Computer Aided Design
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Lecture No. # 20
FE Problems: Solving for Q

We have seen how to derive the potential energy expression for one dimensional finite elements. Today what we will see is if we have the expression for the potential energy, how do we use that to solve Q that is the displacements.

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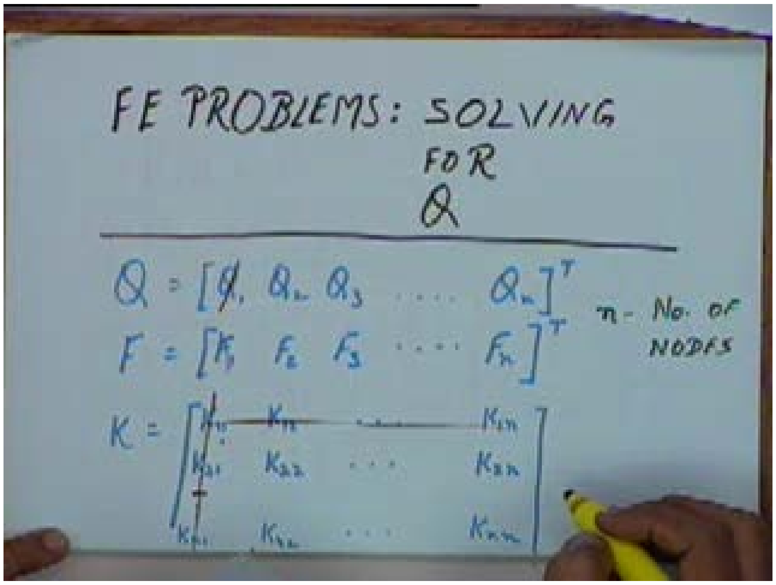

We know that once I have the displacements, I can calculate the stresses, strains and the variation of stresses and strains in the body. So we have seen that, we have been representing Q by a vector which is given by Q_1 Q_2 till Q_n if there are n nodes in the body. The n is the number of nodes, a number of nodes and F which is a force vector which is F_1 F_2 F_3 till F_n both these are global vectors or the global matrices. Q is a global displacement matrix, F is a global force matrix and k is the global stiffness matrix. This global stiffness matrix will be n by n matrix and the terms of these matrix, I am writing them as K_{11} K_{12} till K_{1n} in the first row and so on. So this is a global stiffness matrix.

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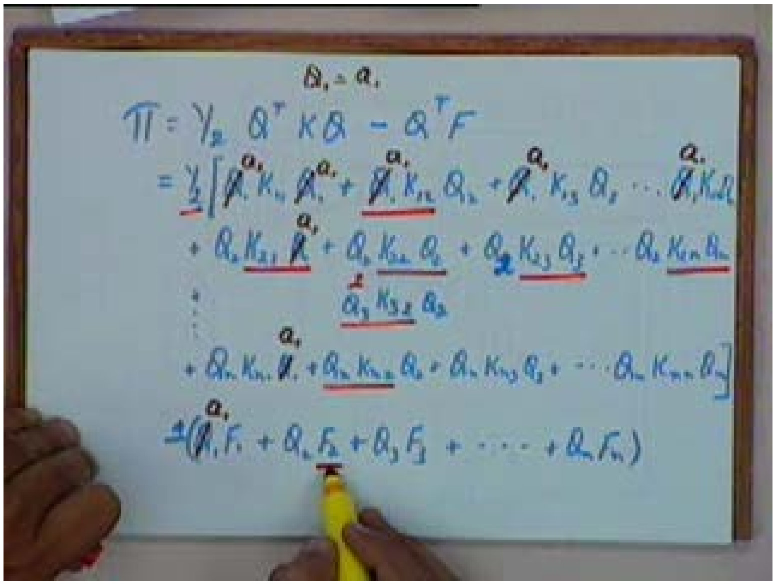

And we have seen that the potential energy pi is equal to half of Q transpose KQ minus Q transpose F. If I expand out this term, what I will get will be an expression of this type. Let's say half, let's say I will put this in brackets Q_1 K_{11} Q_1 plus Q_1 K_{12} Q_2 plus Q_1 K_{13} Q_3 so on up to Q_1 K_{1n} Q_n. That is a first row plus it will become Q_2 K_{21} Q_1 plus Q_2 K_{22} Q_2 plus Q_3 K_{23} Q_3 so on up to Q_2 K_{2n} Q_n plus we will have other terms Q_n K_{n1} Q_1 plus Q_n K_{n2} Q_2 plus Q_n K_{n3} Q_3 so on Q_n K_n and Q_n. This whole thing minus Q transpose times F that will give us Q_1 F_1 plus Q_2 F_2 plus Q_3 F_3 so on up to… This you should be able to see easily where Q transpose is a row vector consisting of Q_1 Q_2 Q_3 Q_n, Q is a column vector and K is a n by n matrix, just look at this.

So if I multiply Q transpose which is a row vector by this matrix, the first term that I will get will be Q_1 multiplied by K_{11} plus Q_2 multiplied by K_{21} and so on. So this row multiplied by this column that will give the first term and that term will then be multiplied by Q_1 when I multiplied by Q. Let's look at that step carefully.

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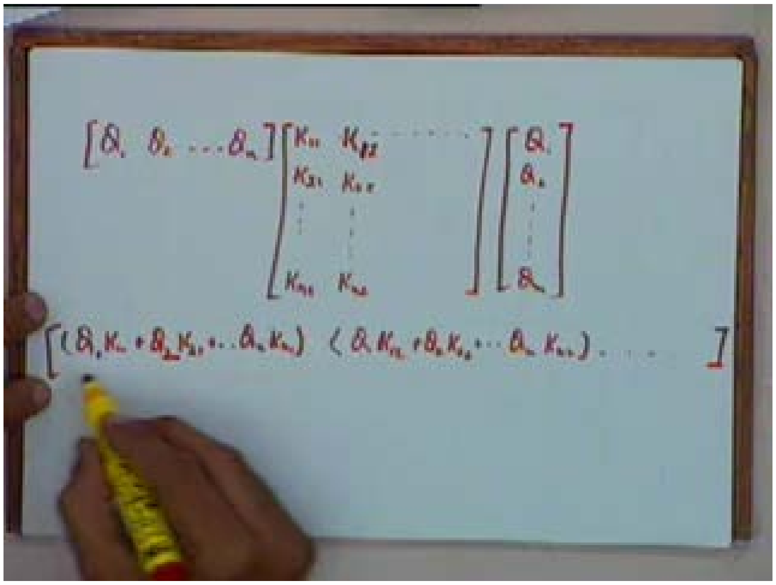

This is the expression we have to evaluate. If I take this row and multiplied by this column, the first term that I will get would be Q_1 multiplied by K_{11} plus Q_2 multiplied by K_{21} plus so on up to Q_n multiplied by K_{n1}. Similarly if I take, this two are multiplied by the second column, I will get the terms Q_1 multiplied by K_{12} plus Q_2 multiplied by K_{22} so on up to Q_n multiplied by K_{n2} and I will get a row of terms like this. If I take this row of term and multiply it by this column, this set of term is multiplied by Q_1 plus this set of term is multiplied by Q_2 so on till the last term will be multiplied by Q_n.

If I multiply this set of terms with Q_1, I will get this first line that I have written. This set of terms are multiplying by Q_1, I will get Q_1 K_{11} multiplied by Q_1, Q_2 K_{21} multiplied by Q_1 which is K_1 K_{12} and K_{21} are the same so it's a same thing, actually it is this column. Q_n K_{n1} multiplied by Q_1 which is this or this, all these terms are symmetric. So it doesn't make a difference. We have already mentioned that this K matrix is a symmetric matrix. So this extra term half of Q transpose KQ can be written in this manner and Q transpose F that will give us this expression. Is that okay?

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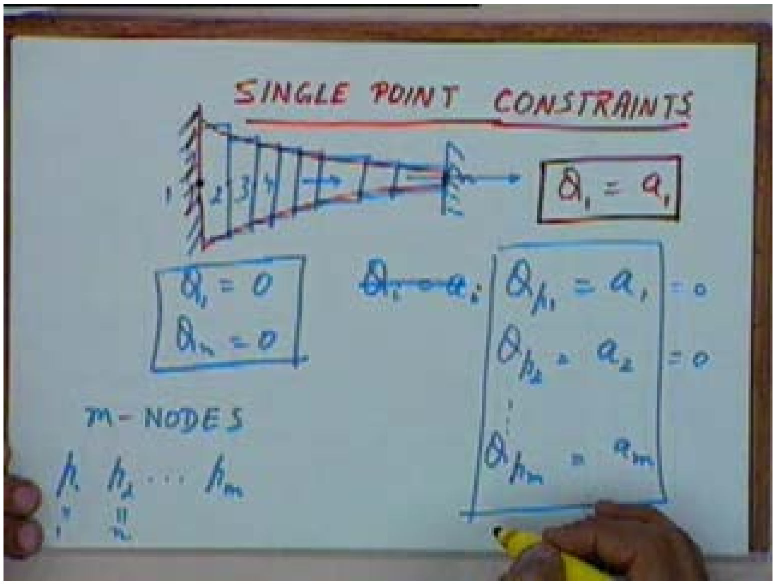

Now if you take any actual problem and this has been divided into a set of finite elements, typically you have some boundary conditions. Let's say this is my node number 1, this is my node number 2, this is 3, 4 and so on up to n. Let's say in this case I have some loading, I am taking a one dimensional loadings or some loads acting in this direction. So if I take this particular problem, a boundary conditions typically would be out of form let's say Q_1 is equal to 0 or if let's say even this end is fixed, I might also have a constraint like Q_n is equal to 0. So let's say we consider boundary conditions of the type with some Q_i is equal to a_i. Meaning, this a_i in this particular case is 0. In some cases we might give it a fixed displacement at one end.

Therefore we will consider boundary conditions of the type Q_i is equal to a_i or instead of Q_i equal to a_i, we will write it as Q of p_1 is equal to a_1, Q of p_2 is equal to a_2 and so on. Let's say Q of p_m is equal to a_m. What I am basically saying is that I have m nodes which are constraint and these node numbers are p_1 p_2 till p_m. So I have got m nodes and these m nodes are constraint by displacements a_1 a_2 a_m. So in this case I have Q_1 equal to 0 and may be Q_n equal to 0. So p_1 will be equal to 1, p_2 will be equal to n, a_1 will be equal to 0 and a_2 will be equal to 0 that will give me these two constraints. So if I consider constraints of this type where some particular node number is constrained to have a fixed displacement.

These type of constraints what we refer to as single point constraints. We are talking of let's say if you have any problem that will have some boundary conditions. We are trying to formulate these boundary conditions. I am saying that right now I will consider boundary conditions which are of single point type that means some particular node is constrained to have a fixed displacement. Value of then. The value of the displacement that we are looking for. That is known at. That is known at some particular nodes because unless there are some boundary conditions, the system cannot be solved. If I have a body which is not constraint and I apply some force to it, it will start moving.

So it has to be constrained in some way. It is constrained means the nodes which are constraint will have fixed displacement. So we are saying that we consider some of these nodes which will have fixed displacement of this type. That is Q p_1 is a_1, Q p_2 is a_2 and so on. So, now in this expression or using this expression of potential energy, this long expression we will see how we can incorporate these single point constraints and then solve the system for finding out the displacements.

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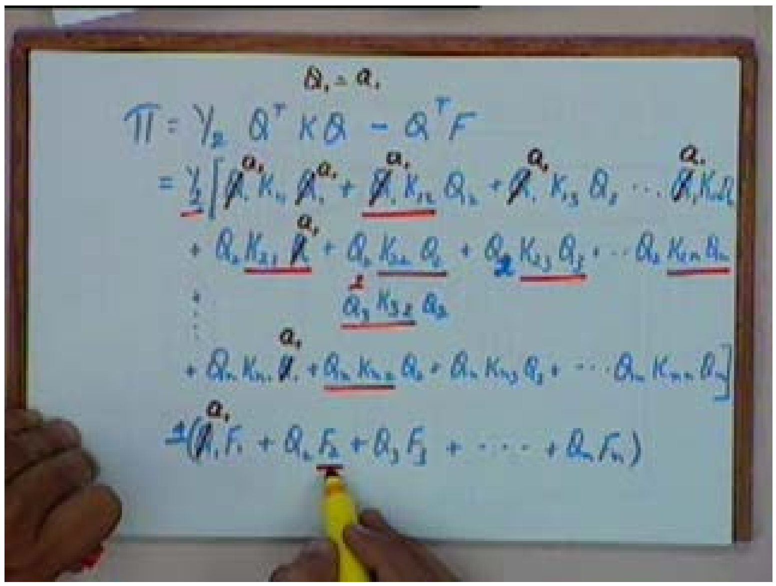

Now we have boundary condition of this type. Let's take as an example, a boundary condition which is Q_1 is equal to a_1 that means the node number 1 has a fixed displacement of a_1. If I consider this boundary condition Q_1 is equal to a_1, we will see how we can solve the system of equation or how we can solve the system for the displacement if this is the boundary condition given to us.

Once we know how to solve the system at one constraint, we can easily generalize it to having a multiple constraints. So we will establish by method for solving the system with one constraint of the type Q_1 equal to a_1. Now if we look at this expression and I add the constraint that Q_1 is equal to a_1, this potential energy expression I can replace all Q_1's by a_1's because my constraint is Q_1 is equal to a_1. So let's say in this I will just change it over here, all Q_1's I will change them to a_1's. If we notice Q_1 will appear only in the first row and in the first column, Q_1 will not appear anywhere else.

Similarly Q_2 will appear only in the second column and in the second row, Q_2 will not appear anywhere else, Q_3 will appear in the third column and in the third row. So when we are replacing Q_1 in the first row and in the first column, we will replace all occurrences of Q_1 by a_1 and this is the expression we have for potential energy, when the constraint is Q_1 is equal to a_1. Now we will make use of fact that we have stated earlier that, we have mentioned earlier that if we have a potential energy expression, this is the potential energy expression we have.

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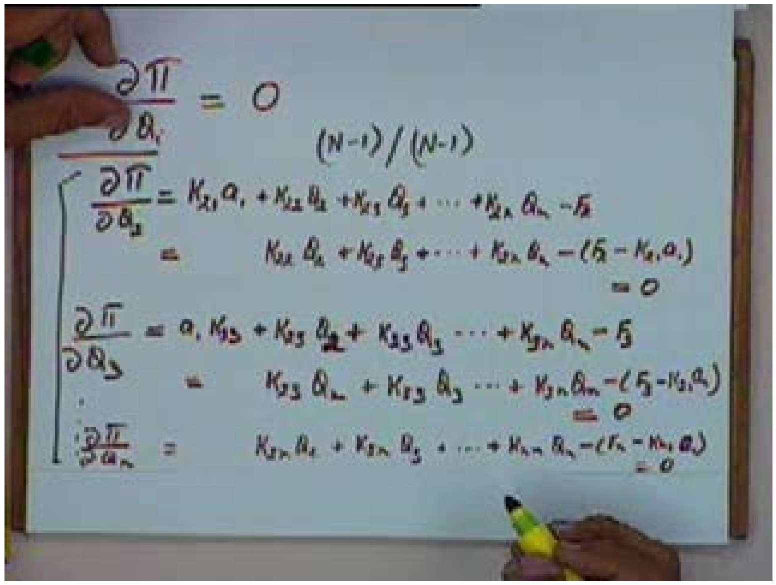

We have mentioned earlier that in equilibrium del pi by del Q_i will be equal to 0, Q_i are the different parameters that we have. In fact when we stated the Rayleigh Ritz method, we mentioned that the partial derivative of the potential energy with respect to each of the parameters should be equal to 0. So if we use this constraint, the first expression that we will get Q_1 in this case is a constraint though I said Q_1 is equal to a_1. I said Q_1 is equal to a_1, so first derivative will be with respect to Q_2, del pi by del Q_2 and if I have to differentiate with respect to Q_2 let's look at this expression. As I said Q_2 will appear in the second column and in the second row, if I differentiate this term with respect to Q_2 I will get these two terms. If I differentiate this with respect to Q_2 I will get these two terms multiplied by 2. If I differentiate this a sorry two times Q_2 (Refer Slide Time: 18:27). If I differentiate this with respect to Q_2 I will get this term, if I differentiate this with respect to Q_2 I will get this. If I look at the second column, if I differentiate this with respect to Q_2 I will get this.

Similarly the next time that I will have over here will contain that would be K_{32}, that will be multiplied by Q_3 and Q_2. So when I differentiate this with respect to Q_2, I will get this term. If I differentiate this, I will get this term and all this is multiplied by half and Q_2 is also appearing in the first term, if I differentiate this with respect to Q_2 I will get this term. Now from symmetry I know that Q_{12} is as same as K_{21}, K_{23} is same as K_{32} and K_{2n} is the same as K_{n2}. So this column is the same as this row, so you basically take two times the row and half of that will again cancel out. So when I differentiate this with respect to Q_2, I will basically get this one signal row of the term that I have underlined. Student: That K_{22} is not repeated. I have a factor of 2 anyway, that will get canceled out. So basically I get this term plus this term plus this term plus this term and so on minus I will get F_2. So what will I get? The first term is K_{21} multiplied by a_1 so I will get this will be equal to K_{21} multiplied by a_1. The second term is K_{22} Q_2 half of that, so plus K_{22} times Q_2. This will be K_{23} times Q_3 will be plus K_{23} times Q_3 so on up to plus K_{2n} times Q_n.

This is this last term that I have minus F_2, I have said minus F_2 or I will write this as, I will see this K_{21} times a_1 is a constraint and so is F_2. The only variable here is Q_2 Q_3 till Q_n, so I will write this as K_{22} times Q_2 plus K_{23} times Q_3. Student: Qn is a_2. Again, Q_n is $a_{2.}$ Q_n is a_2. What is a_2? Student: No constraint on this side. I have taken only one constraint right now that Q_1 is equal to a_1. If I have two constraints I will generalize this thing later on. If I am taking only one constraint that Q_1 is equal to a_1. K_{22} times Q_2 plus K_{23} times Q_3, this will go on up to K_{2n} times Q_n and this minus I will say F_2 minus K_{21} a_1 and this is del pi by del Q_2 and this has to be equal to 0.

Similarly if I find del pi by del Q_3, I will carry out the differentiation with respect to Q_3 now. When I differentiate with respect to Q_3 I will get this term, I will get this term so on, I will get a column of terms from here plus I will get a row of terms from here and I will get this term. If I simplify them what I will get finally because this will give me a_1 K_{13} or if I am writing K_2 let me write, a_1 K_{13} plus K_{23} times Q_3 sorry K_{23} times Q_2. Q_2 plus K_{33} times Q_3 so on up to plus K_{3n} times Q_n minus F_3. And again a_1 K_{13} I will take it along with F_3, so this will be equal to K_{23} times Q_2 plus K_{33} times Q_3 so on plus K_{3n} times Q_n minus F_3 minus K_{31} a_1; K_{13} and K_{31} are the same. I can write either, it doesn't make difference and this again has to be equal to 0.

This way I can write down a system of equations, this is del pi del Q_3. My last term would be del pi by del Q_n and that I will get a similar expression which will give me K_{2n} times Q_2 plus K_{3n} times Q_3 so on till plus K_{nn} times Q_n minus F_n minus K_{n1} times a_1 will be equal to 0. So this way I now have a system of equations. In this system of equations has got n minus 1 equations in it and it has n minus 1 variables. Initially we had n nodes out of those n nodes I have constrained one, so my variables are Q_2 Q_3 till Q_n which are n minus 1 variables and a differentiation has been with respect to Q_2 Q_3 till Q_n, this is again n minus 1 equations. So I have n minus 1 linear equations or n minus 1 variable, I can solve them out directly. Student: Sir, but then what was the need for the constraint because if we wouldn't have a constraint, you would have at n equation n variables.

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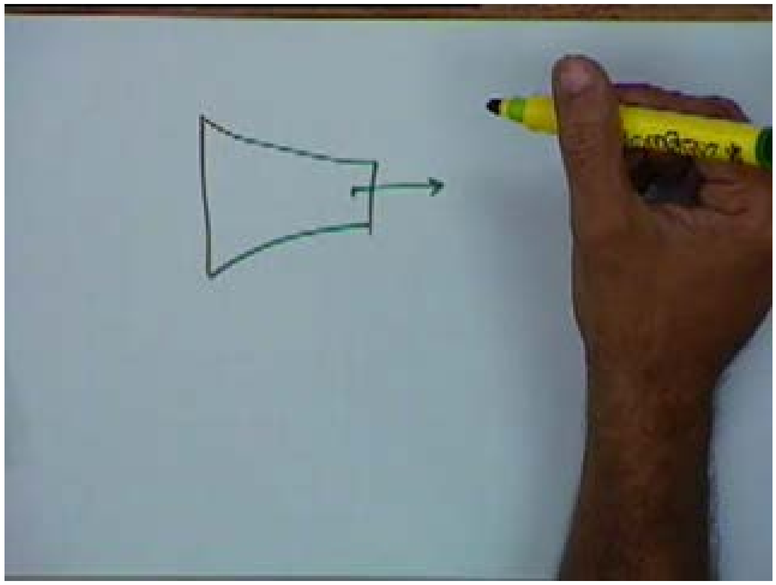

Then you would find out the equations won't be independent and that you can see very easily. If I take any object like this and I give it a force, the object is going to move. If the object is going to move, the deformations they are all the same. That is going to be a trivial solution, just one minute that is going to give me a trivial solution. So, if we try to differentiate and we will get the system of equations to be dependent. They won't be independent on one another. You were saying something? If you have n equation and n variables and the equation is independent and you will always get a unique solution, so only time when you don't get unique solution is when the systems are dependent.

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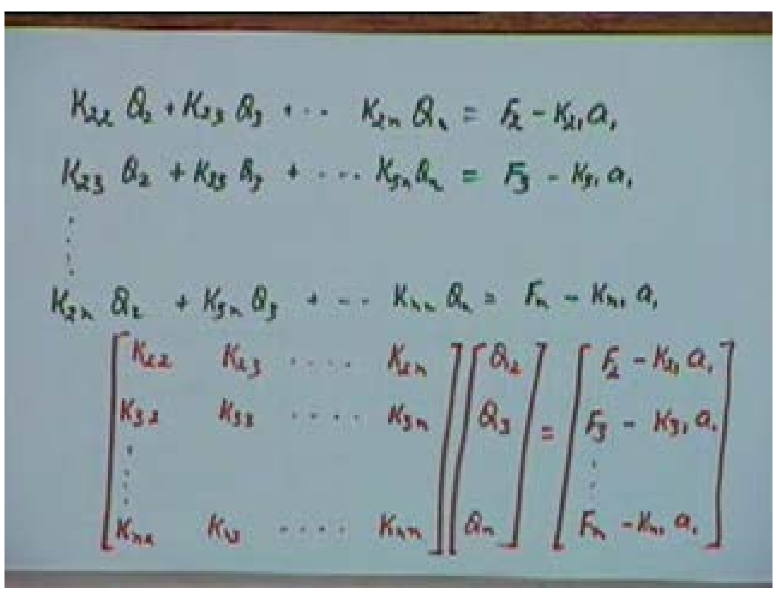

So coming back to these equations, now we got a system of n minus 1 equation. I will just rewrite these equations, this equation I can write that as the first one K_{22} Q_2 plus K_{23} Q_3 so on till K_{2n} Q_n is equal to F_2 minus K_{21} a_1. The second equation that we have is K_{23} Q_2 plus K_{33} Q_3 so on till K_{3n} Q_n is equal to F_3 minus K_{31} a_1 and the last in the series would be K_{2n} Q_2 plus K_{3n} Q_3 so on till K_{nn} Q_n will be equal to F_n minus K_{n1} a_1. Basically I have written this equation, this equation and I just rewritten these equations again. Now I will write this again in a matrix form and this system of equations this can written as, so this system of equations can be written in a matrix form like this. This is the matrix of the K coefficient that I have K_{22} K_{23} so on till K_{2n} and the n equation. The same I have repeated here, multiplied by these deformations Q_2 Q_3 till Q_n that is the column vector I have taken and this will be equal to this right hand side of the equations. Now if you compare this matrix with the stiffness, global stiffness matrix that we had with this global stiffness matrix, you will find that these terms are all the same as these terms except that the first column and the first row have been deleted.

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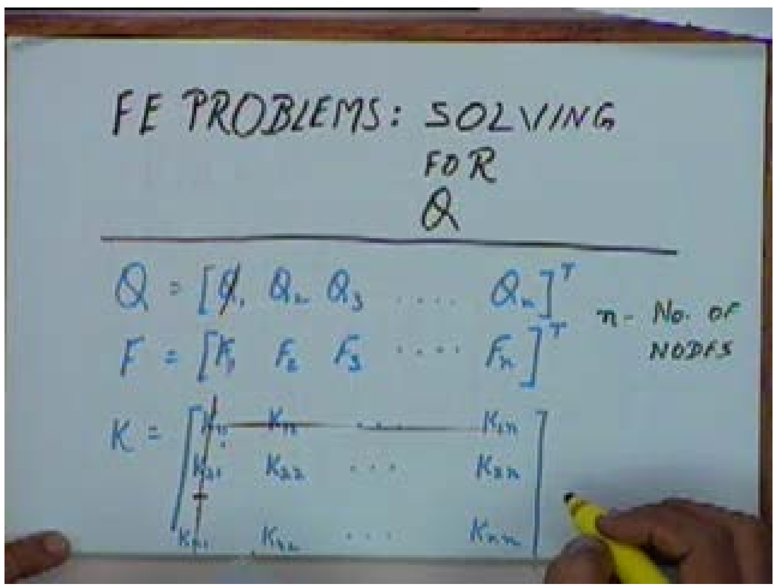

Q_2 to Q_n is same as the Q vector except that Q_1 has been deleted and this force vector is as same as this force vector except that F_1 has been deleted and these terms have been subtracted. What has been subtracted? I have taken the first column of the stiffness matrix, multiplied it by a_1 and subtracted it from each of these terms. K_{21} multiply it by a_1 that is this term K_{31} multiplied by a_1 will be the term here, K_{n1} multiplied by a_1 is a term here. So these individual terms are subtracted from F_2 F_3 so on till F_n.

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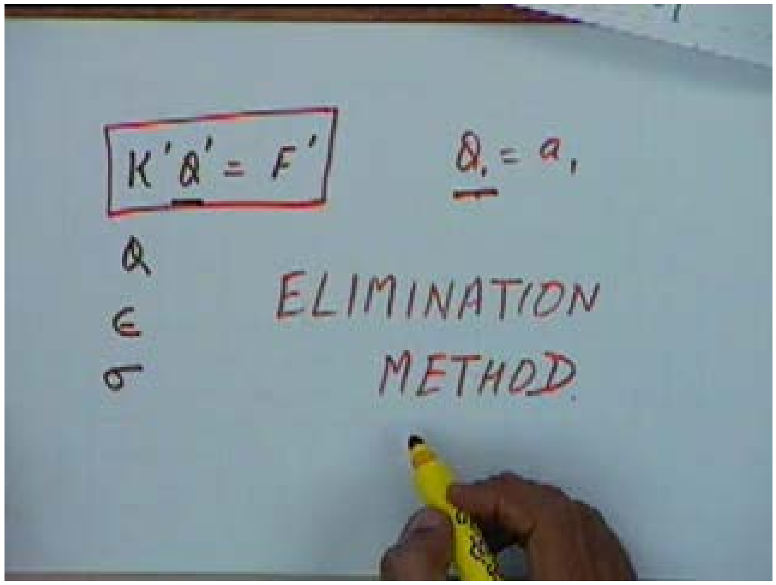

So this system of equations you write them as we will say K prime Q prime will be equal to F prime where K prime has been obtained by taking the stiffness matrix and deleting the first column and the first row. Q prime has been obtained by deleting Q_1 and F prime has been obtained by deleting F_1 and subtracting this column multiplied by a_1 from the other four terms. And this system of equations K prime Q prime is equal to F prime, this we can solve using any method for solving a system of linearly equations.

Once I solve this, I will get the values of Q prime I mean value of the terms in Q prime and then once I know the Q terms, I can find out epsilon and I can find out sigma. So the basic method of solving is that we will first establish the system of, we will first establish the finite element problem by specifying QF and K. We will get the global matrices, add the boundary conditions, get these modified matrices. This is a modified stiffness matrix, I have modified it by removing the first row and the first column. My constraint was Q_1 is equal to a_1, since Q_1 was constraint I deleted the first row and the first column and I got K prime.

I got Q prime by removing Q_1 and I got F prime by modifying F. I modify the matrices KQ and F and I get this system of equations which I can solve to get the different values of Q. So this is how we can solve the finite element problem or one dimensional finite element problem. Any questions up to this point? This method is called elimination method. This is called the elimination method because we are eliminating one row and one column and corresponding terms from Q and F. We initially had n by n matrix in Q in K, we eliminated some terms out of it, we eliminated Q_1, the constraint variable Q_1 is being eliminated. That is why we are saying it as elimination method. This is the elimination method for solving for Q. Any questions up to this point? Then let's take up the same example that we had earlier.

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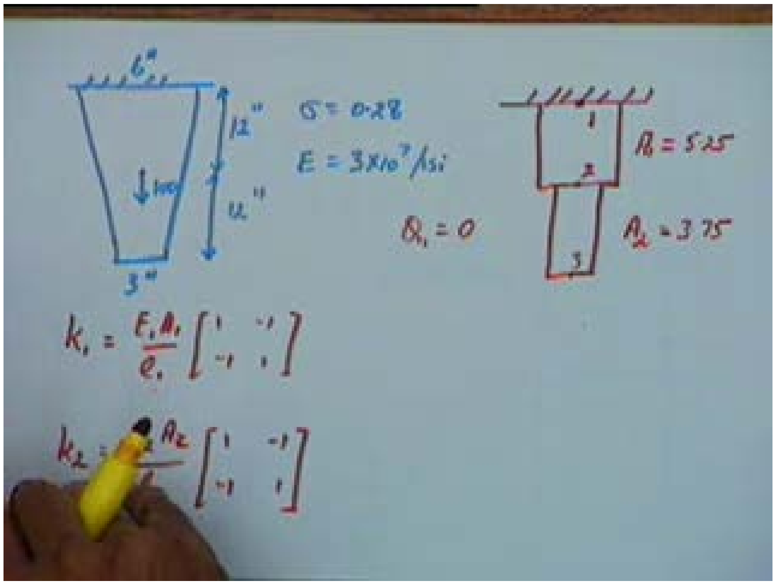

That is in this dimension was 6 inches, this was 3 inches and the load here was 100 and if I remember correctly the density was given as 0.28 and E was given as 3 into 10 to the power 7 psi and this we had formulated as a FE problem by taking 2 elements like this. For the first element a_1 the area of cross section we said would be 5.25. For second element the area of cross section a_2 would be 3.75 square inches and we had found out the local stiffness, the element stiffness matrices K_1 K_2 as E_1 A_1 by l_1 into 1 minus 1 minus 1 1 and this was similarly E_2 A_2 by l_2 multiplied by 1 minus 1 minus 1 1. The constraint this is node number 1, this is node number 2, this is node number 3.

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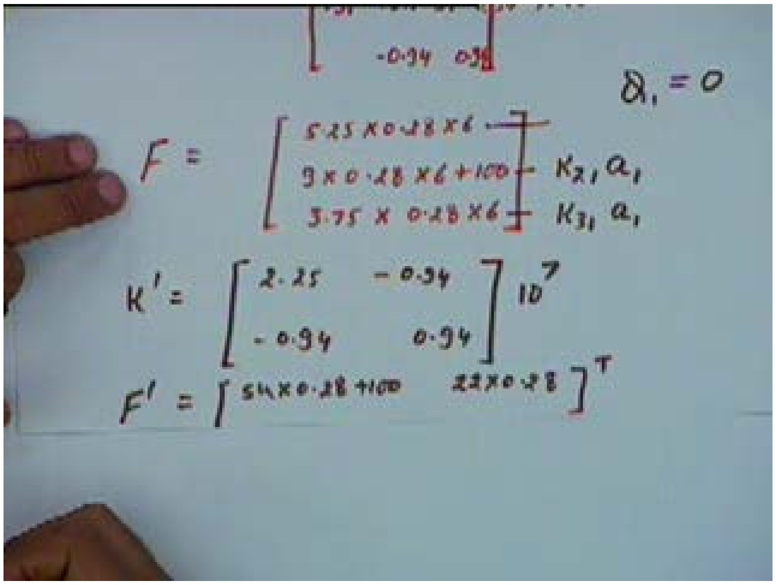

My constraint in this case says Q_1 will be equal to 0 and I will also derive the global stiffness matrix K as 3 by 3 matrix. And thus you can checkup the figures. I think the figures are something like this, I might be wrong at that you can correct me. And the global force matrix F, I think was this is the global force matrix. I think this is multiplied by 10 to the power 7 and now the constraint that we have is Q_1 is equal to 0. If I add this constraint that Q_1 is equal to 0 by the elimination method, what I will do is I have to remove the first column and the first row from the K matrix. So corresponding to that my K prime will be these 4 terms and this would give me 2.25 minus 0.94 minus 0.94 and 0.94 multiplied by 10 to the power 7.

My F prime would be this matrix F with the first term deleted and from the other two terms, I have to subtract. What do I have to subtract? a_1 multiplied by K_{21}, K_{21} multiplied by a_1, here I will have to subtract K_{31} multiplied by a_1 but a_1 is 0. So my F prime will be equal to these two terms that is 54 into 0.28 plus 100 and this would be 22 into 0.28 transpose. So this will be F prime and Q prime is of course Q_2 Q_3. So my system of equations will now become K prime Q prime is equal to F prime where K prime and F prime are given by this and of course Q prime is equal to Q_2 Q_3.

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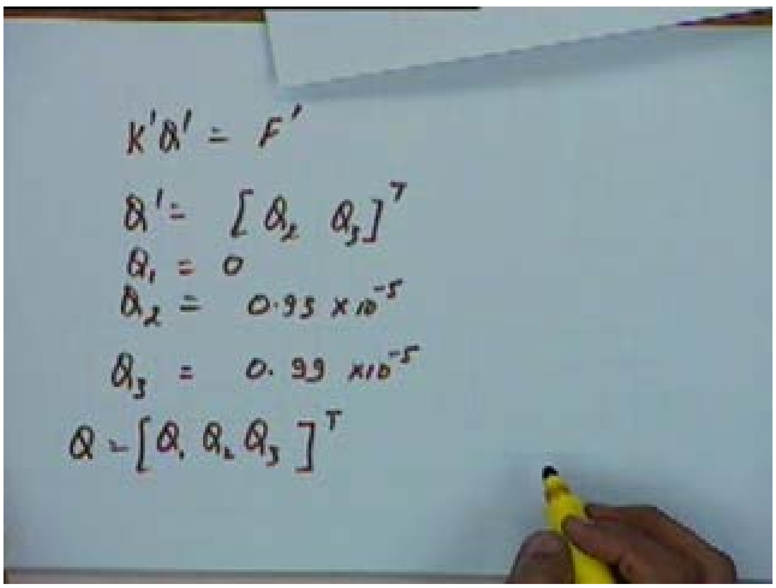

So now I have a system of equations, the two variables two equations I can easily solve it out and I think if you solve it, we get values like Q_2 is equal to 0.93 into 10 to the power minus inches and Q_3 is equal to 0.99 in to 10 to the power minus 5 inches. This means that Q_2 is a deformation here. A deformation here is 0.93 into 10 to the power minus 5 and deformation here is 0.99 into 10 to the power minus 5 inches. These are Q_2 Q_3 and of course Q_1 is equal to 0. So now my Q vector which is Q_1 Q_2 Q_3 transpose is these 3 terms. Q_1 is 0, Q_2 is this much, Q_3 is this much but if I have to find out epsilon that is the strain, the epsilon in the first element is given by B times q.

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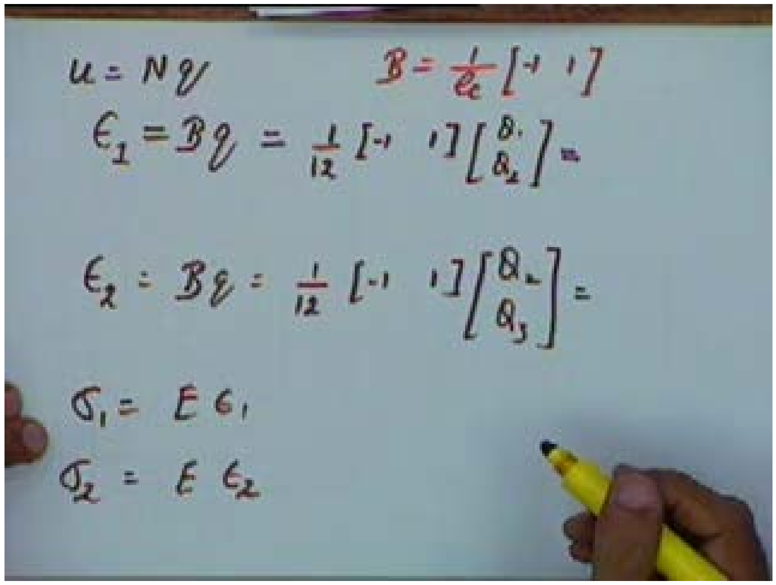

If you remember your earlier formulation, we had said u is equal to N times q and epsilon is equal to B times q. And what is B? B is 1 over length of the element multiplied by minus 1 1, this is equal to B. So B in this case, a length of the element is 12 inches, so B would be 1 by 12 into minus 1 1 multiplied by Q. The q matrix for the first element will consist of Q_1 Q_2. The deformation of the first node is Q_1, deformation of the second node is Q_2.

So this multiplied by Q_1 Q_2 and this we can find out what that would be. We know Q_1 is 0 and Q_2 is 0.93 into 10 to the power minus 5. So from this you can get the value of epsilon in the first element. Similarly epsilon in the second element will be B times q, this will be 1 over 12 into minus 1 1 multiplied by Q_2 Q_3 and Q_2 and Q_3 are again obtained from here. So I can get the value of epsilon in the second element. I get epsilon in the second element. Once I know the two epsilon, I can get sigma_1 to be equal to E times epsilon_1 and sigma_2 to be equal to E times epsilon_2.

I can get the stresses and the strain in both the elements and we have already mentioned that in the beginning that the stresses within the element will be constant. The strains within the element will be constant because we have assumed that there is no variation of strains within the element. These elements are constant strain elements. So the strain within this element will be constant and will be at the level of epsilon_1. The strain within this element will be constant and at the level of epsilon_2. So this is how we can solve a problem like this for finding out the deformations, stresses and strains.

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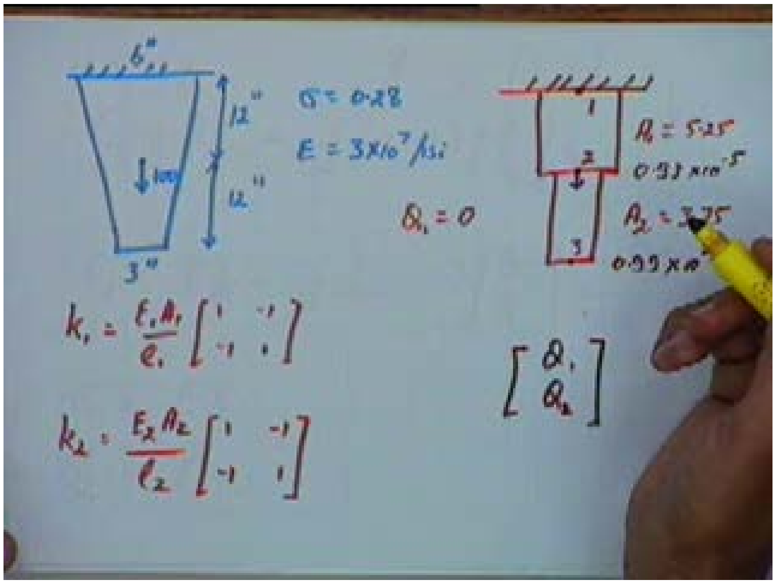

The basic thing we do is formulated as a finite element problem, find out the elemental stiffness matrices and the force matrices, assemble the global matrices and then apply the boundary conditions and solve, use the elimination method to find out the values of Q. Once we know Q, we can go back and calculate epsilons that is the strains and the stresses. What's your question? See basically what we are saying is that if we take the elements small enough, the assumption will be more accurate. Student: How do we decide whether it is small enough? How do we decide whether it is small enough? Typically you are deciding whether it is small enough. One way is that you keep adjusting your element size to see whether you have reached the certain level of accuracy or what should do is if in a certain area, you expect that the stresses would be very high then you take a smaller element size.

If you take a smaller element size, your assumption will be more accurate. The areas where you don't expect very high stresses, you don't want very accurate estimate of stresses and strains there. You can take larger element size but whatever the element size be, the assumption will be 100% accurate. We have method just estimating the error. We try to estimate how much is the error that has been introduced, we won't go into that at the moment. Any other questions? In that case I will stop now. In the next class we will see another method of solving the same system of equation that is a penalty approach and then we will go onto the Galerkin's method.