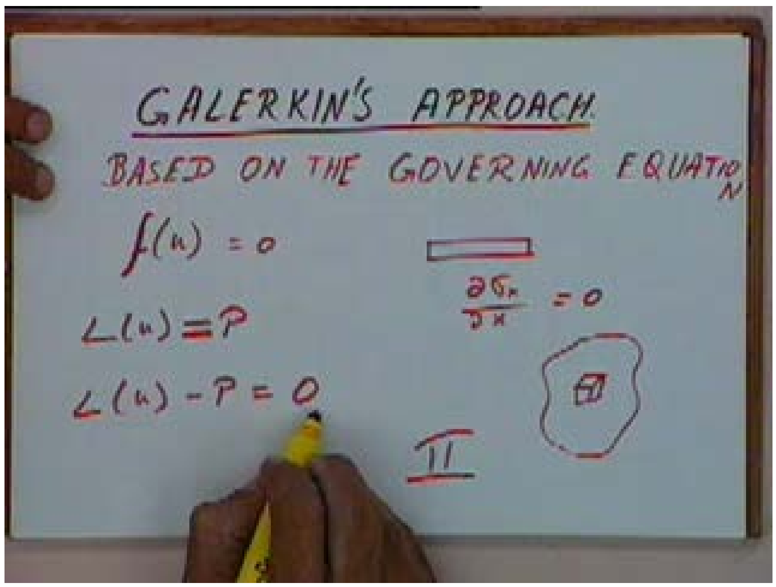
Computer Aided Design
Prof. Dr. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi
Lecture No. # 16
Galerkin's Approach

Today we will be talking of the Galerkin's approach to the finite element method.

(Refer Slide Time: 00:01:12 min)

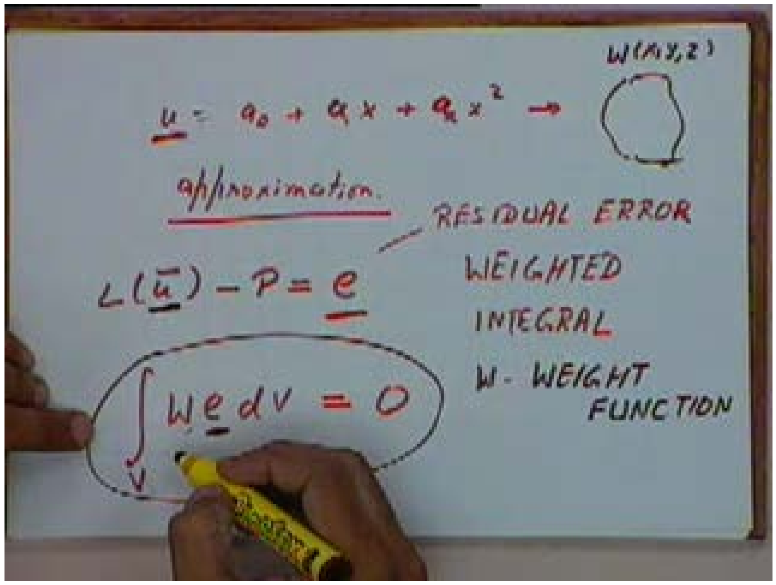


In this method we will basically use the governing equation of the system where the governing equation is typically an equation of the type f of u is equal to zero where u is the displacement. So some function of u will be equal to zero that means if we are talking of a one dimensional case and if you take a one dimensional element and we write down the equilibrium for this one dimensional element, this would be delta sigma$_x$ by delta x will be equal to zero. This you will get simply by looking at the equilibrium of this. In fact if you remember the equilibrium equations we wrote for the general three dimensional case, for the one D case this is the simplified form of that. So if we have the governing equation of this type, we can rewrite these governing equations as L of u is equal to P where P is the load acting on the element at that particular location.

So if we have a general three dimensional element like this and we take a element inside this and we write down the equilibrium equation for this that will typically be of this form L of u will be equal to P or we can rewrite it as L of u minus P is equal to zero where L is some function or some operator upon u. If this is a governing equation, this will form the basis of the Galerkin's approach. This is unlike the Rayleigh's Ritz methods in which we are using the potential energy pi. We are not going to use the potential energy now. What we will use is the, we will use the governing equation that is the equilibrium equation for this system.

So if we have L of u minus P equal to zero, if you remember in the Rayleigh's Ritz method we had taken some estimate for u.

(Refer Slide Time: 00:03:13 min)



Specifically in the example we had considered we had said, we will put u equal to a_0 plus a_1 x plus a_2 x squared. Now this description or this expression for u is not the actual deformation but is the estimate of the deformation or is an approximation. If you use this approximation in this equation that means instead of the actual deformation, we use the approximation we will get let's say L of u bar I am saying, I am using u bar for the approximation of u, so L of u bar minus P but now this will not be equal to zero this will be equal to some error term. L of u minus P is expected to be zero but L of u bar minus P will be some error term involved in that because u bar is an approximation is not the actual deformation.

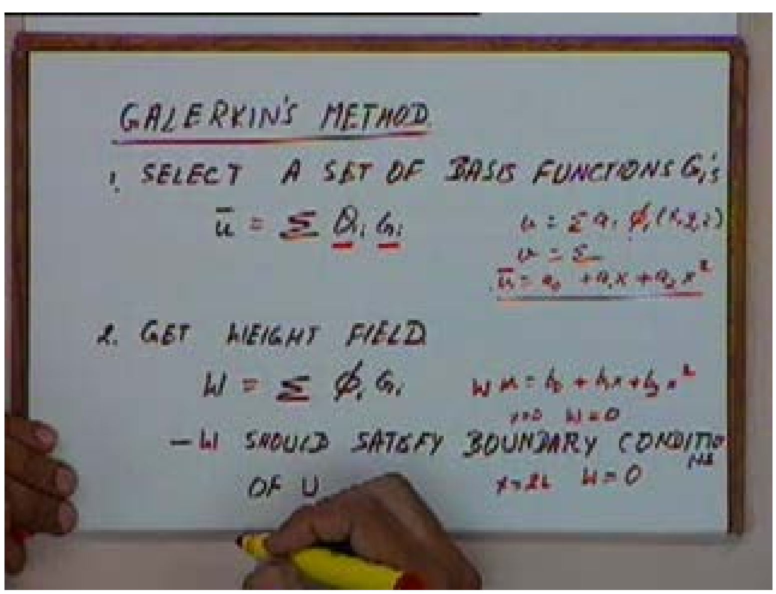
So if L u bar minus P is equal to e which is the error term, an idea will be, the basic idea of this Galerkin's approaches will take this error term and integrate it over the volume, we will carry out what is called the weighed integral. The weighted integral of this error term that we will try to put that equal to zero, so this error term e which is referred to as the residual error and the weighted integral would be let's say we take some weight edv and we integrate it over the volume, we will try to put this equal to zero and here this W is the weighting function or a weight function. So this W is also a, let's say if this is our object, this W is also a function of let's say xyz which is also a field in the three dimensional space.

So we will carry out this, the integral of weight times the residual error and integrate that over the volume and we will try to make that equal to zero. This is what we will do in the Galerkin's approach and in fact this is one of the approximate methods where we are trying to minimize the error term and what we will say is that this integral, this should be equal to zero for any weight function that we take. If we take any arbitrary weight function and this integral turns out to be equal to zero then we will say that this estimate that we have taken of u bar is an accurate estimate is an accurate approximation. That means let's say if I take this weightage to be constant throughout and I get this error to be equal to zero or if I take this weightage to be some other function and I still get the weighted integral to be zero which would mean that this e is almost zero or this u bar is an accurate estimate of u. So in the Galerkin's approach we will be taking this weightage function W as for a certain norm and then we will be taking u bar as for a certain norm and then we will take W as for a certain norm and then we will try to get this integral equal to zero. So let's see the, what's the question?

Sir appreciate P, why after multiplying with a waiting function and integrating over the domain and if you make it as zero a good approximation. Why does is happen? Let's say ideally if I take, if I say that this integral is zero for any arbitrary W, this integral is zero for any arbitrary W, whether W is constant, whether W is varying, whether W is varying linearly or quadratically or in any arbitrary way and this integral is still zero. When can that happen? That can happen only when u is zero, so if I can ensure that this integral is zero for any arbitrary W but normally in this method we take particular W.

So what I am saying is if you can ensure it for arbitrary W then I can ensure that e is zero, e is zero means u is the same as the u bar. That's what I am saying we don't have a issue. Let me complete. So what we will do now is instead of taking any arbitrary W, we will take W as per a certain pattern and we will take u bar also as per a certain pattern and then we will have to say that we will get a accurate estimate of u by ensuring this condition. You are ensuring that it's zero for any W. No. It is just a better equation. There is no way, there is no way of, in any of these approximate methods whenever we are doing an approximate method there is no way of ensuring that the error is always equal to zero. We can see that the u would be better u bar would be an I suppose. u bar really better than the actual estimate of u. We will go through the process, the steps involved in the Galerkin's approach then may be it will become clearer.

(Refer Slide Time: 00:09:15 min)



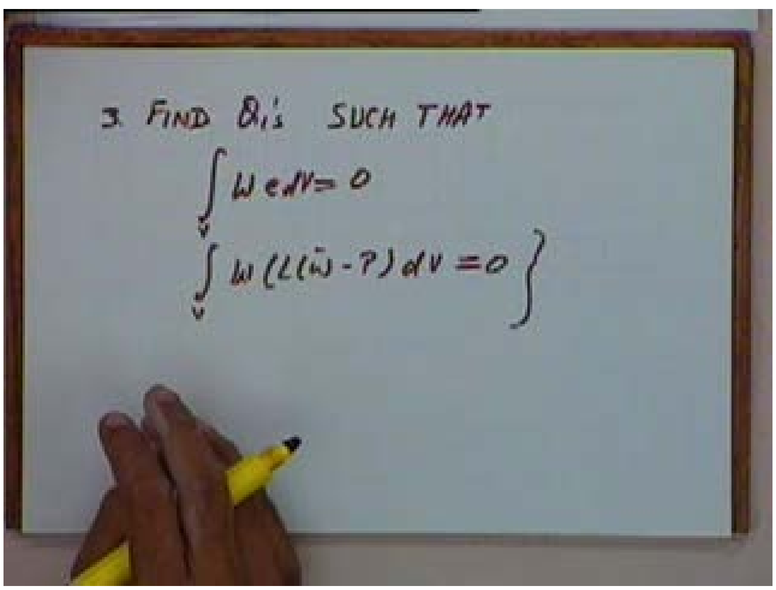
The first step would be that we will say, we select a set of basis functions let's say we call them G_i's. In just like we did in the last method, we will say u bar is equal to sigma let's say Q_i times

G_i. If you remember in the previous method, we had mentioned that we have written something like u is equal to sigma a_i phi_i of xyz and we said v is equal to in similar term and so on. This is something almost the same. We are taking the set of G_i's, a set of functions and a set of parameters Q_i and I am saying that this estimate of estimate u bar is equal to sigma Q_i G_i where G_i's are the basis functions which are the same as the phi_i's we had in the previous method.

And if you remember the one dimensional example we had taken, we had said u equal to a_0 plus a_1 x plus a_2 x squared. This expression is the same as this where the G_i's will be 1, x and x square and the Q_i is will be a_0, a_1 and a_2. So this u bar is equal to sigma Q_i times G_i, the G_i's are the basis functions. Then the next step is to get the weightage field or weight field and for the weight field we will say that this weight field W, we will use this same basis functions that is the restriction in the Galerkin's approach. We will use the same basis functions G_i's and we will say let's say this is sigma some other parameters times G_i. So if we had, if this was the u over here it may be, weightage we will give it as u is equal to b_0 plus b_1 x plus b_2 x squared. So we will give a weight field which will be given by this.

Once you have given it a weight field then we can find out the weighted error and integrate that but on this weight field, we give one more constant and that constant that we give is that this W should satisfy boundary conditions of u. This is an important constraint in this or important condition that is attached that means W should satisfy the boundary conditions of u. For instance we said that when we took the u, we said that this should satisfy the boundary conditions that at x equal to 0, u should be 0 and at x equal to 2 L, u should be 0. So in this case we are saying that u bar is equal to this, so u bar should definitely satisfy the boundary conditions that at x equal to 0 and at x equal to 2 L, u bar should be 0. Similarly W sorry not u this should be W and this W, the weight field this should also meet the same conditions that means at x equal to 0, W should be equal to 0 and at x equal to 2 L, W should be equal to 0. This is an additional constraint that is added in this method and this weightage should also satisfy these boundary conditions.

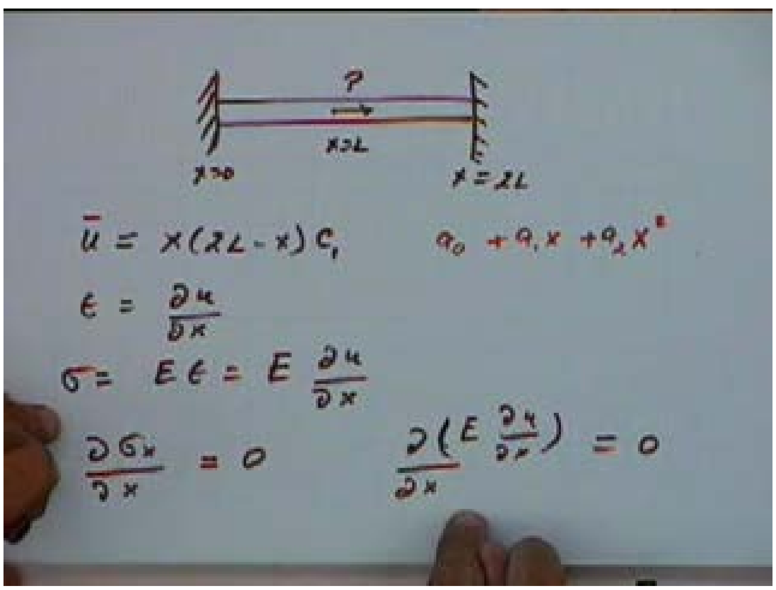
(Refer Slide Time: 00:14:17 min)



Then the next step is now that we have defined the u bar as well as W, the weightage field we will say that we will find, our eventual aim is to find out u bar. The u bar basically we mean that you have find out the Q_i's because G_i's are the function that we have taken and if we can find out Q_i's we found out the deformation field in the body. So our aim is to find Q_i. So we will say that we find Q_i's such that the weighted integral, integral of W times e is equal to dv over the volume is equal to 0 or this is the same as W into L of u bar minus P times dv will be equal to 0. So we will find Q_i subject to this condition.

For this condition will give us this set of equations which we will try to solve, so as to get the Q_i's. So the Galerkin's method will consist of these three steps, first is to select a function or to the select a field for u bar that is to get an estimate or an approximation of the deformation in the body. Then we get a weight field W and then we find out the error and put the weighted integral of the error equal to 0 and as we put this equal to 0, these set of equations will have some variables in that, those are the Q_i's. We will try to get the values Q_i's from this condition. So this is how the Galerkin's approach works. What I will do is I will take a simple example, the same example as we did last time.

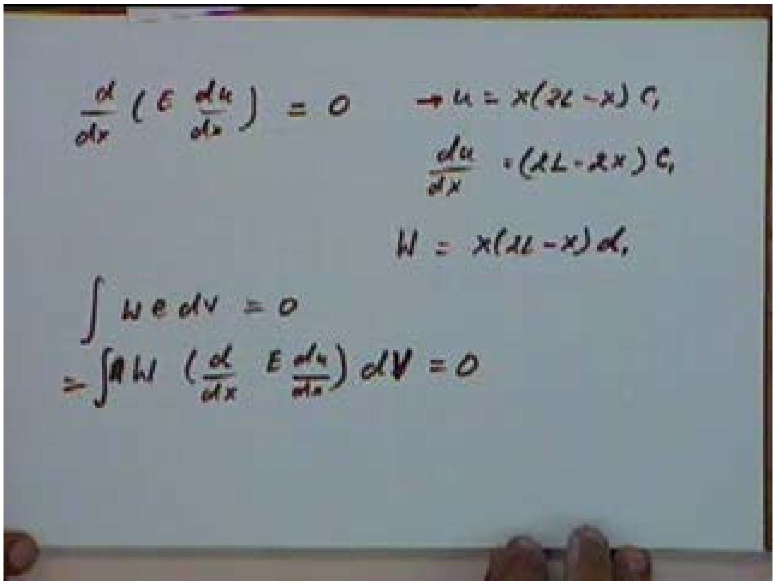
(Refer Slide Time: 00:16:26 min)



We will solve that using the Galerkin's approach and then I will try to make a general set of equations for the three dimensional case. This was the simple example we have taken last time. Now we will take the same example and solve that using this Galerkin's approach. The first thing is we have to take some approximation for u bar, so let's say we work with the same approximation as we had last time and I will write it directly now. We had taken a quadratic expression for u, if you remember what we did last time for that we said that u is equal to a_0 plus a_1 x plus a_2 x square and then we said that u at x equal to 0 0, so a_0 is 0 and u at x equal to 2L is also 0, so we got another condition and we got an expression similar to this. I think instead of c_1 we were using a_2 or something like that, so we will take this expression for u bar.

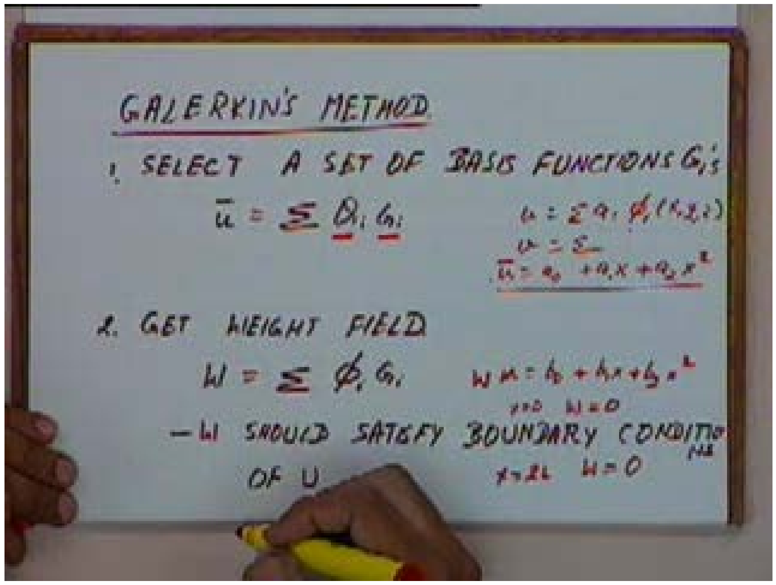
Now if this is a deformation epsilon, we know is equal to del u by del x and sigma we know is E times epsilon which is equal to E times del u by del x and the governing equation for the one dimensional case I just mentioned is del sigma x by del x is equal to 0. If I take this as the governing equation this will become, sigma will replace from here, so del by del x of E del u by del x will be equal to 0. Since now we are talking of a single variable all the del's can be replaced by d's because the partial derivatives will become complete derivatives.

(Refer Slide Time: 00:18:58 min)



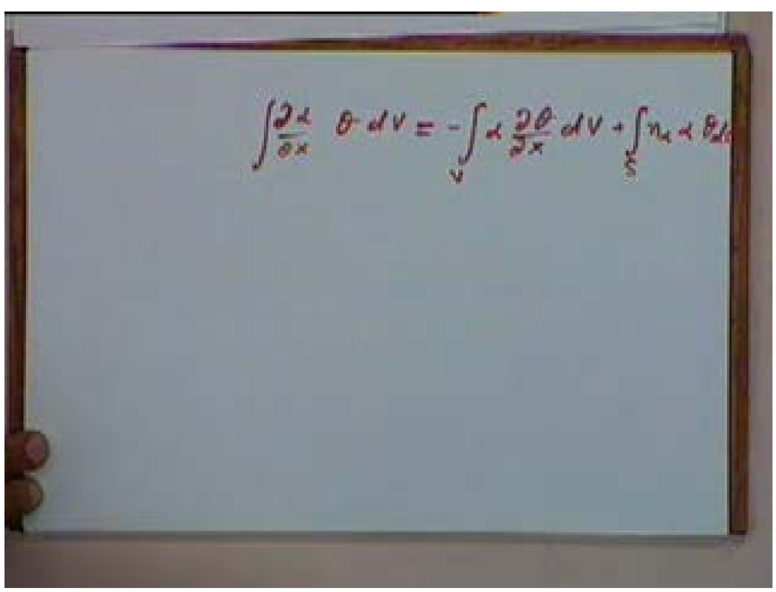
So this will become d by dx of E du by dx to be equal to 0 and except for the k, for the central point where there is the point load. In the point load we will say that this will be equal to P. We will come to that soon. So this is the governing equation that will be valid for the rest of the body. Now if we take this u as what we have taken, if u is equal to x into 2 L minus x multiplied by C_1 then du by dx that will be 2 L minus 2 x multiplied by C_1. Since I have taken this u to be given at this expression, I would take a weight field and we said that the weight field will have the same form as this.

(Refer Slide Time: 00:20:11 min)



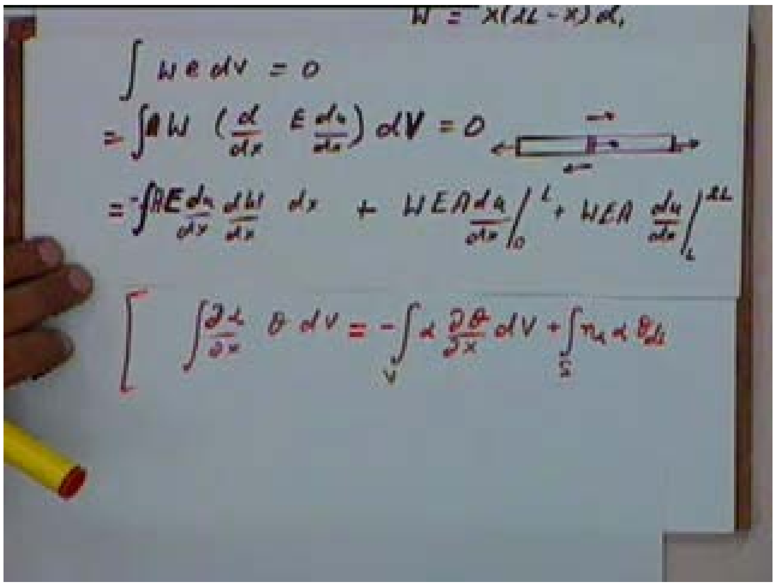
If you remember the second step we said the weight field will be W equal to sigma phi_i G_i where G_i's will be the same and the same boundary conditions will also be satisfied. So we will take W to be x into 2 L minus x multiplied by some other constant d_1 so this is that weight field we will take and now we have to evaluate the integral which is integral of W into e dv and put that equal to 0. So this W we will get from this expression or we can put that W into d by dx E du by dx multiplied by dv will be the same as A times dx will be equal to 0 or does not do that right now, I will just where W is the weight field that we have given. It's going to be a slightly mathematical. Now we want to simply this volume integral.

(Refer Slide Time: 00:21:59 min)



Typically what we will do is of we will make use of the formula that if you have integral of integral of del alpha by del x multiplied by theta times dv. You remember the formula for this? Alpha times del theta by del x times dv, after that plus, plus the surface integral of n alpha n dot alpha theta ds. So we will be using this integral and using this, we will try to simplify the expression that we have over here.

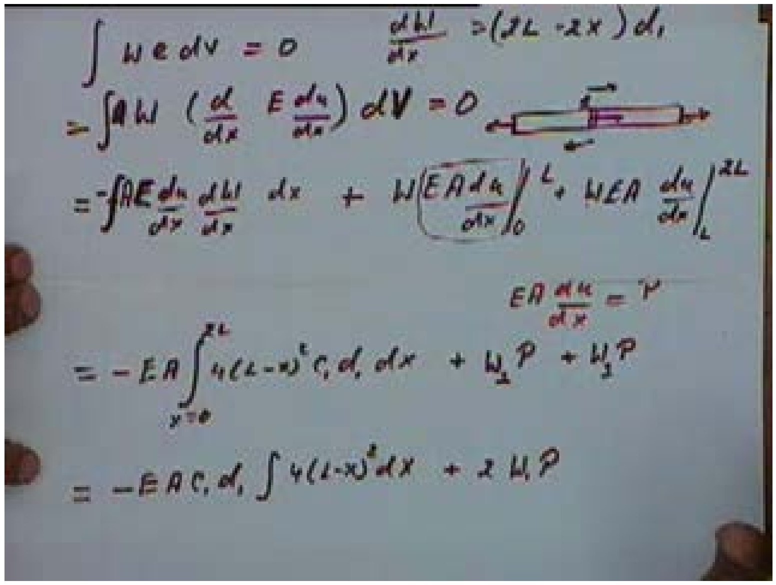
(Refer Slide Time: 00:23:14 min)



We have a volume integral which is of the same form. So if you use this expression over here, your W is the same as theta and this alpha is this expression. If alpha is this expression, we can simplify it now, we get minus of minus, alpha is E du by dx into del theta by del x where theta is the same as W, so into dW by dx, dv I will replace that by A times dx and we will get the integral of this minus, this plus this term. If this is the surface integral and this surface integral will be carried out over, I am talking of a one dimensional object, my surface is here or here because this is normally in this direction and since I am taking a dot with the surface normal these are the two surfaces I have, but in addition to that we have a point load over here. So we have taken in to account the point load, I will essentially have to take a discontinuity here to find out the this surface integral because at this surface I have a normal in this direction, at this surface I have a normal in this direction plus we have a point load acting in this direction.

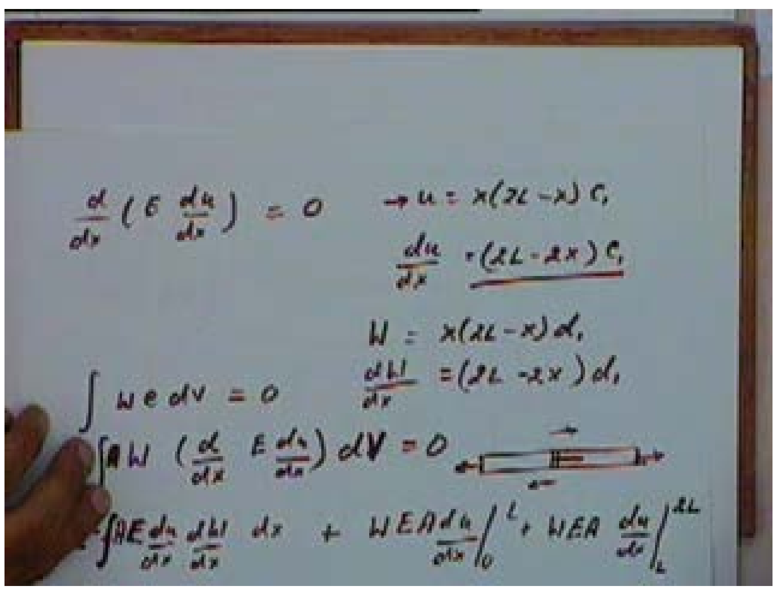
If I take the all that into account, the normal is either in the plus x or in the minus x direction. So this you can check up would become, we would simplify to WEA du by dx from 0 to L plus WEA du by dx from L to 2 L. You can just verify this, alpha is the derivative which is the same as E into du by dx. So E into du by dx is appearing as it is and theta is nothing but W, so theta W is appearing in both the expressions. So to evaluate the integral from 0 to 2 L, I splitted between 0 and L and and L to 2 L. This particular step is slightly mathematical, you can go through it slightly carefully, it will become very simple soon. So this integral will simplify to this. Now we will try to simplify this integral further.

(Refer Slide Time: 00:26:32 min)



Now if I take this first term, we will get minus E times A into integral of du by dx.

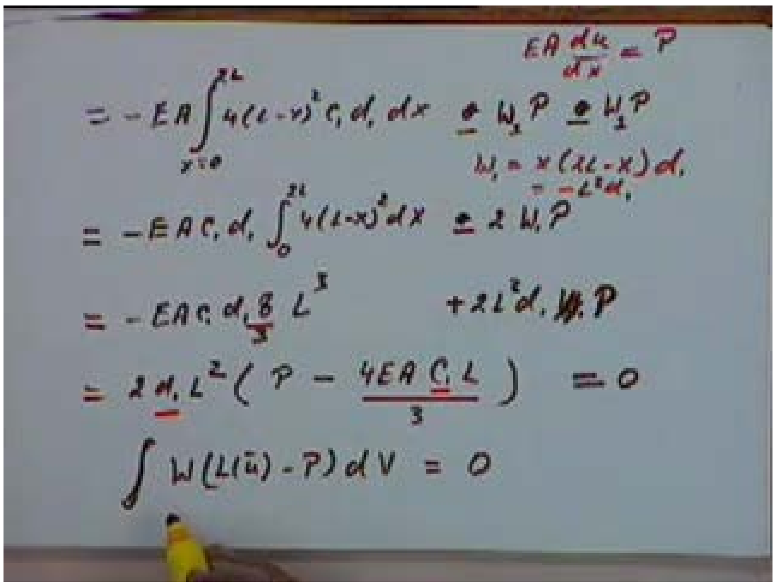
(Refer Slide Time: 00:26:32 min)



If you look at this term du by dx is given here and similarly dW by dx can be found out from here which is 2 L minus 2 x multiplied by d_1. So I will put these terms over here and we will get 4 into L minus x squared into C_1 into d_1 into dx, x equal to 0 to 2 L. These terms du by dx, for these terms if you look at this cross section, at this cross section this compete term that is this term EA into du by dx. What is this? du by dx is the strain, strain multiplied by E is the stress, stress multiplied by the area of cross section is the force. So this term will be equal to P.

So as you value at the integral from 0 to L and from L to 2 L, at x equal to 0 this term will be equal to 0. At x equal L, this term will be equal to P and at x equal to 2 L, this term will again be 0. So this integral we will evaluate to W times P where this W is the integral at this point one. This is my point one, so this integral will be W at 1 multiplied by P and similarly this integral will also be equal to W_1 times P. Excuse me? So this will simplify to minus EA C_1 d_1 integral of 4 into L minus x whole squared dx plus 2 times W_1 P. Sir how is it coming positive in this case? The direction of the normal's are different, as a result of that the minus term minus sign will come into it. We can do a detail steps and you will find this out. The surface normal are in the opposite directions, if we take care of that we will get the, we will get both the terms to be positive. That is basically because of the discontinuity, anyhow. Now this W_1 what is the expression for W_1?

(Refer Slide Time: 00:30:06 min)

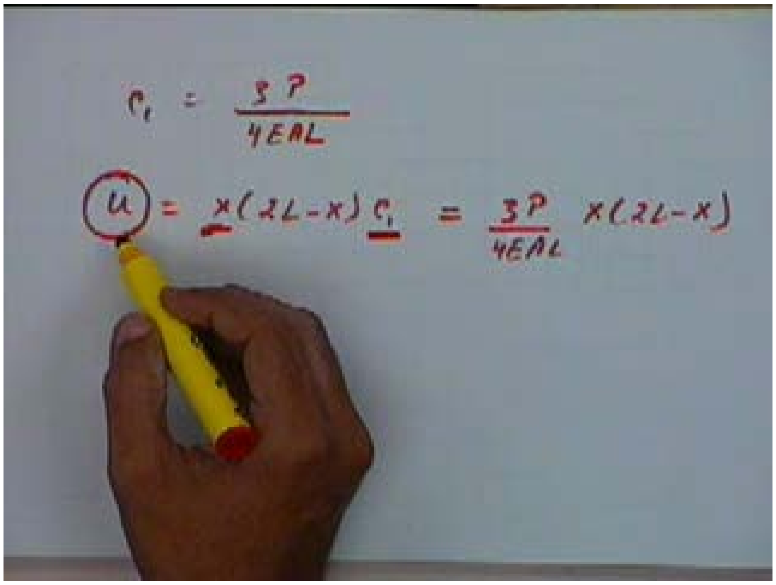


W_1 is equal to x into 2 L minus x multiplied by d_1 at x equal to L, so this will become L into minus L squared d_1. So this term will become minus 2 L squared d_1 into W_1 and this term, just one sec. This integral if you evaluate it from 0 to 2 L, again this will come out to minus EA C_1 d_1 into 8 by 3 L cube. In fact I think there is a, minus sign I have got an extra, I think you will get a minus sign over here we just check it up, where this term comes out to be plus, so this W_1 or P. sir again. This is W_1 or P, it should be P. This will be 2 L square d_1 times P.

Now this is equal to, if you take 2 d_1 L squared as common we will get P minus 4 EA C_1 L by 3. Now this is what the integral of the error term that we have, integral of W into L of u minus P dv is equal to this expression and we said that we want this term to be equal to 0. So if I put this term is equal to zero, now this is 2 d_1 into L square where d_1 is one parameter that we have and we have one parameter over here which is C_1. Now d_1 is the parameter which is specifying the weight and C_1 is the parameter specifying the deformation and we want this to be true for all possible values of W. If this has to be true for all possible values of W that means it has to be true for all values of d_1.

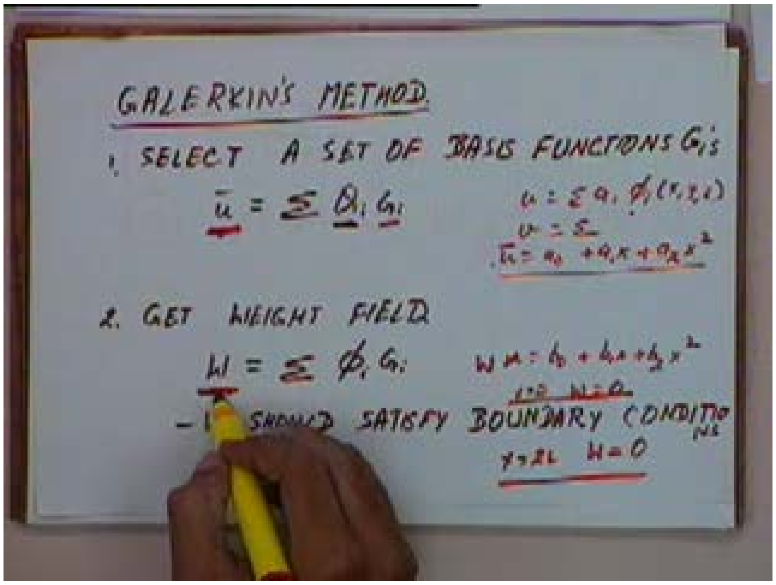
If this has to be true for all values of d_1 then this term has to be equal to 0. If this term is 0, we can get the value of C_1 so if you put this term equal to 0 you will get C_1 to be equal to 3 P by 4 EAL and we have taken u to be equal to x into 2 L minus x into C_1 that will become equal to 3 P by 4 EAL into x into 2 L minus x.

(Refer Slide Time: 00:33:30 min)



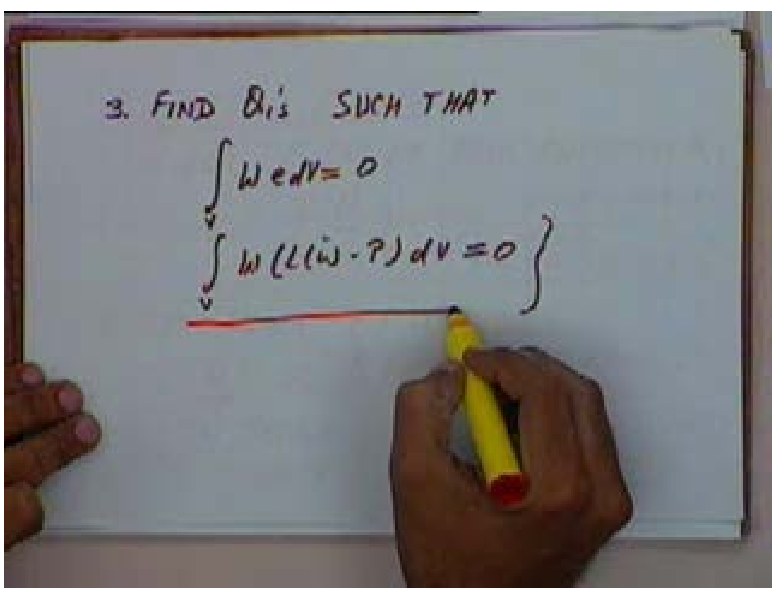
This way you were basically able to get the value of C_1 as a result of that we are able to get a value of the deformation. What we assumed initially that we will take a weight field W and we tried to get this integral equal to 0 for all possible weight fields that is why this weight field has a parameter d_1. So we want this integral to be equal to 0 for all values of d_1. If it has to be 2, if it has to be 0 for all values of d_1 this term has to be equal to 0 and if this term is equal to 0 we can get a value of C_1 from this and we are thus able to solve for a deformation. So this is how we can solve for deformation using an Galerkin's method in a one dimensional case. The steps involved are these.

(Refer Slide Time: 00:35:09 min)



We first select a set of basis functions and take make some assumption for u bar, the u is the same in assumption for W. We ensure that this u bar satisfies the boundary conditions, we also ensure that W should satisfy the boundary conditions. If this W also satisfies the boundary conditions then we can go on to the next step and that is we evaluate the weighted integral, the integral of the weighted error and we put that equal to zero.

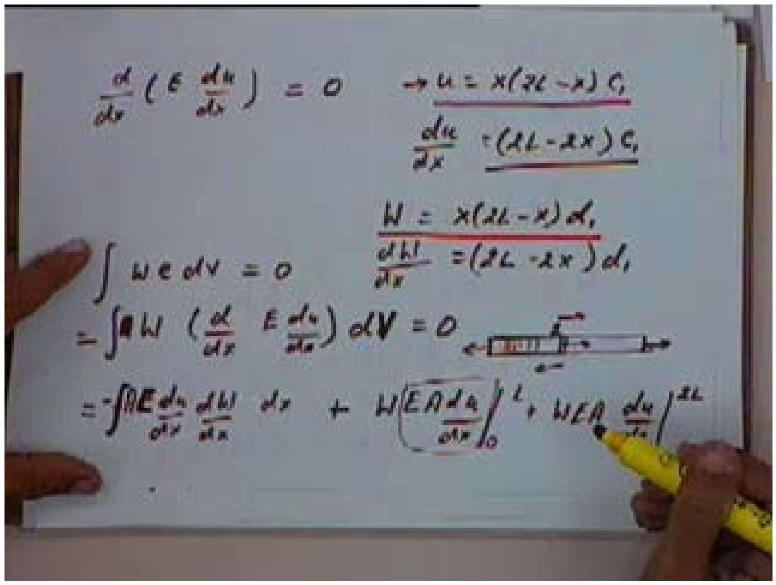
(Refer Slide Time: 00:35:32 min)



As we put this equal to zero we will get a set of equations which we will have to solve for these parameters Q_i. If this is the basic method that is followed in the Galerkin's method, Galerkin's approach. Excuse me sir.

Student: For the center point the value of u will be different will be minus P also, so this was for the other point beside the center point. You are talking of the value of e, you put in the integral WE EDD.

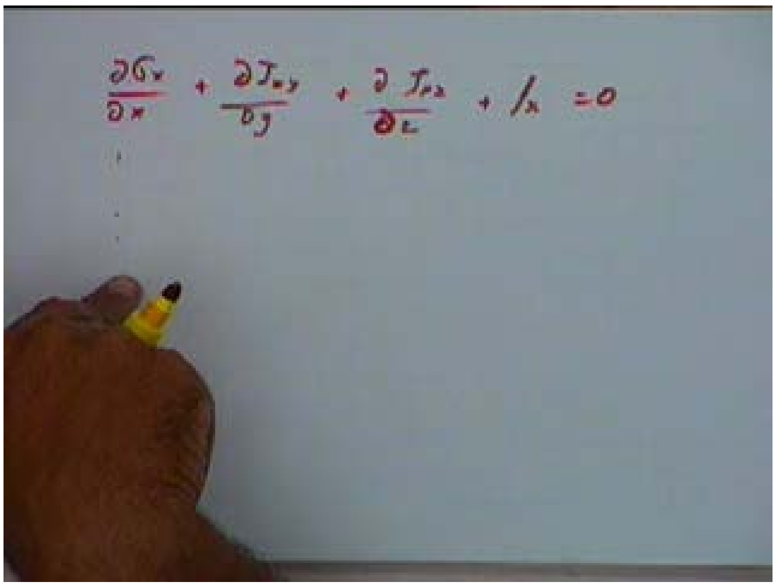
(Refer Slide Time: 00:36:12 min)



The value of P? e, small e error, P that is the point load. Sir e. The value of e, yeah the error. The value of e we are trying to evaluate at across this body, we are trying to evaluate this integral, this error at all the different locations and integrate that over the volume of the body, so the value of the error will be different at all locations. Student: This value d by dx of e times there, e by dx so wont this there is a P term in this for the center point or the way we have defined error itself. So you are talking of this term? No, sir in that integral where W ed this step. The main integral which we equate to 0, here the first step. The error e, sir you defined e as L of u minus P equal to 0 that means for the center point the error will be this L of u whichever is du by dx or d by dx of e times du by dx minus P. minus P and for the rest it will be. For the rest it will be that, that's right because that point load P will be occurring only at the center point in fact that is why I said we have a discontinuity in the error functions and that is why we took it as a discontinuous manner to evaluate the integral.

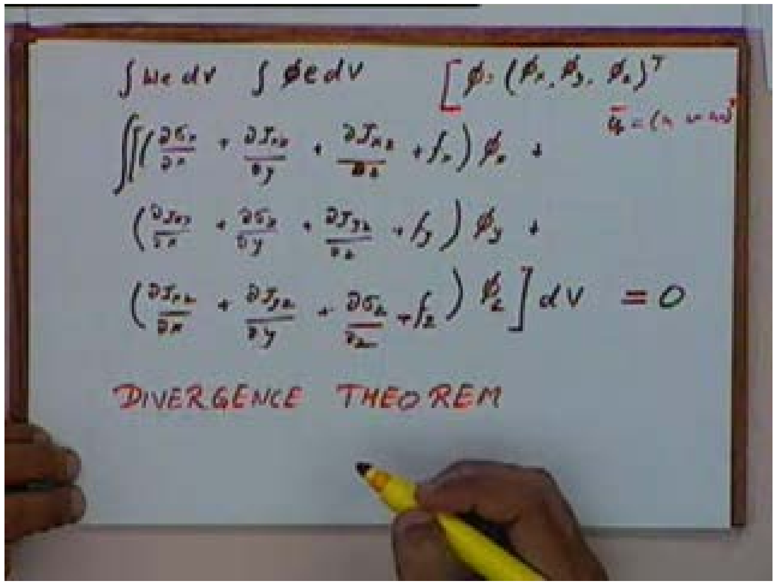
If you do it very rigorously, you will be able to get that. That means we will get the value of C_1 to be different for the center point. Value of C_1, this. No C. What is C_1. C_1 is giving us the distribution that I have assumed for u. So C_1 is the constant that we had assumed, we assumed that the deformation will follow this pattern. So the value of this c_1 will not vary along the length of the body or along the volume of the body. No for the center point if we do it with P also, the point load, won't it come different. No, that's actually we have done it here. In this case we have already incorporated that. We have already incorporated that and calculating these integrals. Fine. The error at the center might be different but what we have to do is that we integrate the error over the volume of the body, once you integrate that only then you will get the value of C_1. Sir, I had been like you have got at two different places, we had point load P at the first half. if you had a point load here and the point load here there will we do it in 4 steps, 3 steps 3 steps because at the point load you have a discontinuity. What we will be doing now is that using this method we will write down the general system of equations for a general three dimensional case and then we won't have to do these steps in detail. We will get a uniform method which will be much more simple, all the complexities will go into the proof of that. So we will take that up and then may be this confusion will not come. Any other questions? What I will do now? If we take a general three dimensional elastic material, we say that for the elastic material if you remember the equilibrium equations I mentioned last time those equations were of the form del $sigma_x$ by del x plus del tau_{xy} by del y plus del tau_{xz} by del z plus fx equal to zero and so on.

(Refer Slide Time: 00:39:42 min)



That's just three equations for this, three equations for the tractive forces plus we will have another three equations of point loads. If you take up those equations we can incorporate all these equations into the Galerkin's approach in evaluating these integrals. Now just see how that can be done.

(Refer Slide Time: 00:40:32 min)

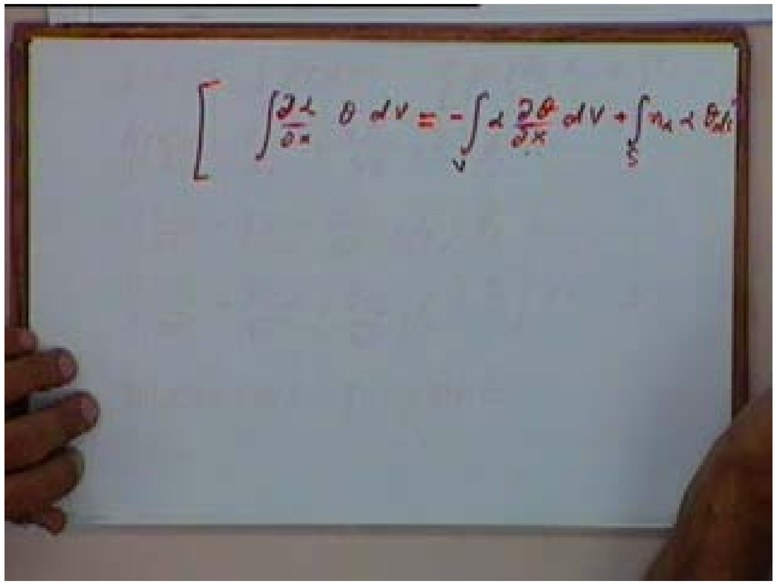


For an elastic material what we have to do is we have to find out the integral of We dv. Now this W we will call that as phi, phi e into dv. If any in the book and other places also, this phi is a more standard notation for this weightage function so integral of phi times e into dv. If you want to evaluate this integral, this will look like del $sigma_x$ by del x plus del tau_{xy} by del y plus del tau_{xz} by del z plus fx this thing multiplied by phi_x plus we will have another such term multiplied by phi_y plus we will have a third such term multiplied by phi_z and this complete thing multiplied by dv this should be equal to 0. The second term will look like del tau_{xy} by del x plus del sigma y by del y plus del tau_{yz} by del z plus fy and this should be del tau_{xz} by del x plus del tau_{yz} by del y plus del $sigma_z$ by del z plus fz.

Now these three give us the equilibrium equations for a three general three dimensional case. This is the equation in the x direction so this would give us the error term in the x direction multiplied by phi_x that is the weightage we are giving in the x direction. Similarly this term multiplied by phi_i and this term multiplied phi_z but this weightage phi that we are giving that will consist of terms phi_x, phi_y phi_z. So this phi will be equal to this and again in the Galerkin's approach we have to follow the same method, the same constrains that for u, we will be assuming a certain field will assume a certain approximation for u. We will say u bar let's say is equal to u v w transpose. Whatever approximation or whatever formulation we choose for u bar, we will use the same formulation for phi with the additional constraint that even for phi the same boundary condition should be satisfied.

Now the importance of satisfying the boundary conditions for phi that we will be seeing later where that ensures a particular type of condition for phi, we will come to that in the end but if this phi is meeting this constraint and this u bar is the estimate of u then this integral term will try to put that equal to zero and again in order to simplify this integral, we use the same formula that we had earlier that is the Divergence theorem and this Divergence theorem is what I wrote down over here.

(Refer Slide Time: 00:44:33 min)



The del alpha by del x times theta dv will be equal to minus of integral alpha del theta by del x dv plus n alpha alpha theta ds surface integral of that. So we will take this Divergence theorem, apply it to this integral and then simplify the integral that we get. That will give us a simple method of applying the Galerkin's approach for all elastic materials. In the next class we will be applying, we will be using this Divergence theorem to simplify this integral and continue from that point and that's all we will be doing today.