Computer Aided Design
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Lecture No. # 19
1 D Finite Element Problems (Contd.)

Today we will be talking of the one dimensional finite element problems and in the last class we had seen how to get this potential energy expression.

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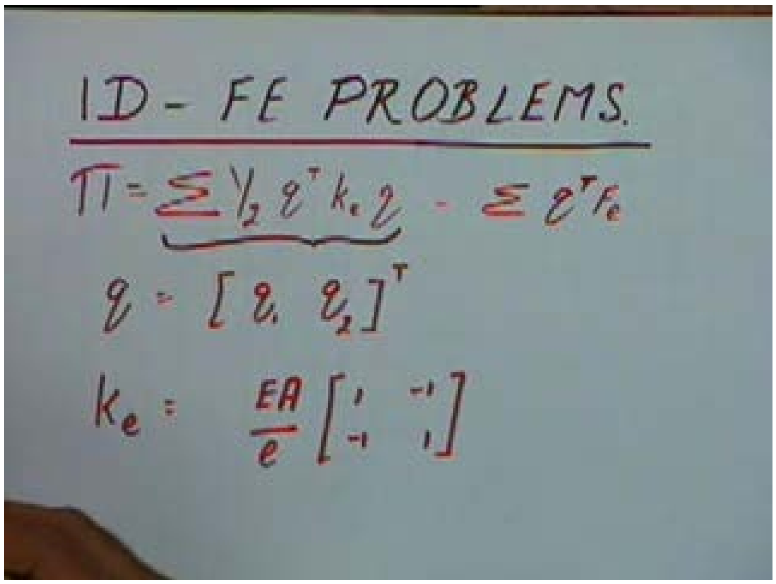

We have shown that the potential energy of a set of one dimensional finite elements is given by this expression where q is a deformation vector given by q_1 q_2 and we had shown that K_e that is element stiffness matrix is given by the expression EA by l multiplied by 1 minus 1 minus 1 1 and the force matrix for the element, this was obtained by summing of the body forces, attractive forces and the point loads. Now what we will see is this expression, the summation of a product of matrix terms. We will try to simplify this and get it directly as a product of matrices. For doing that what we will do is let's say if we take any object and this object has been divided into a set of finite elements like this.

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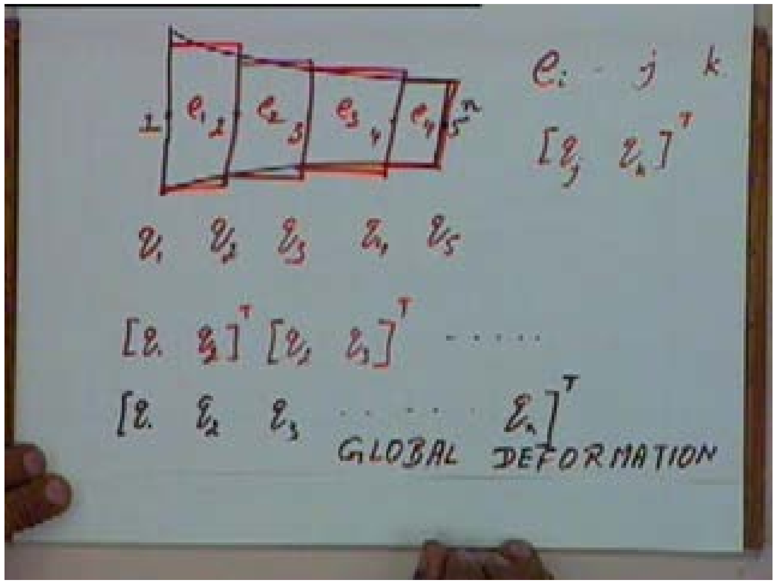

Let's say this is an element 1 2 3 and 4 and these are the load numbers 1 2 3 4 and 5. The deformations let's say we call them q_1 q_2 q_3 q_4 and the deformation of this node is q_5. If you write down the deformation matrix for the first element that will be the matrix q_1 q_2 transpose. Similarly the deformation matrix of the second element will be q_2 q_3 transpose and so on. So if you have an element e_i which is defined between nodes j and k then the deformation matrix for this element would be given by q_j and q_k. So what we will do is instead of having these individual matrices, we will write down one global matrix and that will be the matrix like this q_1 q_2 q_3 so on till q_n where n is the total number of nodes. Let's say this is my node number n. So it will have one vector or one matrix which we call as the global deformation matrix. This matrix we will call as the global deformation matrix.

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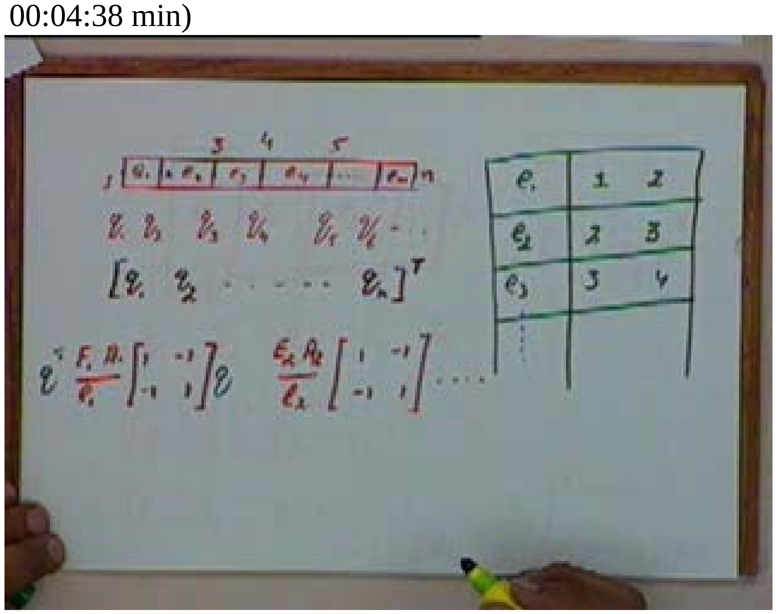

So if we have a set of elements like this that is e_1 e_2 e_3 e_4 so on and let's say if we have m elements and n nodes, our global deformation matrix would be given by q_1 q_2 so on till q_n, this is our global deformation matrix. Now for this matrix for this set of finite elements, we can say that the element e_1 is connected between nodes 1 and 2, the element e_2 is connected between nodes 2 and 3, e_3 is between 3 and 4 and so on. So this let's say table is defining the adjacency relationship between elements and nodes, is defining the position of different elements and that of the different nodes, we will be making use of this table later on.

If we take of the first element and we write down the stiffness matrix for this, the stiffness matrix for the first element is E_1 A_1 by l_1 multiplied by 1 minus 1 minus 1 1. Similarly the stiffness matrix for the second one would be E_2 A_2 by l_2 multiplied by 1 minus 1 minus 1 1 where E_1 is the Young's modulus for element 1, A_1 is the area for the cross section for the first element, l_1 is the length of the first element. Similarly if I take the second element E_2 A_2 and l_2 are the Young's modulus, area of cross section and lengths for the second element and so on. Let's say we can write down the elemental stiffness matrices for each of these elements. And if you look up this term, the first term q transpose K_e q, this term if I expand it out that is I multiply this matrix by q transpose and q, if I multiply this by q transpose that is q_1 q_2, the terms q_1 and q_2 will be multiplying these two terms and then these two terms where this will become a row matrix multiplied by this column and then multiplied by this column.

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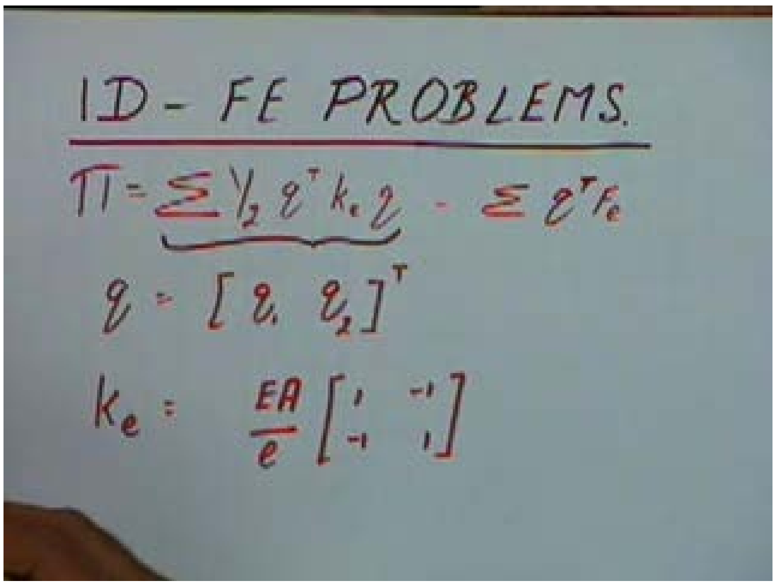

The product of these two will give me a row vector and that row vector when multiplied by this column vector q is going to give me a single potential energy term or the strain energy term.

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So now what we will do is instead of writing the individual matrices like this, I will write a global matrix which will look something like this. What I have done is if I look at this table, the element e_1 is defined between nodes 1 and 2 and I have a total of n nodes. So I will take n by n, I will take an n by n matrix and in a locations 1 and 2 both in the rows as well as in the columns that mean the first column and the second column and the first row and the second row. I will put the four terms, these four terms are the terms that correspond to this. That is the stiffness matrix for element number 1, I have taken first and the second rows and the columns and placed the four terms corresponding to this stiffness matrix. I have got four terms here, that is E_1 A_1 by l_1 multiplied by 1 minus 1, there is a minus sign over here, so 1 minus 1 minus 1 and 1. So the four terms of the elements stiffness matrix for the first element comes or come in these four locations.

Similarly I will take the second element. The second element is defined between nodes 2 and 3. So I will take up row number 2 and row number 3 and column number 2 and column number 3 and corresponding to those 4 locations, I will put these four terms. These four terms are obtained from this matrix that is E_2 A_2 by l_2 1 minus 1 minus 1 1. So these four terms I will put in these locations and so on. I will take out the third element, the third element is defined between nodes 3 and 4. So I will take locations 3 and 4 of this matrix, locations 3 and 4 of this matrix and place the four terms for the elements stiffness matrix for the element number 3. This way I will fill up all these elements. The remaining places will get a 0 and this matrix is what we will call as the global stiffness matrix. Any question up to this point, how I have made this global stiffness matrix?

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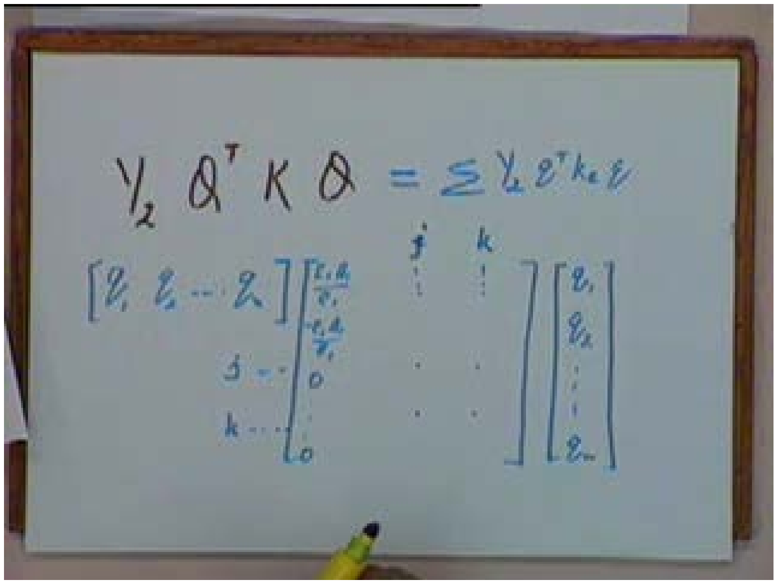

So, now if I look at the global displacement matrix which is this one and the global stiffness matrix which is this one and if I write down the expression half of Q transpose KQ where Q transpose is the global displacement matrix, K is the global stiffness matrix. Basically what I am doing is I am taking this matrix and I am pre multiplying it by Q transpose and post multiplying it by Q. What is Q transpose? It is a row vector containing q_1 q_2 q_3 till q_n. So this is going to be a row vector which looks like q_1 q_2 till q_n. This is multiplied by this huge matrix that I have written. In this place I will get this complete matrix, the global stiffness matrix and this is multiplied by a column q_1 q_2 till q_n.

Now if I look at this, my first two terms here contain E_1 A_1 by l_1 and minus E_1 A_1 by l_1, these two terms and all the other terms are zeros. So when I multiply this row by this column, I will get terms corresponding only to these two. Similarly if I take up the next column, I will get terms corresponding to this, this, this and this. Q_1 will be multiplied by this, q_2 will be multiplied by this as well as by this and q_3 will be multiplied by this and then the product of this row with this column that is going to be multiplied by q_1.

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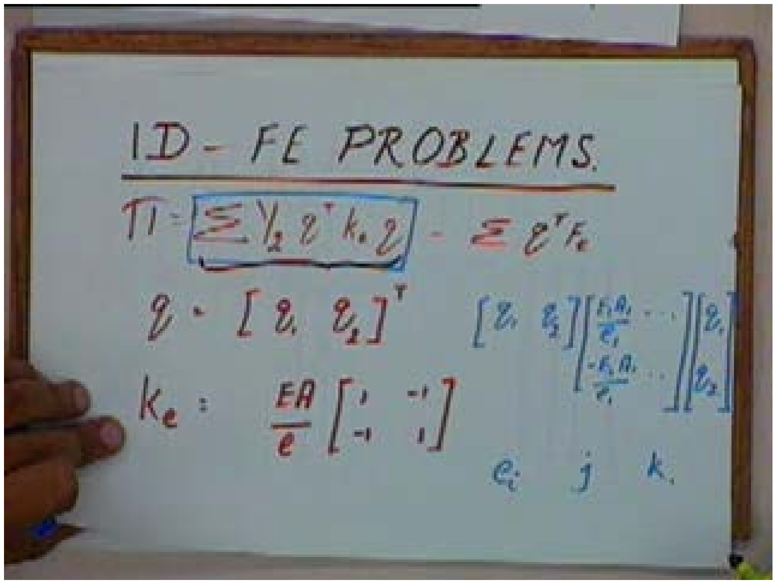

You have to look at this operation slightly carefully. And if I compare this with this expression that I had, half of q transpose into K_e into q. That means half of this row vector multiplied by this, multiplied by this column vector. I will again get q_1 q_2 multiplied by these two terms, again that term will be multiplied by q_1. You can look up the q_1 q_2, this would be multiplied by E_1 A_1 by l_1 minus E_1 A_1 by l_1 and we have another couple of some terms over here and this will be multiplied by q_1 q_2 as the column vector. So these two terms multiplied by these two again multiplied by q_1 that is what will be my first term and we will get the same term when I multiply this row by this column and then multiply that by q_1. Are you able to visualize that? And then what we are doing is I am summing it up over all the elements.

So effect of this summation, I am able to get directly by getting this global matrix because if I take this row and multiply it by any column, let's say I multiply it by the i th column. In the i th node will be a part of some particular element. I will get the corresponding terms from this summation. Let's say if I take the element i which is defined between nodes j and k then the j th row over here and the k th row and so if the j th column and k th column and the j th row and the k th row, these four terms will give me the terms corresponding to the j th element matrices. So what I am saying is this thing will be equal to sigma half q transpose K_e q. Again this e_i, the element e_i? The element e_i is defined between nodes j and the node k.

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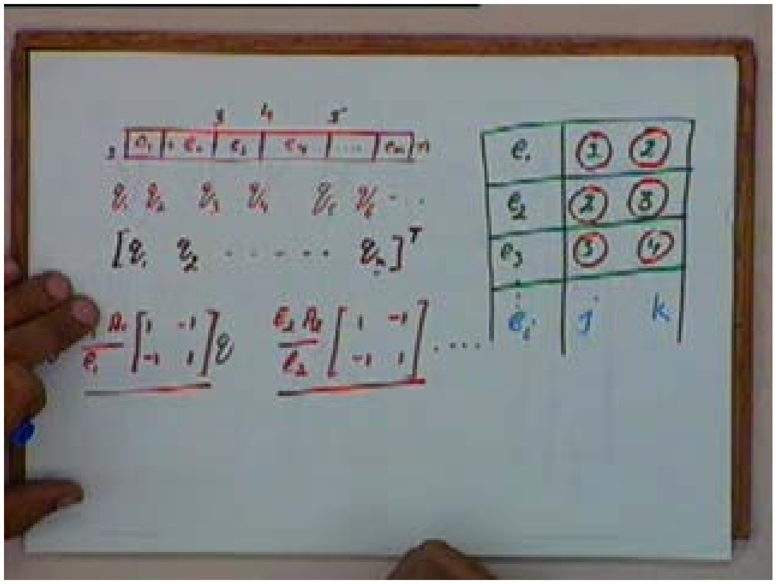

If you look up this table, the element e_1 is between nodes 1 and 2, e_2 is between 2 and 3 and so on. If I take e_i that is between nodes j and k. Is that okay? So this way if I assemble my global stiffness matrix and I define by global deformation matrix, I will get this expression instead of this summation. This summation is equivalent to saying half of Q transpose K Q where K and Q are the global matrices. So if I take any finite element formulation, any problem which I have defined as a set of finite elements, I can take each of these individual elements, get their elemental matrices, the element stiffness matrices and the deformations and from these I can assemble my global equations or I can assemble my global expression by writing this term.

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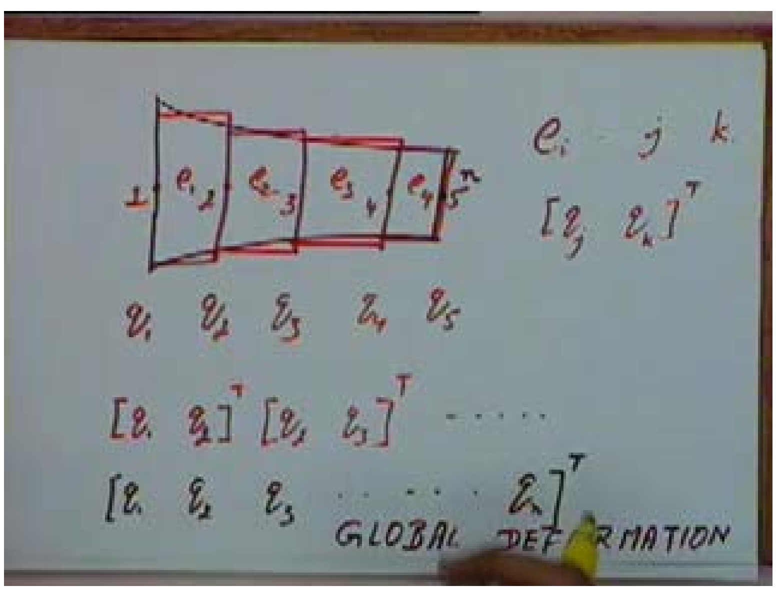

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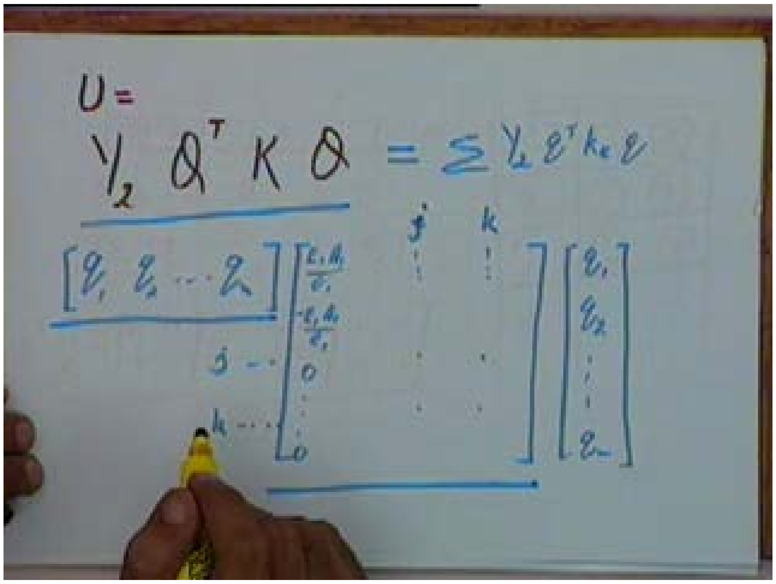

And this is equal to the total strain energy in the system, u is equal to half of Q transpose K Q. This is the total strain energy in the system. Is that all right? Now the potential energy term consisted of another expression which is this q transpose into F_e.

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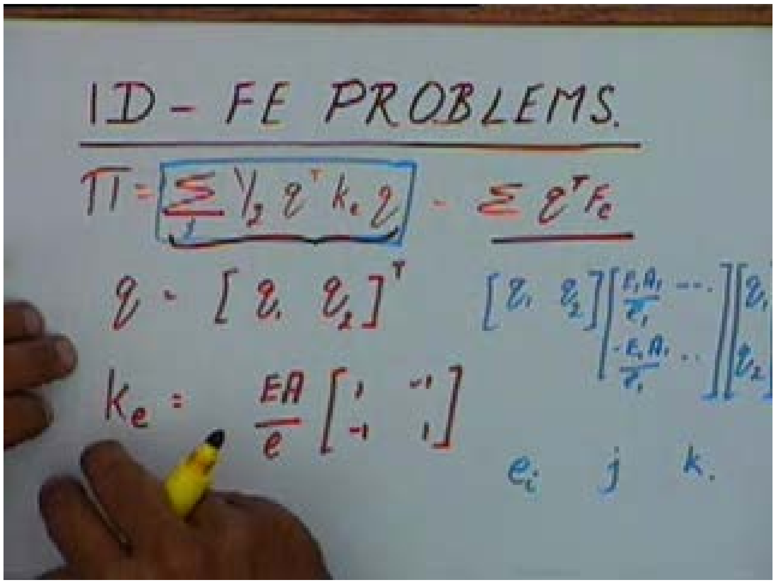

What we will do is just like we have defined the global deformation matrix, we will define another which is a global force matrix.

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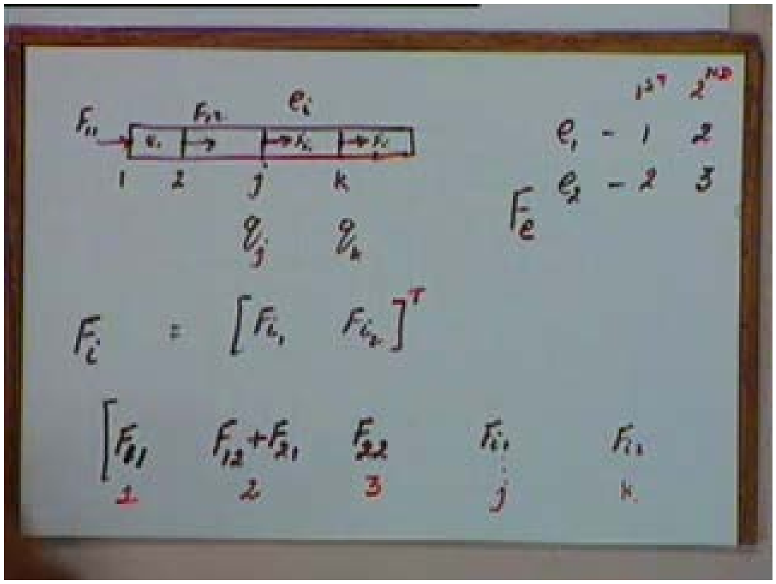

So if we have a set of finite elements like this let's say this is my element e_i between nodes j and k. The deformation of this is q_j, the deformation of this is q_k. We have written expressions for the F_e matrix that is the element forced matrix. What we said essentially was this element force matrix, it consists of two forces acting at the two nodes. So this is let's say some force F_1 or let me say F i_1 and this is F i_2. So the element force matrix for this would look like this F i_1 and F i_2. This is going to be the element force matrix for the i th element. So what we will do is we will define a global matrix in which again if I take the first element e_1 that is let's say defined between nodes 1 and 2. I will take the corresponding force terms for the first element and place them in locations 1 and 2. So, F_{11} and F_{12}, this is my element number 1, this is my node 1 and this is my node 2.

This force is what I am calling as F_{11} and this force is what I am calling as F_{12}. F_{12} is the force on the second node of element number 1, F_{11} is the force on the first node of element number 1. F i_1 is the force on the first node of element number i and F i_2 is the force on the second node of element number i. So if my element 1 is between nodes 1 and 2 and element 2 is between nodes 2 and 3, my second term here I will add F_{21} and third term here we will have F_{22}. This is location number 1, this is location number 2 and this is location number 3. Forces are also 1 D. In the 1 D formulation we have assumed everything to be one dimensional. If you have a two dimensional element we will see how to formulate that later on. We have always been doing that, the first node here is node number j and the second node is node number k.

If my first node is node number j, what I will do is I will take up the j th term over here and in the j th term, I will add F i_1 and in the k th term I will add F i_2. See the first and the second node, whereas this is my first node and this is my second node. This is my first node and this is my second node, this is what we referred to as elemental numbering. Element have a local numbering which is nodes 1 and 2, these are the global node numbers. This 1 2 j k and so on, these are the global node numbers but this global node number j corresponds to element node number 1 for element number i. That means that this element, this node number locally is 1 and this is local load number 2 but global number is j and k.

So when I look at the local matrices, in the local matrix this is my first force element, this is the second forced element. In the global numbering this will go in to the j th location, this will go in to the k th location. This way what I will do is I will take up all my elements and the force is coming on each of the nodes. The forces which are a part of the elements force matrix, those forces I will place at these global locations. So at each of these n nodes, I will get a force which will look something like this. So if I have a set of forces like this, this matrix is what I call as my global force matrix. What I have basically done is that I have taken up the force elements on each of the, forces on that two nodes of each of the elements, I place them on global locations and this matrix is what I call as the global force matrix.

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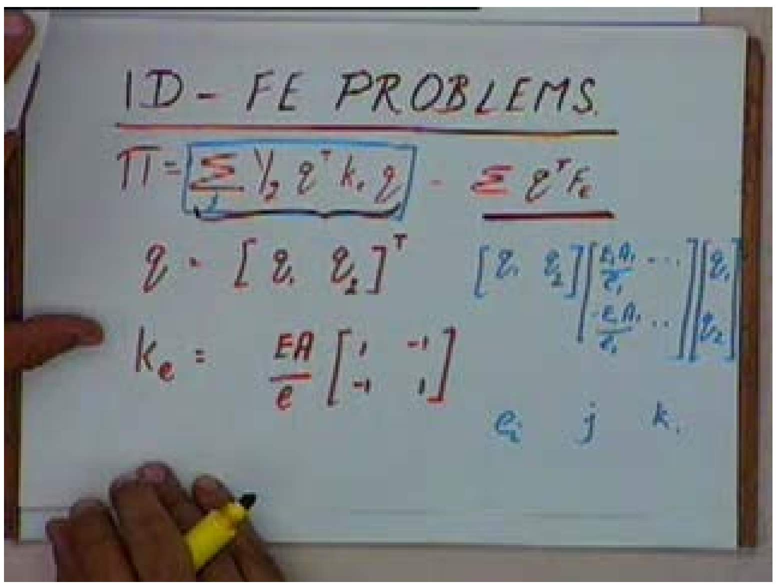

And now if I look at this term that is sigma q transpose F_e. You will find that this will be equal to Q transpose multiplied by F.

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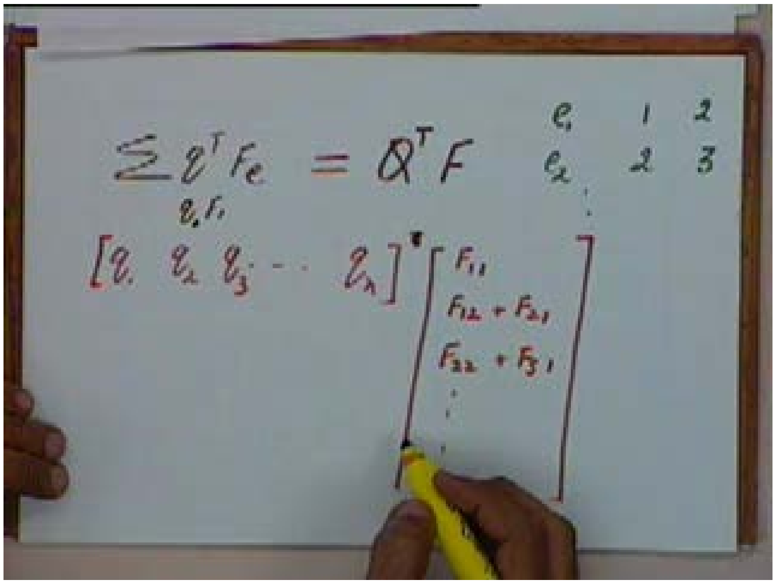

How do we get that? This Q transpose is nothing but q_1 q_2 till q_n transpose and this F would consist of F_{11} F_{12} plus F_{21} F_{22} plus F_{31} and so on. What I have taken is I have element e_1 defined between nodes 1 and 2 and element e_2 is defined between nodes 2 and 3 and so on. So if I take this row sorry if I take this row and multiplied by this column, q_1 is multiplied by F_{11}, q_2 is multiplied by F_{12} and these two terms are what I will get if I take the matrix q_1 F_1 that means the q matrix for the first element and F matrix for the first element. If I take these two and multiply them, I will get q_1 F_{11} multiplied by q_2 $F_{12.}$

Similarly if I take up the second element, I will get q_2 and q_3 multiplied by F_{21} and F_{22}. Similarly if I take the third element q_3 and q_4 will be multiplied by F_{31} and F_{32} and so on. Summation of all these terms is nothing but the Q transpose multiplied by F. So this way by writing the global equation in this manner, the potential energy expression that I had earlier that is this expression I can simplify that to, at pi equal to sigma half of q transpose K_e multiplied by q minus sigma q transpose F_e.

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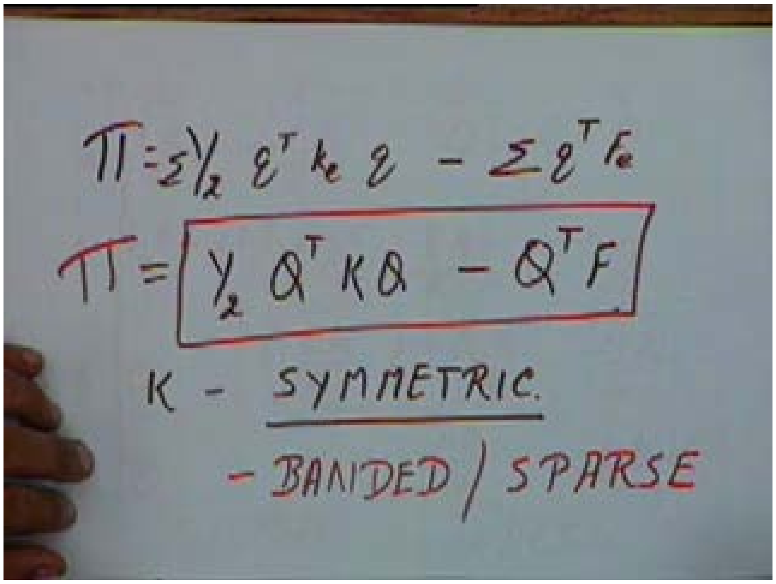

This is nothing but half of Q transpose K Q minus Q transpose F. This is the expression for the total potential energy in the system and this has been obtained by writing the global deformation equation, global stiffness matrix and the global force matrix. Any questions up to this point? A couple of observations about this global stiffness matrix. If you look at these elements stiffness matrices, the first thing that we notice that these elements stiffness matrices they are all symmetric 11 minus 1 minus, 11 minus 1 minus 1.

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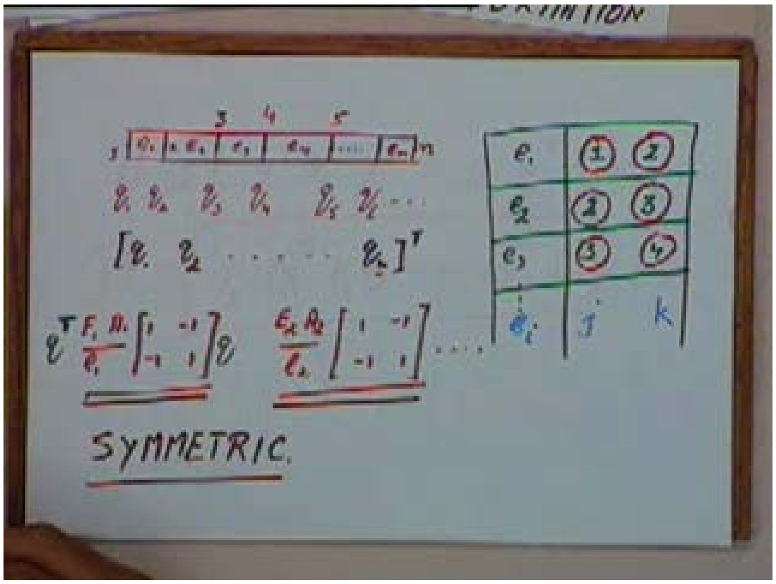

So all the elements stiffness matrices are symmetric and if you see the method by which we have assembled the global stiffness matrix, we have retained the symmetry because the elements stiffness matrices I mean, I have placed both the row as well as column terms.

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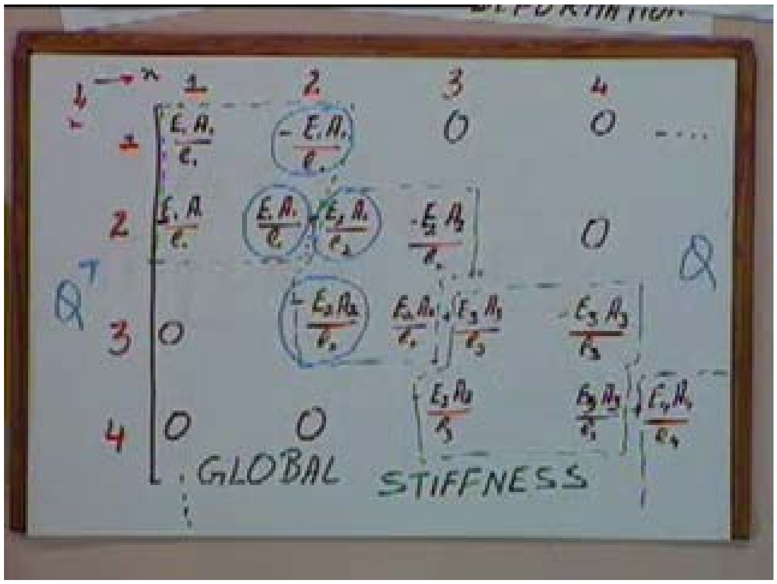

So if I take up the i th column and the j th column and the i th row and the j th row and the element define between nodes i and j will come on the four symmetric terms. So the global stiffness matrix is also a symmetric matrix. The first thing that we say is this k is a symmetric matrix.

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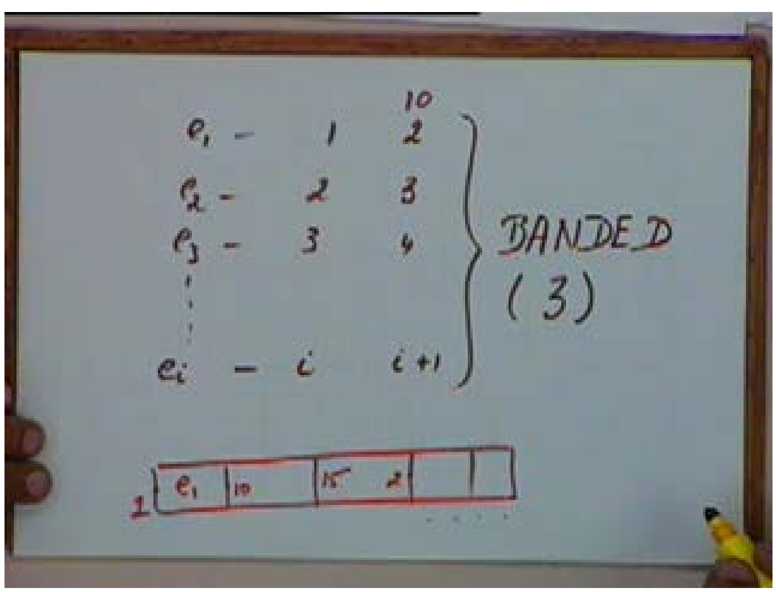

The other thing that we say is that if I have element 1, this is defined between nodes 1 and 2, element 2 which is defined between nodes 2 and 3, element 3 is defined between nodes 3 and 4 and so on. That means the element i will be defined between nodes i and i plus 1. If I follow this sequence then I will get terms only three terms in any row, the three consecutive terms next to the diagonal. If you notice here element 1 is between 1 and 2, 2 is between 2 and 3. So between 2 and 3 I have element 2, between 3 and 4 I have element 3. If I take up the third row, I will find non zero terms only in locations 2 3 and 4. So this global stiffness matrix will be a banded matrix with the bandwidth of 3. So this is a banded matrix. So if I take numbering like this, I will get a banded matrix. The bandwidth will be 3 in this case, the width of the band.

However if I don't do a numbering like this, I will say that my element number 1 is defined between node number 1 and 10 that means I am taking the set of elements like this. This is my element number 1 and instead of saying that this is between nodes 1 and 2, I say this is 1 and this is 10 and this is may be some other number 15 and this is node number 2 and so on. Then what would happen is the terms for the stiffness matrix of element 1 will go in location 1 and 10. These 4 terms won't come like 1 2 and 1 minus 1 and minus 1 1. 1 will come here and this term will go in the tenth column and similarly this term will go in the tenth row and this term will go in the location 10, 10.

If that happens this will no longer be a banded matrix. This matrix will then become non-banded but it will still be symmetric but this will become a sparse matrix. There will be a term here, a term in the tenth row in the tenth column that is, but all the term in between will be 0. So the global matrix that I have, this k either it is banded or it is sparse. And whether it is banded or sparse that depends on the numbering that we have done, how we have numbered the nodes. So the numbering becomes very critical whether it is a banded matrix or sparse matrix. Basically critical because for banded matrices, we can get very efficient methods for solving banded matrices. While if it is not a banded matrix and we have a very large system of equation; the system can become complex.

So the numbering that we do, this numbering, the numbering of the nodes that is very critical. Student: Sir midterms is zero. What do you mean by midterms? Student: I mean between 2 to 9. Between 2 to 9 let's say this is my node 2 and this is my node number 9. Student: No, E_1 is defined between nodes 1 and 10. So corresponding to E_1 for the nodes 2 and 9, 2 to 9 all the elements will be 0 that's why it's sparse. Corresponding to e_1 and to me. So if I look at this stiffness matrix, I will get a term in this location and I will get a term in the tenth location but I won't get any term in any of these locations because this element 1 between node 1 is defined between nodes 1and node 10 between nodes and between node number 1 and let's say node number 2 there is no element. If there is an element between node 1 and node 2 then I will get a term in this location. Is that okay? No.

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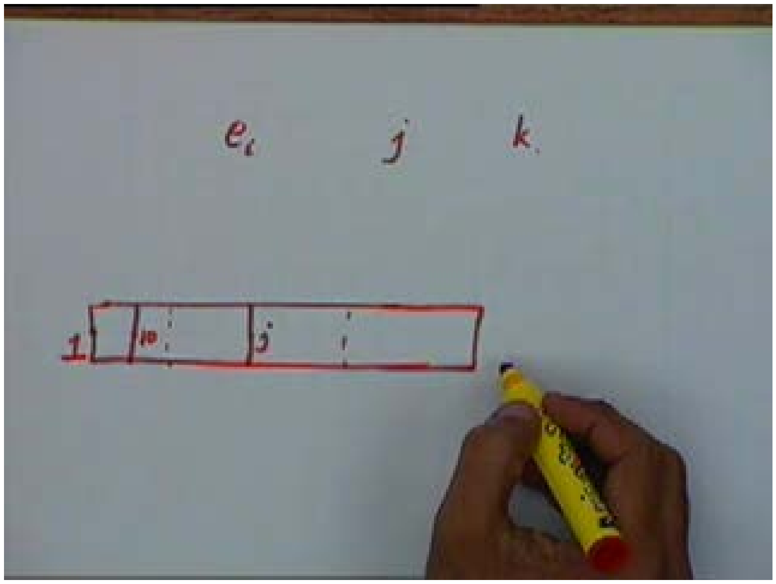

See if I have an element i between nodes j and k, in that case I will take the j th row, k th row, j th column and the k th column. In a one dimensional case like this, one node can appear only between two elements. If this is my let's say node number j, it can either appear with this element or with this element. So, one node can appear only in two elements.

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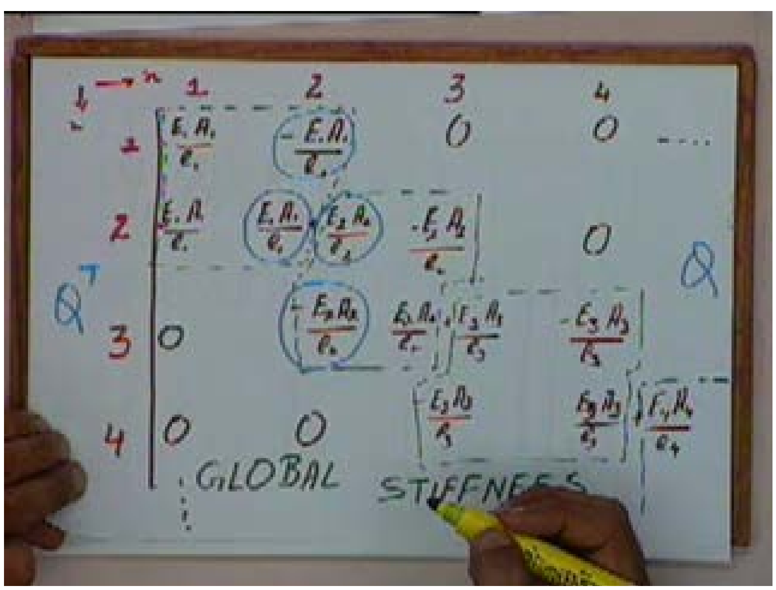

If I take the first node and let's say if I am calling this a node number 1 and this is connected to let's say node number 10, it cannot be connected to any of the other nodes, since this is the first node I am taking. If 1 cannot be connected to any of the other nodes, all these locations will have to zeros.

When will this term being nonzero? Only when there is an element between node 1 and node 3, only then this term can become nonzero. That is why between locations between 2 and 9, all of them will become 0 because I have taken the first element between 1 and 10. This I have taken to be my element number 1. That is why all the other terms will become zero. Is that okay? j equal to 2 and k equal to? Student: Somewhere in between I will get 10 also. So what's the difference between that and this? I didn't get this. I have taken some element which is between nodes 2 and some other number let's say 9. Terms with this element will be defined between locations 2 and 9. So if you take up second row and the ninth row and second row and the second column and the ninth column, I will get a term here and I will get a term in the ninth column. I will not get a term here, I will get a term here only when I have an element between 2 and 1. Is that okay?

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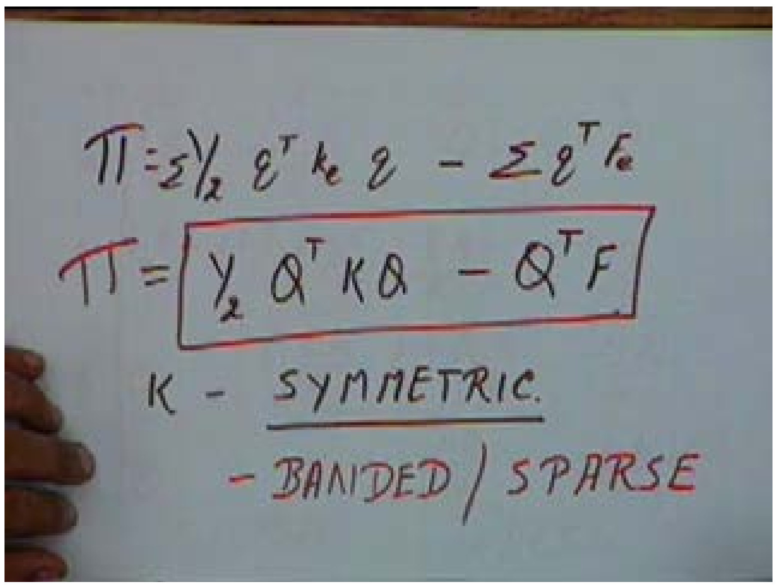

That is why we say that this global matrix is always symmetric and is either banded or sparse depending on the numbering that we have done. If a numbering system is efficient then it will be banded matrix with the small bandwidth otherwise it will be a sparse matrix. In fact when you work on let's say a system like patron or any of the finite element package, you will find that the system first assembles the global matrices and then it tries to optimize on the numbering because the numbering that you might do might be very inefficient. So you try to optimize the numbering and aim at getting the minimum bandwidth. So at some stage you will find a message saying that minimizing bandwidth or optimizing bandwidth that is basically done with the idea of making it a banded matrix with the small bandwidth. So this is how the global matrices are assembled. Let's take a small example for this.

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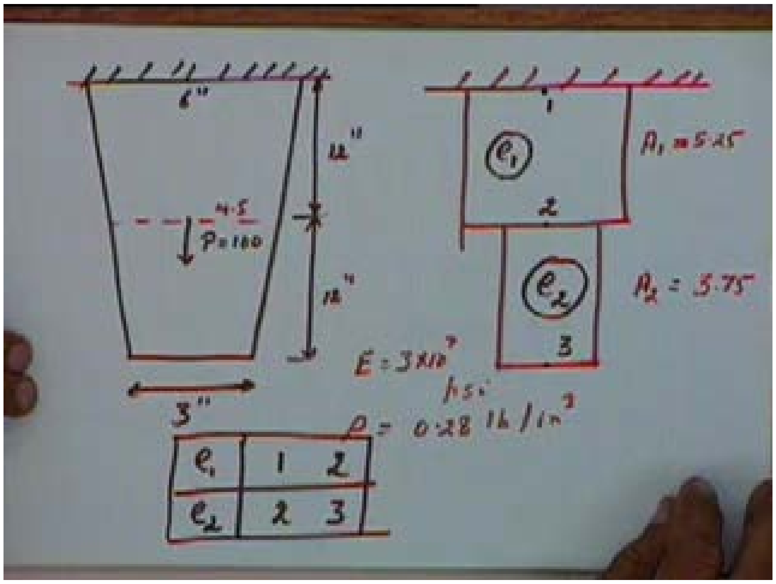

Last time we have taken an example where we said that, we have taken a body like this and then we have modeled this by a two finite elements and for this first element we said we will take the area of cross section to be 5.25 square inches, for the second element we said it will be 3.75. We basically at this point it is 4.5, this 5.25 is a mean of 6 and 4.5, 3.75 is the mean of 4.5 and 3. And I think we have taken E to be 3 into 10 to the power 7 psi. This is also in pounds and we had taken rho density to be 0.28 pounds per cubic inch.

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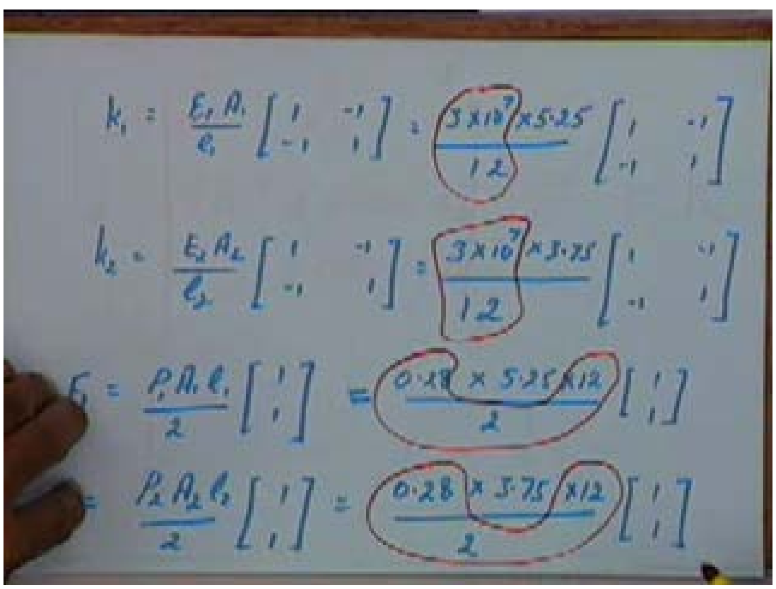

And for these we have written down the elemental matrices that is k_1 is equal to E_1 A_1 by l_1 multiplied by 1 minus 1 minus 1 1 and k_2 will be equal to E_2 A_2 by l_2 multiplied by 1 minus 1 minus 1 1 and this will be equal to E_2 A_2 by l_2, so we had derived these matrices. Now if we take the element first matrices, the total body force is the total weight acting which is rho_1 A_1 l_1 that is A_1 l_1 is the volume, rho_1 is the density multiplied by 1 1. This will be a first matrix on the first element and the second element will give us rho_2 A_2 l_2 by 2 multiplied by 11. Rho_2 A_2 l_2 these are the total weight of the second element divided by 2, so we will have half the total weight acting at each of the two nodes. Here also half the total weight acting at each of the two nodes and we put in the values. We will get this to be 0.28 multiplied by 5.25 multiplied by 12 by 211. This would be 0.28 multiplied by 3.75 multiplied by 12 divided by 2 and this will be 1 1.

Now from these matrices let's derive the global matrices. If we take this, I will call this node number 1, this is my node number 2 and this is my node number 3. This is my element number 1 and this is my element number 2. So I have element 1 defined between nodes 1 and 2, element 2 is defined between nodes 2 and 3. So now since I have got 3 nodes, my global stiffness matrix will have 3 terms in it. It will be a 3 by 3 matrix.

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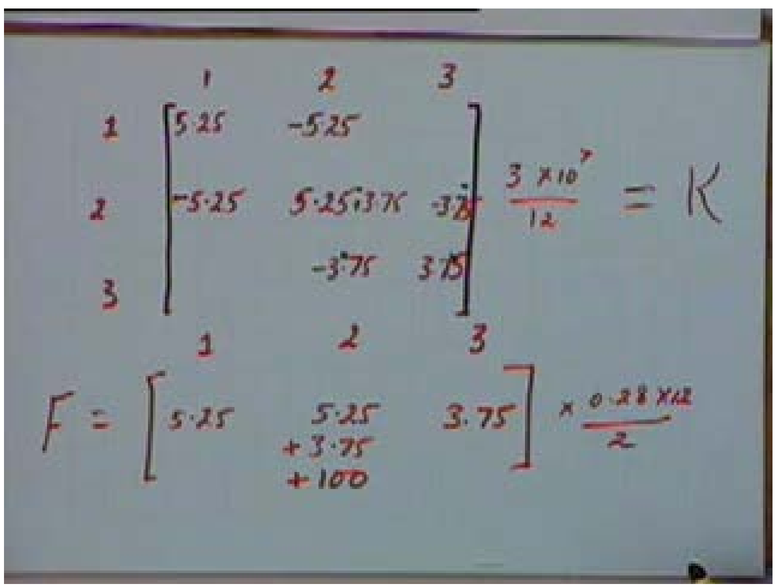

So if I write down a 3 by 3 matrix 1 2 3, 1 2 3, the stiffness matrix for the first element. For this first element we will take locations 1 and 2. So locations 1 and 2 would mean this location, this location, this and this. So these 4 terms, these 4 terms will come in these 4 locations. If I see both these terms, this part is common. So I will just take, leave this outside 3 into 10 to the power 7 by 12. I will get a term of 5.25 minus 5.25 minus 5.25 and 5.25. So if I take the second element that is defined between nodes 2 and 3, so the second element will take locations 2 and 3 in the second row and the third row and the second column and the third column and these 4 locations I will put these 4 terms. So I will get this plus 3.75, here I will get a minus 3.75, minus 3.75 and a plus 3.75. So this matrix will be my global stiffness matrix.

My global force matrix will also consist of 3 terms in locations 1 2 and 3. Between locations 1 and 2, I will get the forced terms coming from first element that is this and again between these two, I will take this as common and I will multiply it by 0.28 into 12 by 2, between 1 and 2 I will get 5.25 and 5.25 and for the second element between nodes 2 and 3, I will get this term of 3.75. So it will become this plus 3.75, 3.75. And in addition to this, we have a point load of 100 pounds acting at node number 2. So this point force will also be added here, it will become this plus 100. Student: I will take the factor inside fine. This term will come inside that, change you can make. So this will be our global force matrix and this will be our global stiffness matrix. So this is how we assemble the global matrices. Once you assemble the global matrices then how to solve them that we will see in one of the subsequent classes.