Computer Aided Design
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Lecture No. # 18
1 D Finite Element Problems

We had a discussion on 1 D finite element problems. In the morning we had seen that if we have a one dimensional rod, we can split it into one dimensional finite elements with uniform cross section that means the variation is only along the length, the area of cross section would remain the same across the length of the element. In this case let's say if the area of cross section is A and we said that lets say the position of this starting point of the element is x_1, position of the end point is x_2 and I said the deformation at this point is q_1, the deformation here is q_2.

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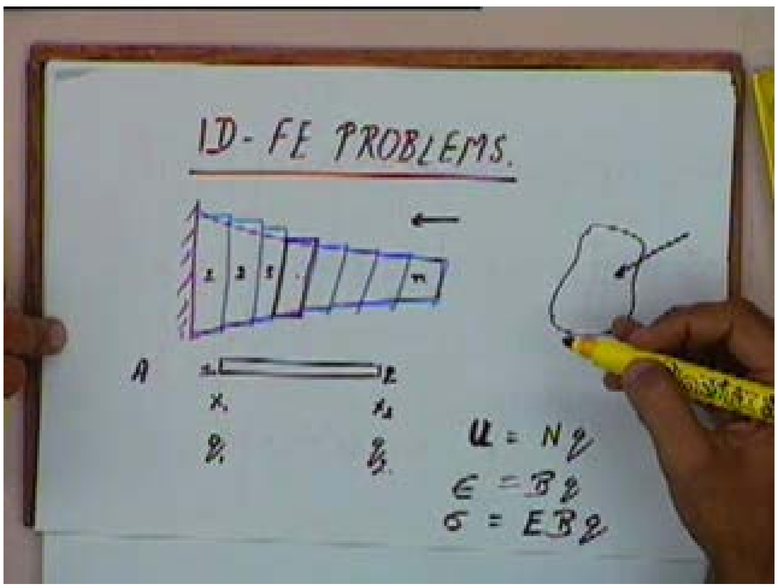

We had said that U, the deformation at any point inside this element will be equal to N time's q. We also derived the fact that epsilon which is the strain at any point in the element will be equal to B times q and the stress that is sigma that will be equal to E times Bq where E is the Young's modulus of the element. So if we take this formulation then we will now try to derive the expression for the potential energy in this object.

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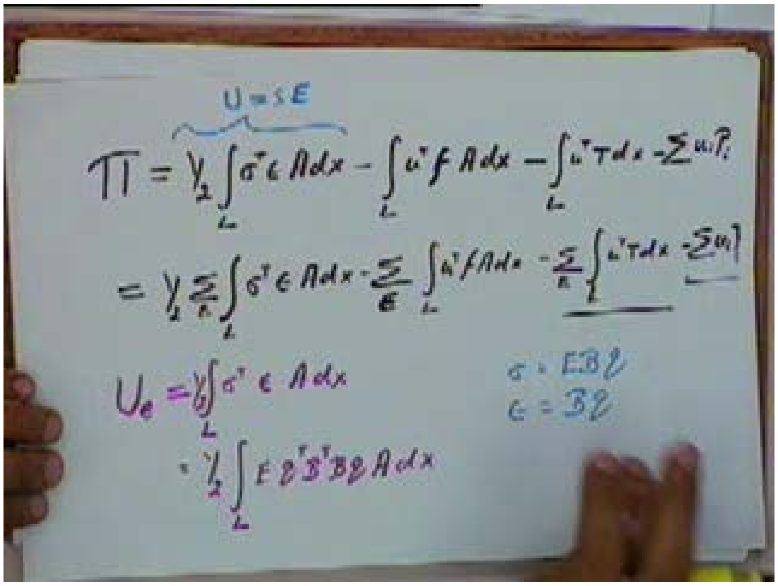

In this body the potential energy expression pi, the total potential energy we know is equal to half of the volume integral of sigma transpose epsilon dv, dv will become A times dx that is the integral along the length minus integral of u transpose f that is body forces into A into dx. Again I am replacing dv by Adx, again this integral is along the length of the element minus integral of u transpose T into dx. So again this integral will also become over the length of the element in this case and minus the potential energy contribution because of the point forces that will be sigma u_i $P_{i.}$ So this will be the expression for the potential energy in this problem. This first term is what we call as the strain energy U or the strain energy.

Now this potential energy in this object, the potential energy in this complete object will be the sum of the potential energy in the individual object, in the individual elements. So if I take this element as element number 1 2 3 and so on, there are n elements then the total potential energy of this body will be the sum of the individual potential energies. So if I use that I can say that the total potential energy will be half of sigma integral over the length or sigma transpose epsilon Adx minus sum over all the elements of integral of u transpose fAdx minus sum of all integral of u transpose Tdx minus sigma u_i multiplied by Pi and this summation is over all the elements that means summation over all the elements, only the capital we have used that.

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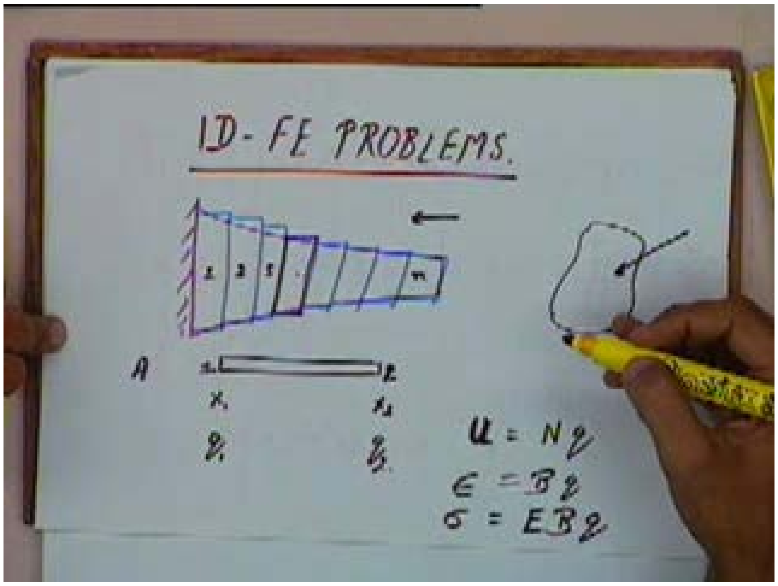

Again, see if you have this of the element, right now we have assumed that all the forces are one dimensional forces. So if I am talking of one dimensional forces over this element, acting on the surface that force has to be along this. So we really only be talking of some shear kind of forces. So instead of, that force instead of being on a per unit area basis, I can treat that as a force on a per unit length basis. So in this case it is T that I have used is force per unit length and this is a one dimensional case. With the three dimensional case, if I have a three dimensional surface patch like this I can have a force acting on a per unit area basis that is a force acting on the surface but that is a three dimensional force but now I have a one dimensional force acting along the length. So I can effectively take that to be force on per unit length basis.

So this tractive force is a force per unit length and not per unit area that is that length term is not visible. So if I take each of these terms separately, let's first take this first term that is the strain energy U in the element. So this strain energy in the element is equal to the integral over the length half of the integral over the length sigma transpose epsilon Adx. Let's try and evaluate this integral. This will be equal to half of integral of, we know that sigma is equal to Ebq and epsilon is equal to BQ. So sigma transpose would be nothing but where E is a constant so E multiplied by q transpose B transpose, epsilon is nothing but Bq so Bq into Adx and this integral is over the length.

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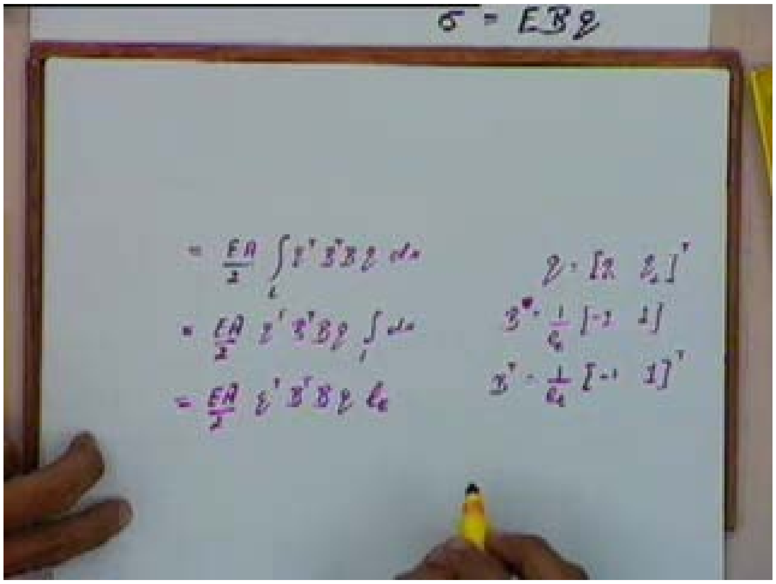

If I continue with this, this will become EA by 2 integral of q transpose B transpose Bqdx. Now q transpose and q they are both constant matrices, if you remember q, q is equal q_1 q_2 and q_1 and q_2 are constants. Similarly B transpose we said was equal to 1 over le where le is the length of the element multiplied by minus 1 1. So B transpose is also a constant, it doesn't vary with x. Now this integral over the length all these four terms are constant terms. So this will be equal to EA by 2 multiplied by q transpose B transpose Bq integral over length dx. Now this B transpose is this multiplied by sorry I think this is B and B transpose would be 1 over le multiplied by minus 1 1 transpose. So integral over the length of dx is nothing but the length of the element, so this will be equal to… (Refer Slide Time: 10:00)

Student: Integral we can actually count for the shape of that figure, this pieces even if you take a 5 we put it here.

See when you are evaluating the integral, you can take into account the profile but our objective right now is that we want to take, make an approximation such that we have simple elements by which we can account for everything.

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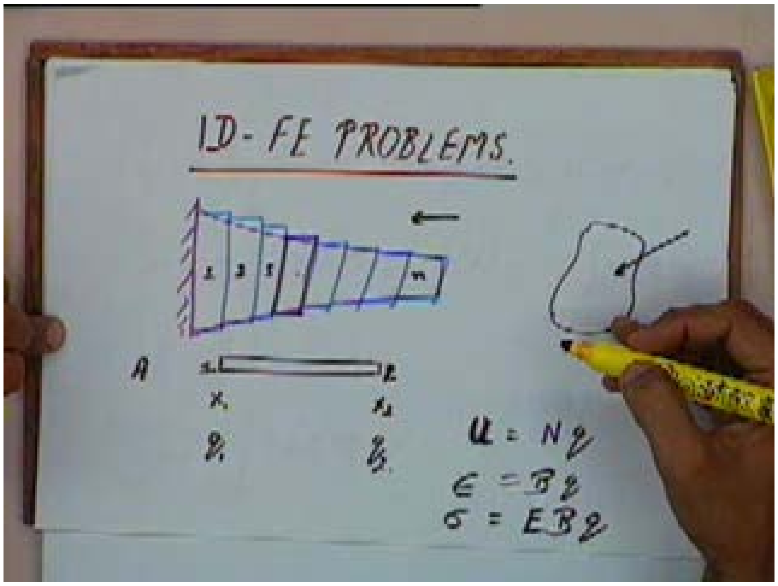

Our objective is that if we can do the analysis of one small element like this, we will be able to repeat that over the complete element and then do that. If you want to…

Student: Then linear approximation for N_1 N_2 will be correct for the profile.
Professor: Either quadratic or higher order interpolation depending on the profile. The moment you start doing that you cannot extrapolate the analysis of one element to the complete body and then you have any arbitrary shape possible. You are right, we can incorporate that in the integral but that will only going to make things much more complex for me.
Student: Other things get complex.

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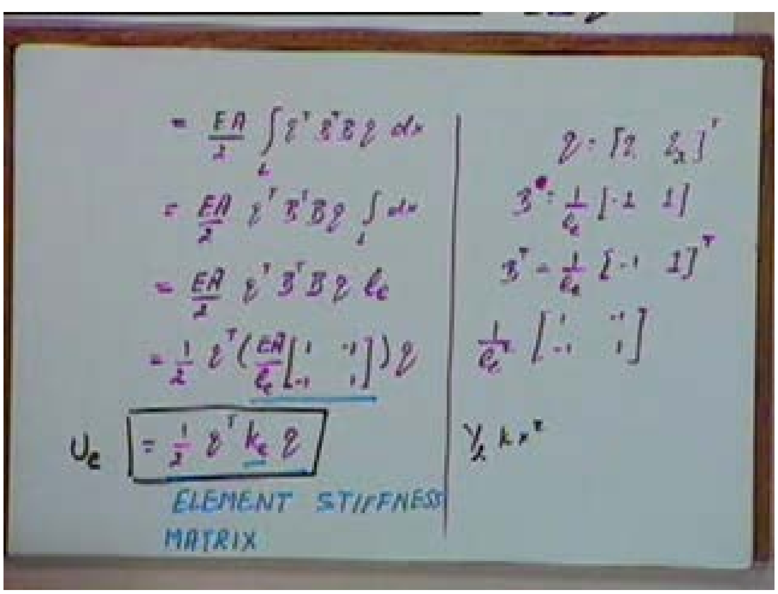

So this is what we get, a B transpose into B. If I multiply these two matrices, I will get 1 over le squared multiplied by 1 minus 1 minus 1 1 where B transpose is a column vector of size 2, B is a row vector of size 2. If I multiply by 2, I will get a 2 by 2 matrix. So if I put this back over here, this is equal to half of q transpose into EA multiplied by 1 minus 1 minus 1 1. Is that okay? So this I will say is half of q transpose k_e multiplied by q. This matrix is what we will call as the k_e matrix or we will call this the element stiffness matrix. This element stiffness matrix is what we are getting over here which is EA by l_e multiplied by 1 minus 1 minus 1 1 and why do we call it a stiffness matrix that you can see from this expression. This expression for potential energy, this you can compare let's say with the potential energy of a spring which is half k x squared. So this x is same as the deformation over here and this k is the stiffness. So this k_e is called the element stiffness matrix, it basically relates the deformations with the strain energy in the object.

In fact we will see when we go in for higher order elements and more complicated elements, our aim will always be to get this expression. We will always get U that is U_e the element strain energy to be equal to this expression and we will try to get some expression for this stiffness matrix. This is my first step in our attempt to solve the finite element problem. So this is what we call the element stiffness matrix.

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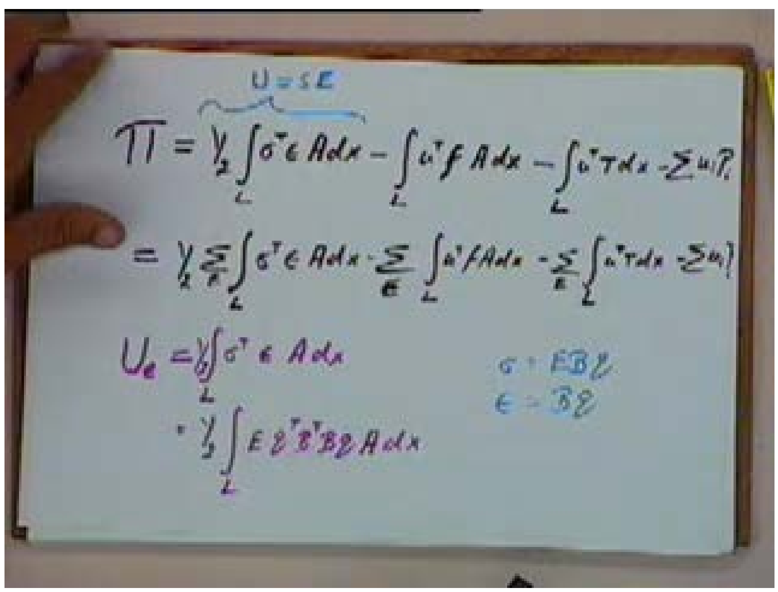

Now if we take up the potential energy terms, the second term over here that is the potential energy contribution without the body forces. If we take this then the integrals for this would look like, would look something like this, integral over the length u transpose fAdx. We know u is equal to N time's q and we know this f is a body force, so this integral what will that become? u transpose will be q transpose times N transpose.

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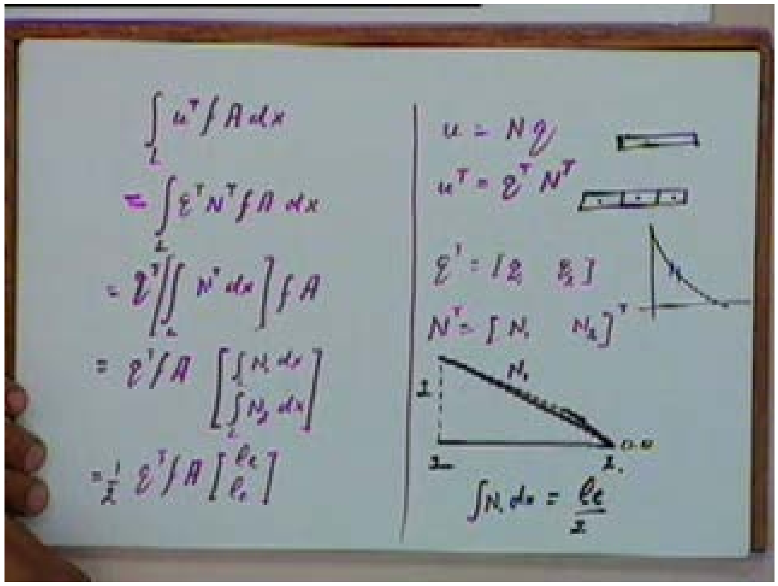

So if I replace that over here, I get q transpose N transpose multiplied by fAdx. Now f is the body force acting in the element. We said initially if we take an element like this, over the length of this element we will assume the body force to be constant. If you have a number of elements connected together and so on, the body force in this will be different from that and this will be different from that and this but over the length, the body force is going to be constant because you are taking elements to be small enough. So in that case f will be a constant and A, again area of cross section that is also constant. q transpose into N transpose, q transpose will be equal to q_1 q_2. N is equal to matrix N_1 N_2, so N transpose is this. So this will become if I say q transpose into the integral of N transpose dx multiplied by f into A, f and A are also constants. N transpose is this, so this will become equal to q transpose fA into…

Now this N transpose of this matrix, so I will put that as integral of N_1 dx integral of N_2 dx both these integrals are over the length. Now what is the integral of N_1 dx? If you remember the shear function that I had drawn, we said that this is my point one, this is my point two. N_1 is equal to 1 at this point and is equal to 0 here and it varies in a straight line. At this point this is my shear function for N_1, N_1 is 1 here and is equal to 0 here.

Student: Sir, if the integrate N then isn't this against the of finite elements methods because finite elements we mean that it has to be something infinitely small. When we take an integration we are not able to find the, what the, it's not finite.

Professor: just a sec, just a sec I didn't get your question at all.

Student: sir, like when we say finite element method, we divide the structure into small finite elements. If we integrating over them then we are not really defining any finite point, it's a like this it is finite.

Professor: See what we will do is we will take the, this say individual element within this element whatever variation that we assume or whatever variation that we get, we will find out our equation corresponding to that but variation over elements that we will not take care of. So over the elements we will get some analysis for this, some analysis for next one and then we just sum it up.

Our aim is that we will take the elements small enough such that whatever assumptions we have made about this element, those assumptions will be valid. We have made the assumption that our displacements are moving, are changing linearly because that assumption will be valid only if my elements are small enough. If I take very large elements that is probably not going to be true. For instance if I take a large element my strain distribution might actually be like this, it can be any kind of curve but within a small gap I will assume it to be linear that is not the finite element method will tell us. But within this element we have to do the integration according to our assumptions. So this N_1, this is varying along the length of the element in this manner and what is the integral of N_1 dx? Integral of N_1 dx is nothing but the area of this curve, the area under this curve. So integral of N_1 dx is nothing but it will be the length of the element divided by 2 because this is a triangular area, we are taking linear shear functions so N_1 is a straight line. So integral of N_1 over the length of the element is going to be the area of this triangle and that is nothing but the length multiplied by 1 multiplied by half which is l_e by 2. So we will use that expression over here and this will become equal to half of q transpose fA l_e l_e.

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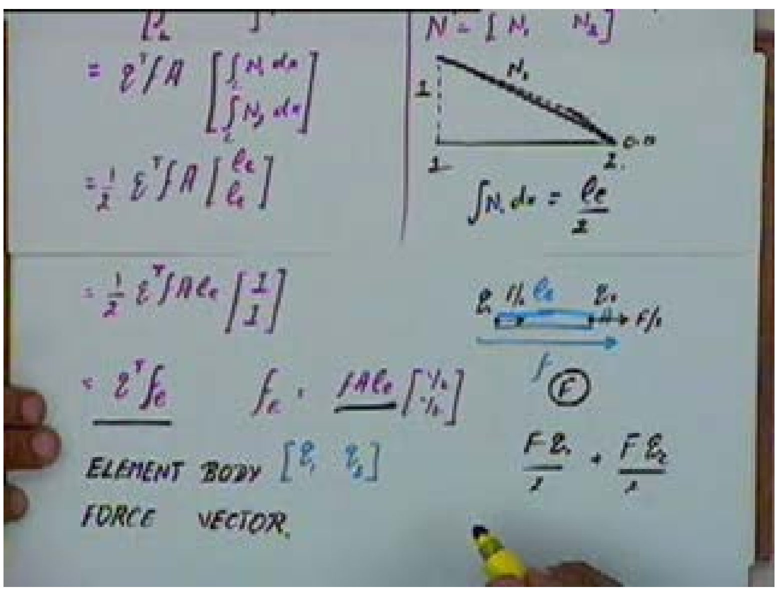

So this is going to be equal to half of q transpose fA l_e multiplied by 1 1. Now this we will put that as q transpose multiplied by a vector f_e or a matrix fe where f_e is equal to fA l_e half half. Now what is fA multiplied by l_e? f is the body force acting on a element, A is the area of cross section, l_e is the length, so A times l_e is the volume of the element. What is f times Al_e? That is the total body force acting on that element. So if I have an element like this that has some body force acting in this direction given by f, its area of cross section is A and length is l_e. So the volume of this element is A multiplied by l_e, the total force will be f into A into l_e and this f_e I am multiplying by q transpose, q transpose is nothing by q_1 q_2. So when I am multiplying q_1 q_2 by this essentially what I am doing is whatever the total body force acting on that multiplied by half multiplied by q_1, q_1 is the deformation of this point, q_2 is the deformation of this point. The total force acting on this is let's say f that is this term. So the deformation here is q_1, deformation here is q_2. I am saying that this potential energy is equal to f into q_1 by 2 plus f into q_2 by 2 that is what this multiplied by this will give us.

So f into q_1 by 2 is nothing but if I split this force f into two parts, one part acting over here and another part acting over here then this force is f by 2 and this force is also f by 2 because of this force the contribution to the potential energy will be f by 2 multiplied by q_1 and because of this force, the contribution will be f by 2 multiplied by q_2. So effectively what I am saying here is that the contribution to the potential energy can be computed by taking the total body force that is f splitting it into two parts, one part acting on the first node one part acting on the second node. So this vector f_e or this matrix f_e is only telling me that half the force is acting on the node one, the other half of the force is acting on the node two. So effectively I am splitting this force into two parts, two equal parts one acting on the first node, the other acting on the second node and I say that the total potential energy contribution because of the body force is given by this expression and this matrix f_e is what we call as the element body force vector.

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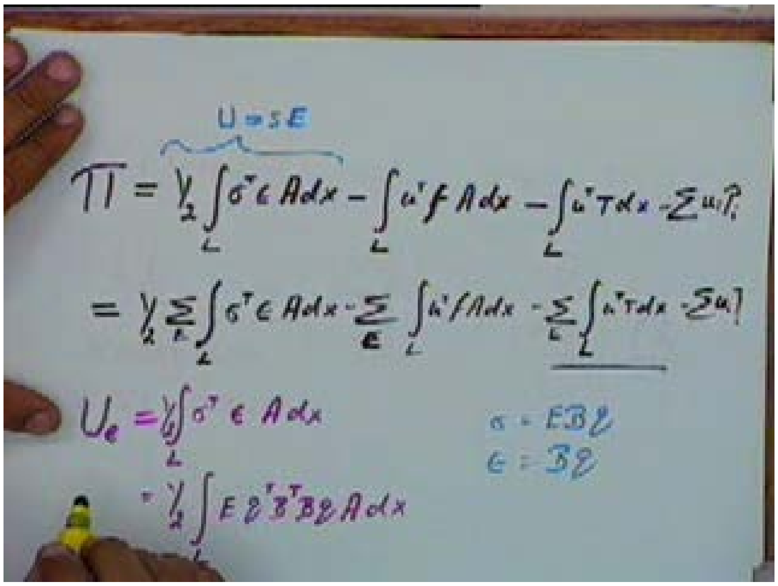

Now similarly we will do a, we will evaluate the next term in our potential energy expression that is this term.

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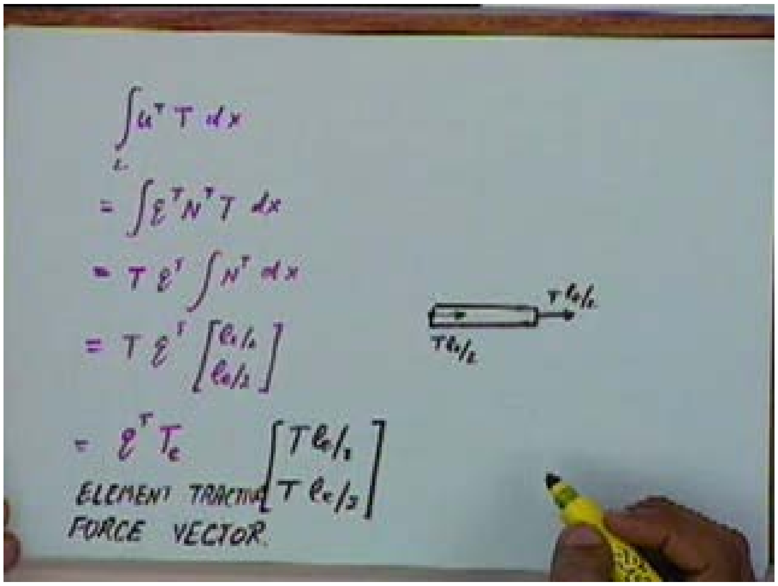

And from this term what we will get is a similar relation integral of u transpose Tdx over the length. This is equal to integral of q transpose N transpose multiplied by Tdx because I know u is equal to Nq so u transpose is q transpose N transpose, we just did that. This is equal to T into q transpose into integral of N transpose into dx. This will be equal to T into q transpose into again integral of N transpose dx will give us the same thing that is l_e by 2 and l_e by 2. This is the same as what we have just derived here.

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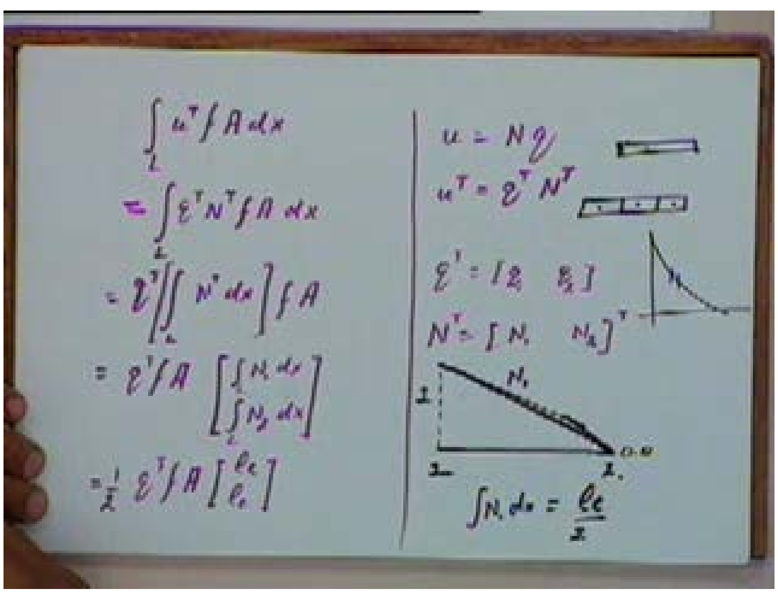

Integral of N transpose dx is what we get here. The same expression I am using there. So this let's say, I will say this is equal to q transpose into T_e where T_e is nothing but T into l_e by 2 and

T into l_e by 2. Again T is a tractive force acting on a per unit length basis, so T into l_e is the total tractive force on the body on the element, Tl_e by 2 will be half of that, Tl_e by 2 is another half of that. So effectively what we are saying is that this tractive force can also be treated to be acting as if it is split equally between the two nodes and we have half the force acting here and the other half of the force acting here and this vector we can call it lets say the element tractive force vector. Any question up to this point? This derivation is very similar to the derivation we just did. Now, the last term that we have in the potential energy expression that is this term sigma u_i P_i.

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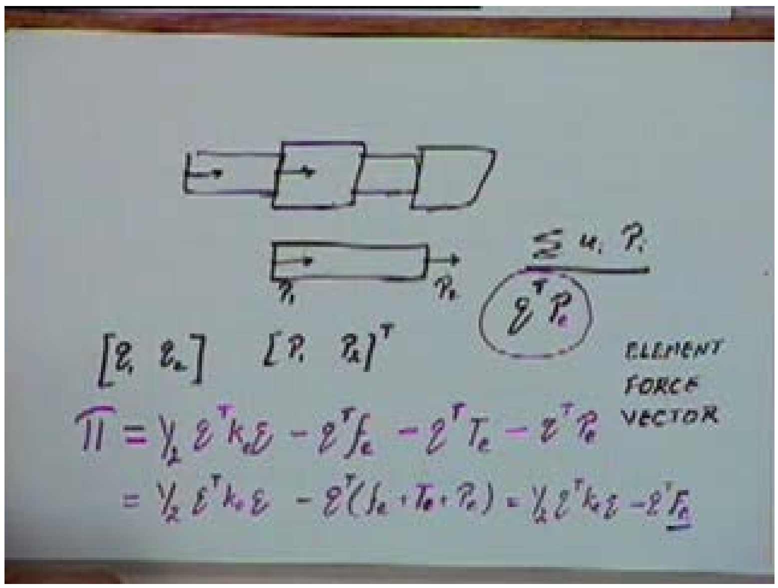

In this if we have a set of elements and so on, we can assume that all our point forces will be acting on some of the nodes. So the point forces are acting only on the nodes. So if we have some element then we can have some force acting here and we can have some force acting over here. So the contribution of the last term sigma u_i P_i, I can again rewrite this as q transpose multiplied by let's say P where P is let's say this force is P_1, this force is P_2. P is a vector P_1 P_2 transpose and q transposes will be q_1 q_2. The product of these two will give me this term because the u at this point is nothing but q_1 and u at this point is nothing but q_2 because this u is the deformation.

So the potential energy contribution because of point loads in this element, I can write it like this. So now if I look at the complete equation that I have, the potential energy pi, the first term is a strain energy term and I said that term is equal to half of q transpose k_eq. The second term that is this term is minus q transpose into f_e, this term is minus q transpose into T_e and this term is minus q transpose into P, let's say P_e. Now this, I will again put that as half of q transpose k_eq minus q transpose into f_e plus T_e plus P_e and this is equal to half of q transpose k_eq minus q transpose into F_e where this F_e capital F_e that I have used is what I call as the element force vector.

Now this element force vector is found out by taking the body forces, the total body force acting on the element dividing it equally into two parts, one part at the first node the second part at the second node. Taking the total tractive forces dividing it into two parts, one part at the first node the other part at the second node. I am taking all the point loads both for the first node and for the second node, summing up all the forces on the first node summing up all the forces on the second node and making a vector out of that or a matrix out of that. This is how we get the element force vector.

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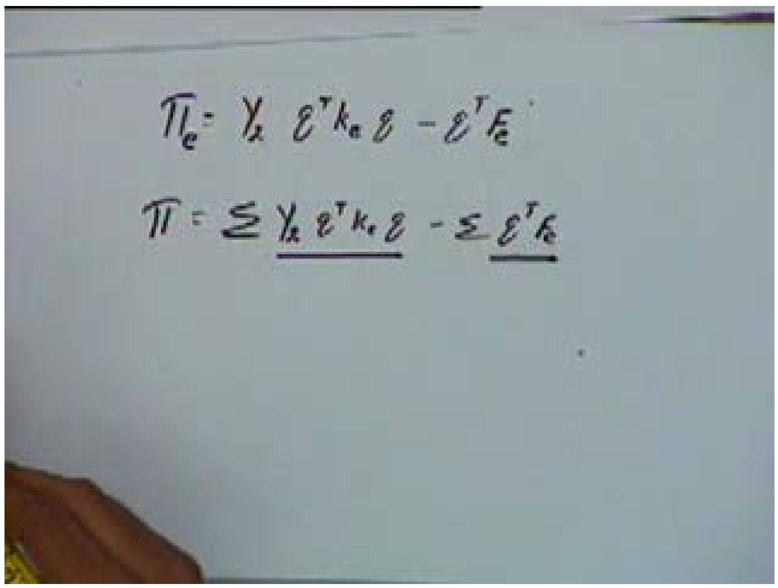

So our potential energy of the element is given by pi which is equal to half potential energy, this is energy of the element q transpose k_e multiplied by q minus q transpose F_e and the total potential energy in the system pi that will be equal to sigma half q transpose k_e q minus sigma q transpose f_e. Any questions up to this? Then the next thing is that these expressions are further potential energy in one single element. Similarly this is also further potential energy contribution in one single element. When I do a summation over all the elements, I can simplify these equations further. We can get them into one single matrix element, matrix and then remove the summation. Before going into that let’s take a simple example and see how we can get the k_e in F_e matrices.

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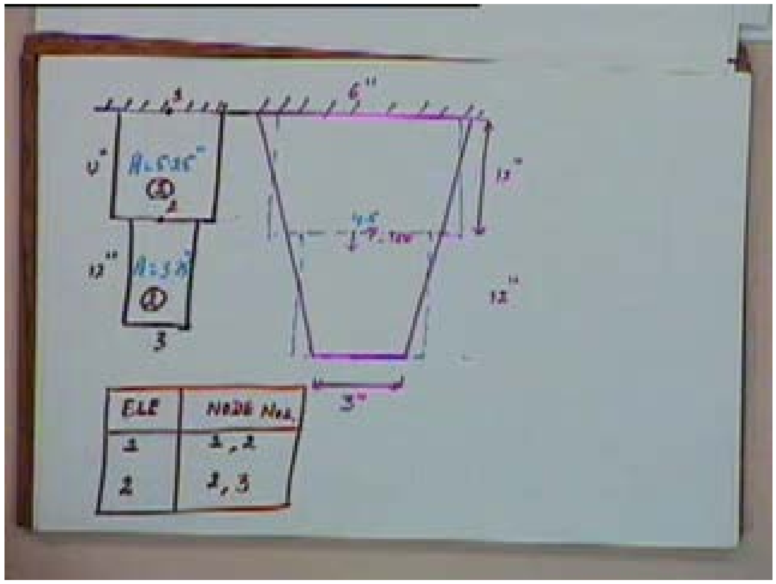

So if we take a simple example, let’s say this width is 6 inches. At the center point there is a force acting P equal to 100 units. This length let’s say if I take that to be 12 inches, the remaining length is lets say another 12 inches, let’s say this part I take it to be 3 inches. If I take an object like this I can approximate it by two elements which looks something like this. The length of the first element is 12 inches, the second element is also 12 inches.

Essentially what I have done is at the place where this point force is acting, I split it like this. I have taken one element here and the second element somewhere here and lets say for this element my area of cross section A, this is 3, this is 6, 9, I have taken a linear variation. So at the center how much will I get? 4.5, this would be 4.5, so for this element I will get between 4.5 and 6 if I take an average, I will get the area of cross section to be 5.25. Again?

Student: Area of cross section would be…
Professor: Area of cross section is in a perpendicular direction.
Student: Suppose depth is 1
Professor: So if I take a unit depth, I will get that to be 6 plus 4.5 and a half.

Similarly the area of cross section here is 4.5 plus 3 that is 7.5, I will get it to be 3.75 inches squared. Now let’s say this is my element number 1, this is my element number 2. Let’s say I call this my node number 1, I call this my node number 2 and I call this my node number 3, so my element one is between node numbers 1 and 2 and element number 2 is between node numbers 2 and 3.

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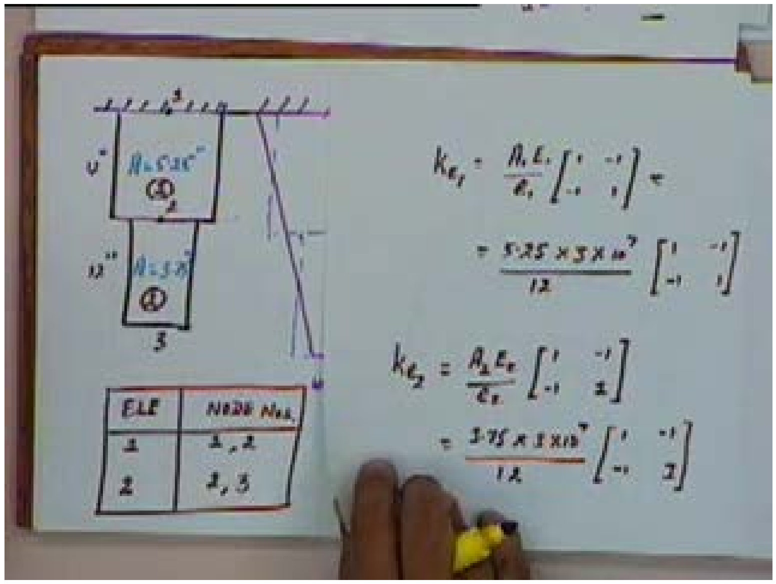

Now if I take the first element, for my first element if I want to write down the k_e term. How much is k_e, what is the expression for k_e? It is A_1 E_1 by the length of the element multiplied by 1 minus 1 minus 1 1. So this will be equal to area of the first element, this will be equal to, area of the first element is 5.25. Let's say the value of E, we take it to be 3 into 10 to the power 7 psi divided by the length of the element which is 12 inches multiplied by the matrix 1 minus 1 minus 1 1, this is for element number 1. Similarly k_e for element number 2 is equal to I will get A_2 E_2 by l_2 multiplied by 1 minus 1 minus 1 1 and this will be equal to 3.75 into 3 into 10 to the power 7 divided by 12 into 1 minus 1 minus 1 1. So these will be the two element stiffness matrices.

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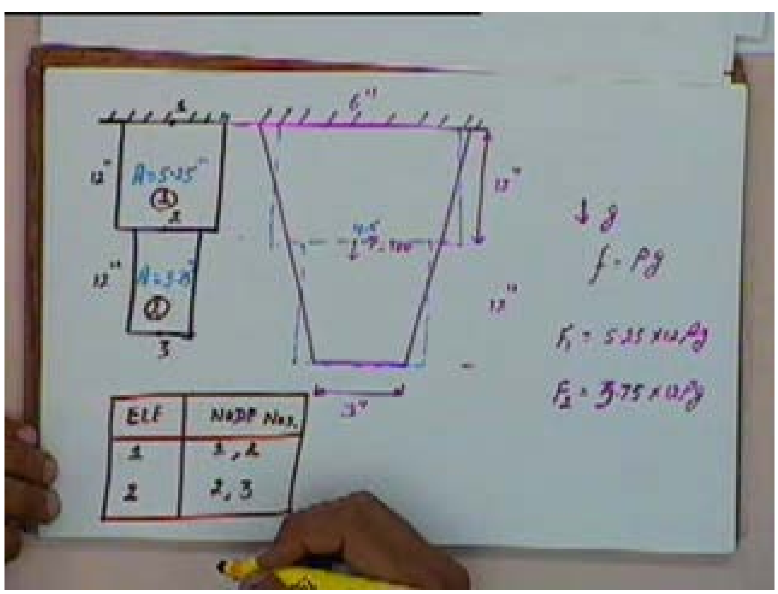

Now if in this object I assume that there is a vertical force due to gravity. I will get a force which is per unit volume f to be equal to rho time's g where rho is the specific gravity. The total force acting on this element that would be total force acting on the first element f_1 that will be equal to its volume 5.25 into 12 rho g. The total force acting on the second will be 3.75 into 12 rho g.

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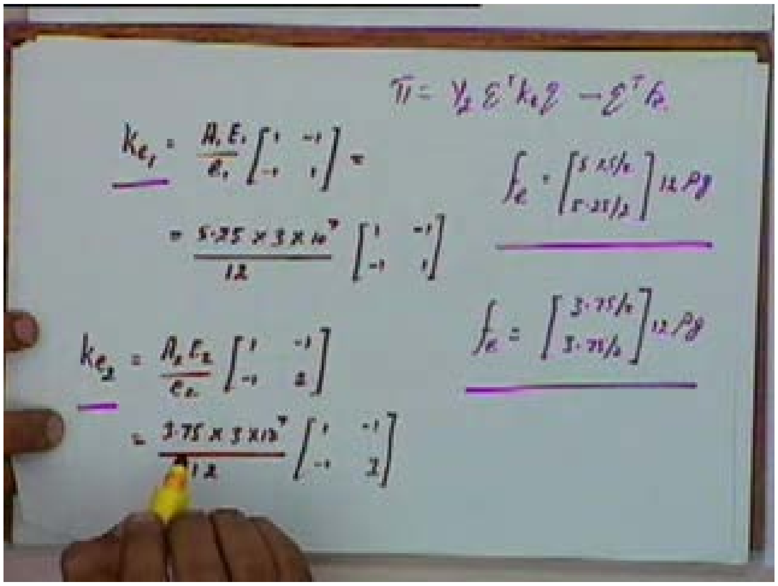

So if I take up my first element, for the first element f_e will be equal to f by 2 acting at both the nodes so that will come out to be 5.25 by 2, 5.25 by 2 multiplied by 12 rho g. No, that is the f_e 2. See I am looking at this element and whatever the total force acting half of it will act here, the other half of it for this node will act here. Similarly if I take up the other element I will get, the body force will be 3.75 by 2, 3.75 by 2 multiplied by 12 rho g. And if I take the point loads, I have a point load of 100 units given at the node number 2, at this node. We will incorporate that later on when we talk about the global equations, we will just see how that can be incorporated.

So as far as the elemental equations are concerned, we will get body forces given by this expression, element stiffness matrix given by this and by this and this. If I have this as the element stiffness then the potential energy we have just said is half q transpose k_eq minus q transpose f_e. So the k_e is this expression for the first element, for the second element it is this expression. You were saying something.

What about the force that is acting due to weight of the lower element on the upper? See right now what we have not done, we have said that the deformation here is let's say q_2 and the deformation here is q_1. We haven't yet found out the expressions for q_1 and q_2. When we do that even that weight will be taken into account. When we solve the global equations, your question will get answered, we haven't yet come to the global equations. Right now I am only telling you how to find out the different matrices. In the next class we will be talking of how to get the global equations and how to solve the equations. Once I answer that then your question will automatically get answered.

Any other questions? Now I will stop here now. In the next class we will talk of how to assemble the global equations and how to solve these equations.