quadratic elements. So far we were having only linear elements, two nodded linear elements. We will go onto quadratic shape functions and we will see how they can be solved.