Computer Aided Design

Prof. Anoop Chawla
Department of Mechanical Engineering
Indian Institute of Technology, Delhi

Lecture No. # 15

Finite Element Method: An Introduction

Today we will be starting, today you will have your first of a series lectures on finite element method. What I will first do is I will first give the basis that goes behind this finite element method and then we will take specifically one dimensional, two dimensional and three dimensional analysis using finite elements. First let us do a quick recap of some of the solid mechanics fundamental that you have done earlier.

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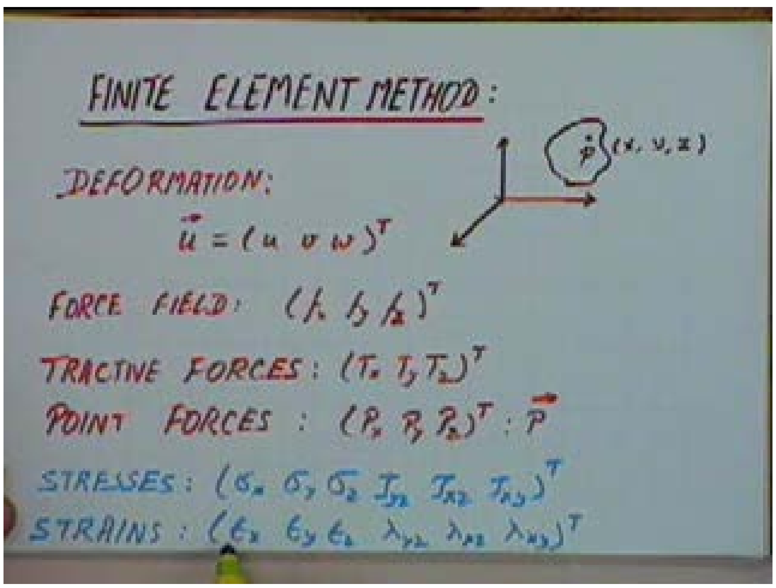

In terms of rotation if you have any three dimensional entity and in that there is a point p given by xyz then the deformation at this point p that is a vector U that will be consisting of three deformation that is u v and w and we will represent it as a column vector. If have a column vector u v w that will be a notation we will use for the deformation at any point p. So this deformation vector u will have three components, u for deformation in the x direction, v for deformation in the y direction and w for deformation in the z direction.

Similarly if there is a force field that force field we will represent by f_x f_y f_z that again a three dimensional vector which should be again a column vector and this force field is essentially the force field in an object and this will be force per unit volume that means the force per unit volume acting at each element in this body that is what we referred to as a force field. Then if there can be some tractive forces on the body, these tractive forces we will represent as T_x T_y and T_z transpose. So these tractive forces in a case of three dimensional body, they are the forces per unit area acting on the surface.

If we have any three dimensional object, we can have a volumetric force that will be force per unit volume typically something like gravity or magnetic forces or something like that. On the surface we can have pressure fields or we can have a uniform distributed load or similar force field acting on the surface. These will be called tractive forces and we can have point forces. If we have a point force, we will represent point forces by the vector P_x P_y P_z transpose. This should be the vector, we will refer to that as a P vector or the point force vector. Then if we talk of the stresses, the stresses we will represent by, the stress will consist of 6 components, 3 stresses sigma$_x$ sigma$_y$ sigma$_z$ and tau$_{yz}$ tau$_{xz}$ tau$_{xy}$ are the shear stresses. So this vector would represent the stresses and similarly if you talk of the strains, the strain will be represented by epsilon$_x$ epsilon$_y$ epsilon$_z$ lambda$_{yz}$ lambda$_{xz}$ and lambda$_{xy}$. Instead of representing stresses and strains by a three by three matrix, we will represent them by a vector consisting of 6 terms. The three stresses epsilon strain epsilon$_x$, epsilon$_y$ and epsilon$_z$ and we will have the shear strains lambda$_{yz}$, lambda$_{xz}$, lambda$_{xy}$.

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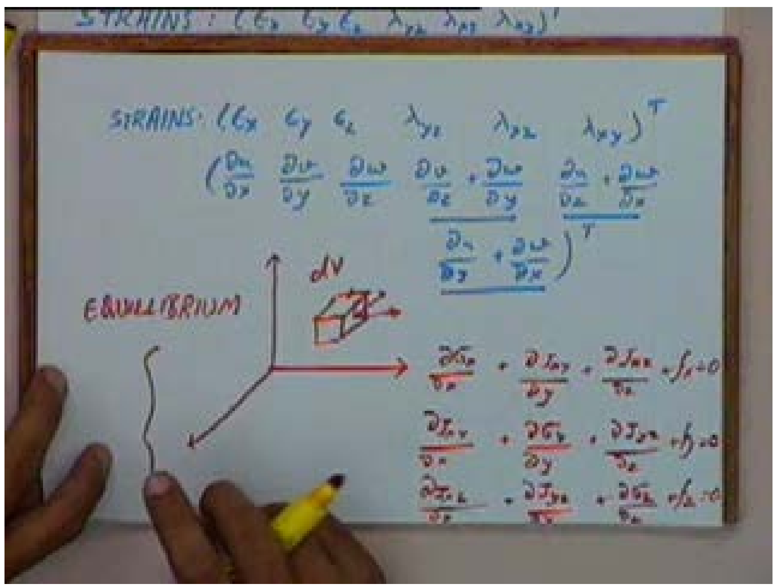

Now these strains epsilon$_x$, if you remember how epsilon$_x$ is defined this will be same as del u del x where u is the deformation in the x direction. Similarly epsilon$_y$ that is the same as del v by del y and epsilon$_z$ is the same as del w by del z. Then the next term is lambda$_{yz}$ and this lambda$_{yz}$ would be the same as del v by del z plus del w by del y. The next term is lambda$_{xz}$ that would be the same as del u by del z plus del w by del x. The last term which is lambda$_{xy}$ that would be del u by del y plus del v by del x. So the strains can also be represented using these partial derivatives. So this term is here, this term is here and this term is here then if we consider any elemental volume, a solid if we consider a small elemental volume and for this elemental volume dv, we write down the equation of equilibrium then we will get a set of three equations which will be given by del sigma$_x$ by del x plus del tau$_{xy}$ by del y plus del tau$_{xz}$ by del z plus fx is equal to zero, this you must you have done in your solid mechanics. You are taking this elemental volume, finding out the forces acting on all the faces.

For instance because of the stress sigma$_x$ there will be a force acting in this direction because of shear forces there will a force acting in these directions and so on and on all the faces plus there will be elemental body force. If I write down the equilibrium of all these forces in the x y as well as the z direction, I will get three equation of this form. This is one, the next would be del tau$_{xy}$ by del x plus del sigma$_y$ by del y plus del tau$_{yz}$ by del z plus fy will be equal to zero. And the third one would be similar it is del tau$_{xz}$ by del x plus del tau$_{yz}$ by del y plus del sigma$_z$ by del z plus fz will be equal to zero. Student: Sir, in all these equations the force sign will be negative because the external force should be equal to the stresses. No, the signs will be taken care of by the direction that we are choosing as this thing. So the magnitude can be negative but because of sign convention we are choosing, you are getting positive tau. So these are three equations we get from equilibrium in an elemental body element or in a cubical body element.

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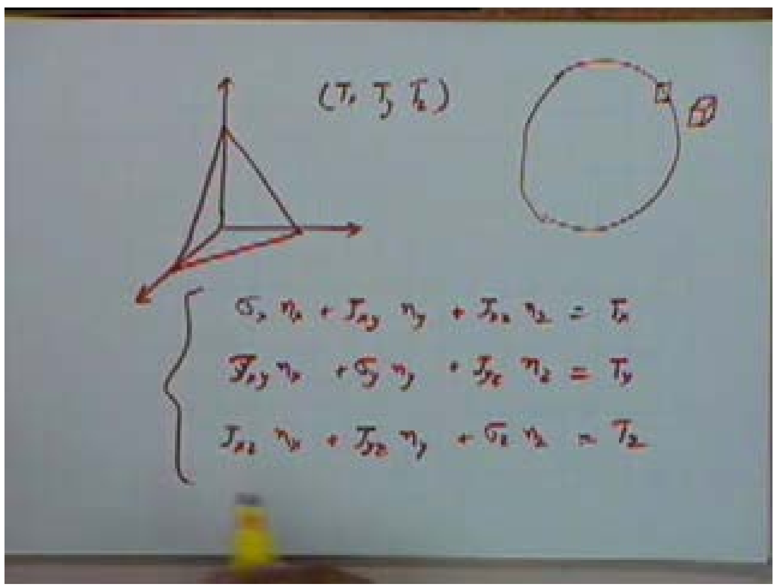

Similarly if you take an element which is on the surface, if an element is on the surface it will look like this. If you have a three dimensional surface or a curvature and you are taking an element which is on the surface it won't be cubical but it will be somewhat pyramidal element like this or a tetrahedral element like this. If you consider an element of this shape and we write down a similar equation, now the equation will be slightly different because on this surface we will have the tractive forces and those tractive forces will be in a direction given by T_x T_y T_z. This is the corner point. No, if you have a body like this and I take any element on the surface, this is a point on the surface so now this is an element on the surface rather. If I have taken a complete element, it would have been something like this in a two dimensional sense or in the three D the element would like this.

I am cutting it by a curved surface that I am approximating by a plane like this. If I write down the equations of equilibrium for this, what we will get would be something like this sigma$_x$ n_x plus tau$_{xy}$ n_y plus tau$_{xz}$ and n_z will be equal to T_x and tau$_{xy}$ n_x plus sigma$_y$ n_y plus tau$_{yz}$ n_z will be equal to T_y and tau$_{xz}$ n_x plus tau$_{yz}$ n_y plus sigma$_z$ n_z will be equal to T_z. So these will be the other three equations of equilibrium in the case of this element. These derivations you must have done in your solid mechanics courses. We will be using some of these equations when we come to the finite element method. Yeah, tractive forces per unit area.

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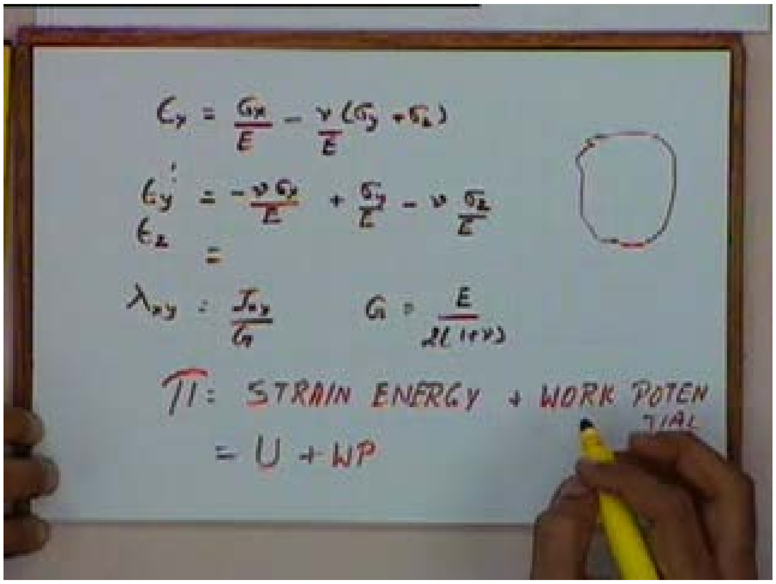

The next set of equations let's say if we are talking of the strain epsilon$_x$ and you want to express the strain in terms of the stresses, epsilon$_x$ will be equal to sigma$_x$ divided by E where E is the Young's modulus minus nu times sigma$_y$ plus sigma$_z$ divided by E where nu is the Poisson's ratio.

Similarly we will have equations for epsilon$_y$ and epsilon$_z$. So for epsilon$_y$ would be minus nu sigma$_x$ by E plus sigma$_y$ by E minus nu sigma$_z$ by E. Similarly you can write for epsilon$_z$. If we consider lambda$_{xy}$, lambda$_{xy}$ will be tau$_{xy}$ divided by G. Here we can show that G is equal to E divided by 2 into 1 plus nu. Now for any three dimensional object which is subjected to a set of strains and set of loading, the potential energy in this object that is the total potential energy pi, this potential energy is equal to strain energy plus the work potential. This will, strain energy we will be using the symbol U plus the work the potential WP and the strain energy if I say pi is equal to U plus WP.

The strain energy would be half of integral of sigma transpose epsilon dv polymetric integral, this is U. That means if we take the stresses, multiply the transpose of the stresses with the strain vector and integrate that over the volume then the our half of that will be the work, the strain energy and the work potential that would be minus of integral of U transpose fdv minus surface integral of u transpose Tds minus sum of all u$_i$ transpose P$_i$, i going from 1 to n. This work potential is essentially the work that can, that would be done against the forces that are acting. So if there is a let's say a body force, this is my object and there is a force field here like this. If this object undergoes the deformation and it gets deformed like this then in carrying out this deformation we have done a force against this force that is this component.

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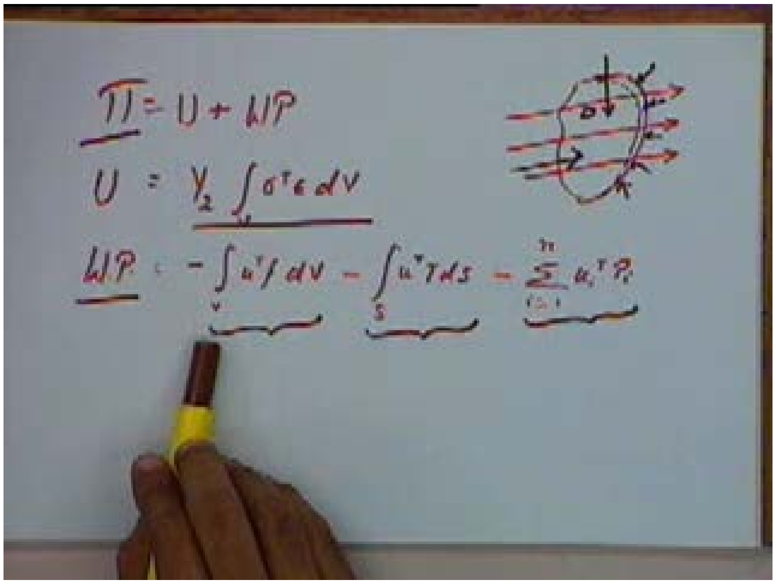

So at each element inside this body, the work done will be u transpose into f, work done per unit volume, u transpose is a displacement, f is the force. The product of force and displacement will be the work done per unit volume, this force is per unit volume multiplied by the volume of this element and integrated over the complete element.

Similarly this will be the work done at the surface. If there is some load at the surface then in carrying out this deformation whatever work is done against these forces that is expressed by this term and if there are some point loads then whatever work is done against these point loads that is expressed by this term. So this is the expression for the work potential, this is the expression for the strain energy and the potential energy is the sum of these two. Is that all right? Any question up to this point? Now sir we use strain energy include the work potential at all which are found in the work potential equation. The strain energy no means work done against deformation I mean work done for it to get deformed that is the first term of the.

No, work potential is the work done against the external forces, the strain energy is whatever energy is retained inside because of the strains in the body. The two are because of two different terms, the sum of the two will be the total potential energy. If we have an object we keep it compressed the strain inside the body, there will some internal forces which are causing those strains, they will cause some potential energy that is the strain energy part of it. The work potential is that in holding this, I have applied some forces from outside so that is the work that has been done from outside so that part is the work potential.

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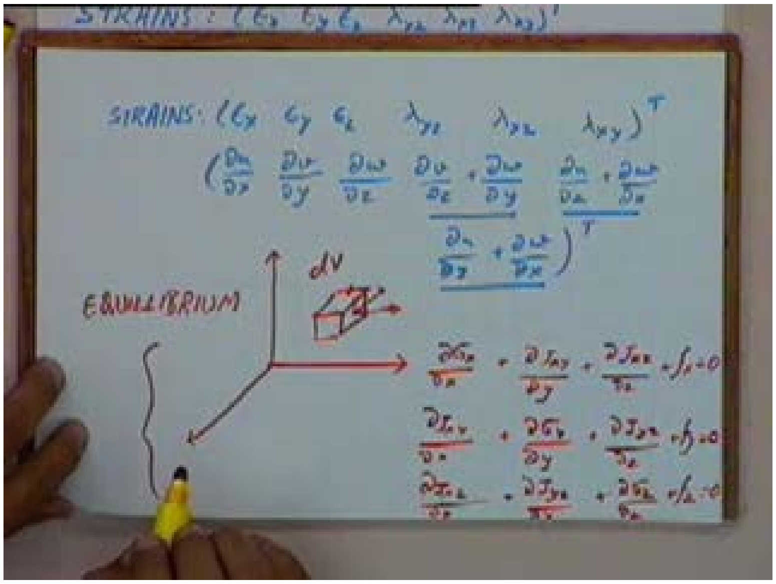

Now if you look at these equations, these express the strains in terms of the deformations uv and w.

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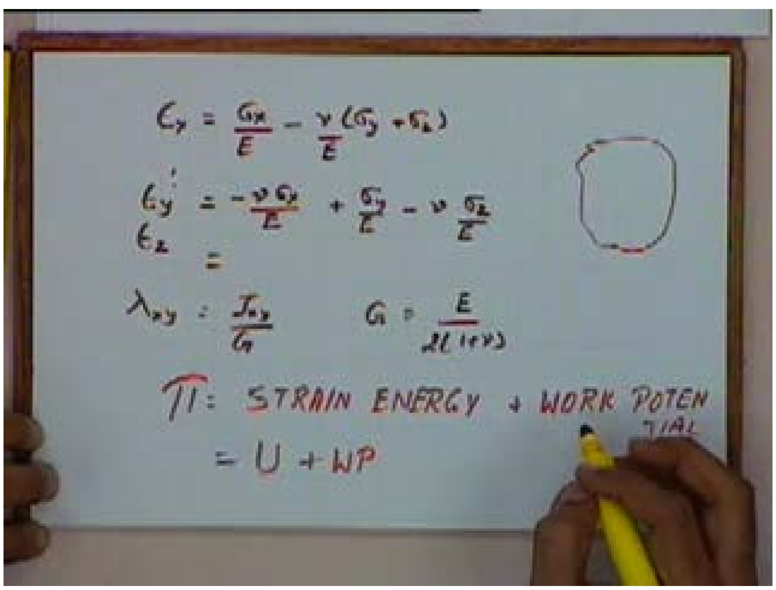

Similarly we have expressed here strains in terms of the stresses. So if I know the deformations I can get the strains from here. Once I know the strains then using these equations I can get the stresses. If I know the strains I can get the stresses from these equations.

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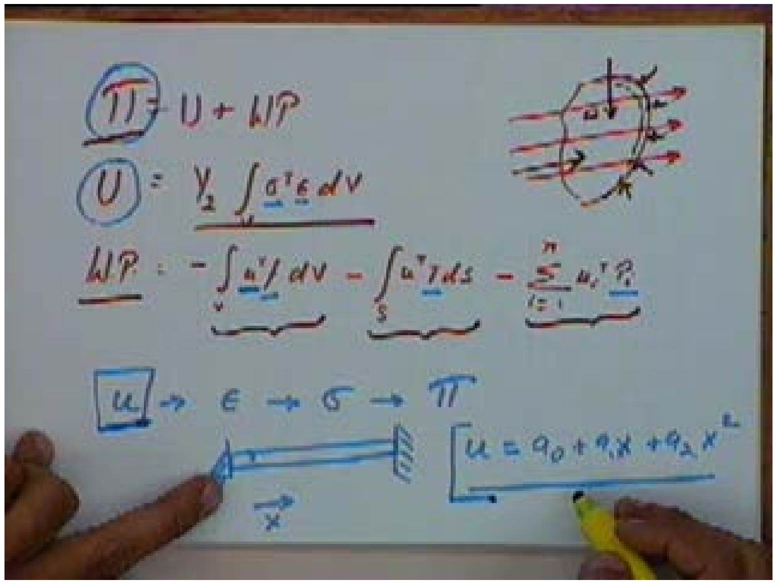

Once I know the stresses then I can get the potential energy from this equation because if I know u which is the deformations. If I know u I can find epsilon, if I know epsilon I can find sigma and I know the loading T and P_i's and the f's. So basically if I know the deformation, I can find out all the other terms in this and thus I can find out the total potential energy. So from u I can get epsilon, from epsilon I can get sigma and once I know epsilon and sigma I can get the total potential energy pi. So in any object to find out the total potential energy, all that I need to know is an expression for the deformation vector u. If I know this deformation vector u, I can compute the total potential energy and in this in a finite element approach, what we basically do is we try to get an expression for this vector u.

For instance if I have a simple one dimensional element or one dimensional object like this maybe I will say that the deformation in this u is given by a_0 plus a_1 x plus a_2 x squared that means I will say that this deformation as a function of x is given by this. Once I know u, I can calculate epsilon, I can calculate sigma and I can calculate the potential energy and then I know that in equilibrium my potential energy will always be a minimum. For stable equilibrium, potential energy is always a minimum. So if I assume a displacement vector like this, if I assume that the deformation at any point is given by this equation, I can compute the potential energy using this expression and this is the basic approach that are used in the finite element method. We have the set of equations, we will use these set of equations to express pi or the total potential energy as a function of u and our aim is to find out u. If you can find out the deformations, we can find out everything about the object. So we will just see how we can find out the deformation in objects.

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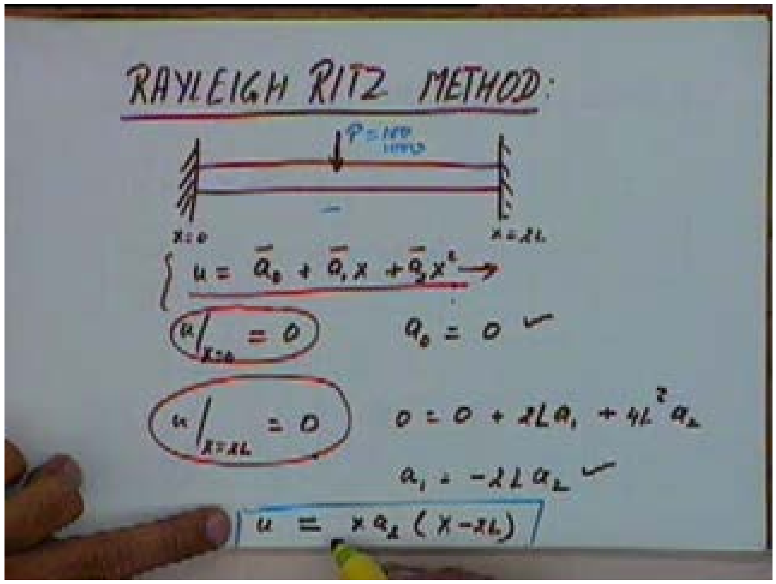

The first method for doing that is what is referred to as the Rayleigh Ritz method. To explain this method let's take the same case. If you have an object like this or a beam like this with a central loading, let us assume that the deformations in the vertical direction are given by this equation because this equation may or may not be correct; this is only an assumption to start with. In fact from your solid mechanics knowledge, you will probably know that this equation is wrong because the deformation in this will not be a quadratic expression like this but right now we are just making an assumption which may or may not be correct that we will come to know later on.

So if you are making this assumption at this point, we have x is equal to 0 and at the end let's say we have x is equal to 2 L where L is the half length of this rod. This beam is fixed at the two ends, so one of their boundary conditions would be that u at x is equal to 0 will be equal to 0 and u at x is equal 2 L will be equal to 0. If you use this expression put it in this, what we will get would be a_0 will be equal to 0. At x equal to 0, these two terms will become 0, u is given to be 0 so a_0 will be equal to 0. Now we put x is equal 2 L and then u is equal to 0 so what we will get will be 0 will be equal to, a_0 is known to be 0 plus 2 L into a_1 plus 4 L squared into a_2. So from this we will get a_1 will be equal to minus 2 L a_2.

So if we put this back over here, we will get an expression for u which will be in terms of a_2 and it will be x into a_2 into unit of x minus 2 L. I have put this and this into this equation. Now in this method so far what we have done is we have assumed the distribution for u and we have assumed this distribution u in terms of the parameters a_0 a_1 and a_2. So in this distribution we have three parameters a_0 a_1 and a_2 and then we have ensured that the boundary conditions are met. These are the boundary conditions; we have ensured that these boundary conditions should be met.

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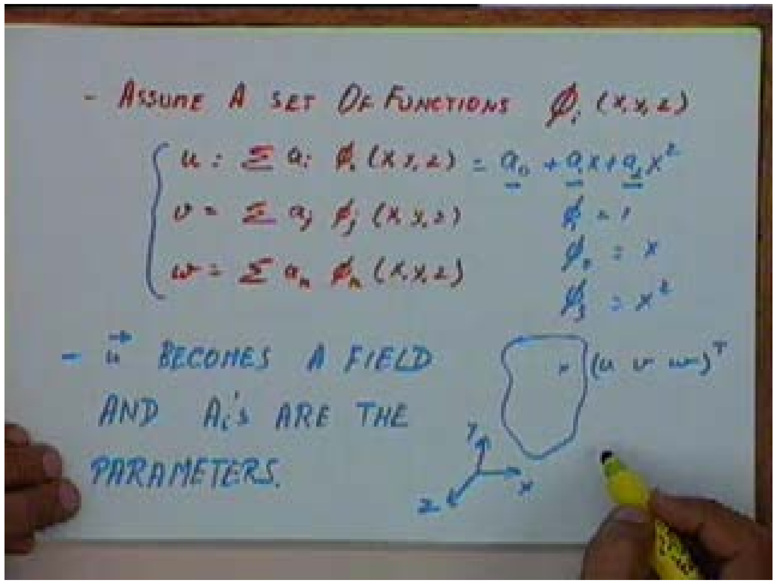

So the first step in the Rayleigh Ritz method is what we say assume a set of functions which are given as let's say phi_i of x y and z such that u is equal to sigma a_i phi_i of x y z, v is equal to sigma let's say a_j of phi_j x y z and w is equal to sigma let's say a_k of phi_k x y z. For instance in this case I have assumed that u is given by this expression, in this expression we have a_0 plus a_1 x plus a_2 x squared which is the same as this. So my phi_i's my phi_1 is 1, phi_2 is x and phi_3 is x square. We will write it here phi_1 is equal 1, phi_2 is equal to x and phi_3 is equal x squared. So my u is given by and expressed by combination of 3 phi_i's which are 1, x and x squared and my constants for these parameter a_i are a_0 a_1 and a_2. So in general I will write down u v w using similar expressions, expressions like this.

I can assume it to be a quadratic function, I can assume it to be a higher order function or I can assume it to be consisting of some other type of function. So I will assume that the deformations would be given as some function of x y and z. Essentially what I am doing is whatever my solid be, throughout the solid I am assuming the deformation field given by this. I am assuming that at any arbitrary point the deformation u v and w, this deformation will be given as a function of x y and z governed by these equations. We are starting with this assumption, as I said this assumption may or may not be correct, we don't know at the moment. The first step in the Rayleigh Ritz method is to start with a set of assumptions like this. And as a result of this, this u vector, this becomes a field. It becomes a field meaning, it is a function of x y and z and this function of x y and z will be governed by these parameters a_0 a_1 and a_2. This has become the field and A_i's are the parameters of this field. So now I have got this deformation vector as a function of x y z and it has some parameters a_0 a_1 and a_2.

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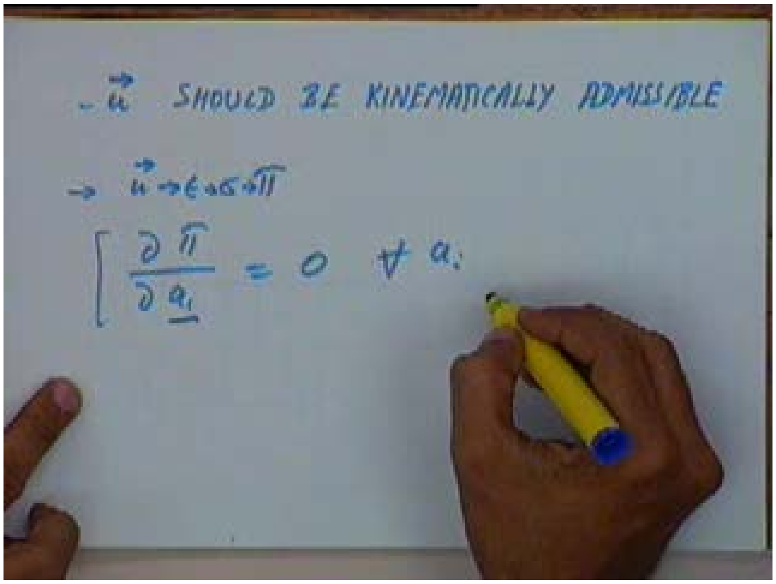

The next thing that we do is that we say that this u bar, for this vector U this should satisfy the boundary conditions or we say that this should be kinematically admissible that way whatever the boundary conditions be those conditions should be met. I said in this example we took this to be the initial field then we said that these are the boundary conditions. We applied these boundary conditions and this field reduced to this expression. So we made an initial assumption about the field and then ensured that the boundary conditions are met or we assumed or we made sure that this deformation is kinematically admissible, kinematically permissible.

Now once I have assumed u as a function of the phi's then I can compute my potential energy because from u I will get epsilon and I will get sigma and then I can compute the potential energy. Once I know the expression for the potential energy, then we will make sure of the fact that under equilibrium, the potential energy should be a minimum which means that del pi by del a_i should be equal to 0 for all a_i because my this field u, this field u is a function of these a_i a_j a_k's or it consists of terms like a_0 a_1 and a_2 but these terms a_0 a_1 and a_2 they should be so adjusted that the potential energy should become a minimum. That is how we say that del pi by del a_i should be 0 for all a_i.

Sir is that necessary that for equilibrium potential energy should be a minimum it can be in a state of unstable equilibrium also. No, we are not concerned with that at the moment because when we were talking of finding out the stresses etc, we will be talking of the case of a stable equilibrium and besides in the Rayleigh Ritz method it can be shown that if we go by this approach and if you have certain set methods for choosing the phi_i's we can get a reasonable solution. If you choose a wrong function over here then obviously we cannot get the correct solution but if you choose this in a systematic manner and so on then we can get the correct solution for that but for equilibrium this condition has to be met.

Essentially what we are doing is we have to find out these values of these a_i's and we know the expression for pi. Since we know pi, in order to find the a_i's we will get a system of equation from this and we will solve those equations simultaneously to get the value of a_i so that is the basic procedure we are going to follow. So this is going to give us a system of equations and this system of equations will be in a_i so from this we will solve for a_i.

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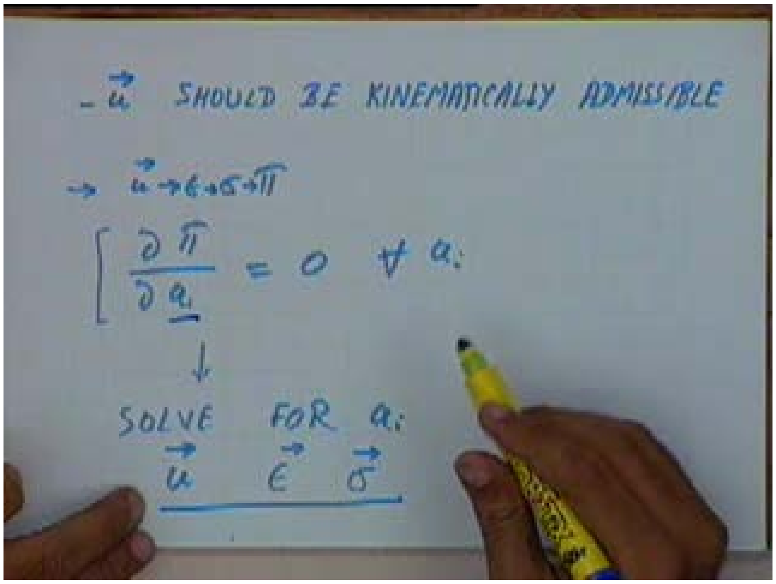

Once we solve for a_i then we can get the u vector, once we know u we can also get epsilon and we can get sigma. So in this way we will be able to find out the stresses and the strains in the body. We have started with an assumption about the deformation field. We assumed that the deformation field is of a particular mathematical nature and if that assumption is valid then we will get an exact solution otherwise we will get an approximate solution and one thing that it should always be kept in mind is that the finite element method will always give the approximate solution. It never gives the exact solution. There will always be a certain amount of error in the finite element approach. Any question up to this point? What is meant by…? The boundary condition should be met? So then already we are calculating a_i's by applying the boundary condition. But all the a_i's will not be found.

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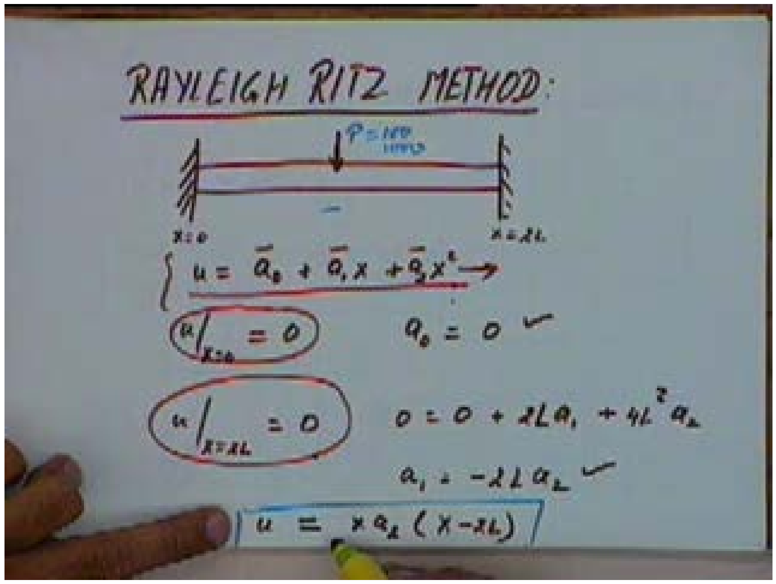

For instance in this case I have applied the boundary condition that at x is equal to zero, u should be zero and at x equal 2 L, u should be 0 but in my earlier assumption there were 3 terms a_0 a_1 and a_2 and if I consider these three terms I cannot solve for all three of them by taking these two boundary conditions. Student: We can apply that the slope in between is 0, in the middle of the slope at x of the values is zero. You can apply that, but is that going to be independent? Nevertheless, right now I have taken a simple equation over here, simple expression for u. This will always be a more complicated thing. So just by applying the boundary conditions you cannot solve the problem because it's quite straight forward irrespective of what the load is boundary condition would be the same.

Even if I applied a load p equal to 100 Newton's or I apply a load p is equal to 1000 Newton's, these boundary conditions are going to remain the same but the deformation will not be the same. So obviously if you are doing it correctly in that case just by applying the boundary condition you shall not be able to solve the u vector, it doesn't make sense. So by applying the boundary conditions you will be able to simplify this expression but we will normally not be able to solve it completely. So we will get an expression like this and in this case it contains one term a_2, since it contains a_2 now we will get the expression for the potential energy in terms of a_2 and then apply this condition to get the value of a_2. We just go through this small example and then maybe it will become clearer.

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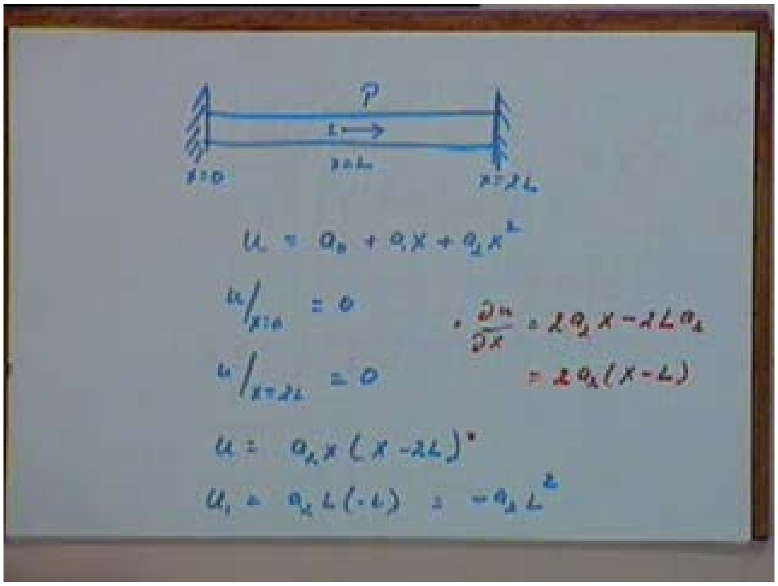

What we will do is we will take the same example except for the fact that we will make it a strictly one dimensional case with the loading also in the same direction. This is x is equal to 0, this is x equal to 2 L and the loading, this loading is P acting at x equal to L, we will make it strictly a one dimensional case. Now if we say u that is deformation in the x direction, again we will assume it to be a_0 plus a_1 x plus a_2 x squared. Again we will say u at x equal to 0 is equal to 0 and u at x equal to 2 L will also be equal to 0. By the same method now we will get u to be, what was the expression we had? a_2 into x into x minus 2 L. Now we have to write an expression for the potential energy of this system.

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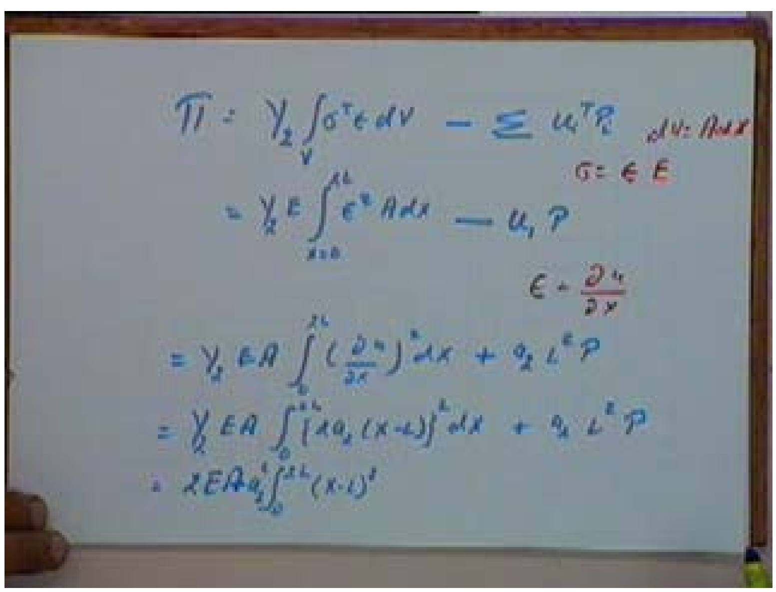

So potential energy pi that is equal to half of integral sigma transpose epsilon dv volume integral of that, this is the strain energy component plus the work potential and the work potential in this case they know body forces, they know tractive forces, we only have a point load. So that will become minus what we had written sigma u_i transpose P_i. Since its again a one dimensional case, sigma will be equal to epsilon times e. In the one dimensional case sigma is equal to epsilon into E. So we will use that over here and we will get half of, E of course which is a constant multiplied by we will get epsilon square then we will say that dv will be equal to A times dx. We will assume this rod to be of uniform cross section, so dv is equal to A times dx so it will become epsilon square into A into dx and so volume integral, so it will be integral from x equal to 0 to x equal to 2 L minus let's say we will call this point as point number one, u_i that means the deformation at this point multiplied by the load at this point, so it's a minus u_1 into P. Is that okay?

Now we will make use of the fact that epsilon is equal to del u by del x, so our expression for pi will become equal to half E into A into integral of del u by del x whole squared dx minus u_1 times P. What is u_1? U_1 is the deformation at this point, so if you will look at this equation u_1 will be equal to a_2 into L into x minus 2 L that is L minus 2 L into minus L, it will become minus a_2 L squared. So it will become, this will become plus a_2 L square times P. del u by del x, again if you look at this from this we will get del u by del x will be equal to, this is a_2 x square will become 2 a_2 x minus this will become 2 L a_2 which will be equal to 2 a_2 into x minus L. Is that okay? Simple differentiation, we will put that back over here.

So we will get, this should be equal to half of EA into integral this is from 0 to 2 L from 0 to 2 L del u by del x is 2 a_2 into x minus L, it will become this whole thing squared dx plus a_2 L squared P. We just simplify this and half EA, so this will become 4, this will become 2 EA integral from 0 to 2 L. It simplifies to it should be E. Again. Anything wrong? No sir. Is that all right? What would be this integral? 4 by 3 EA a square L cube, 4 by 3. What is it? 4 by 3 EA.

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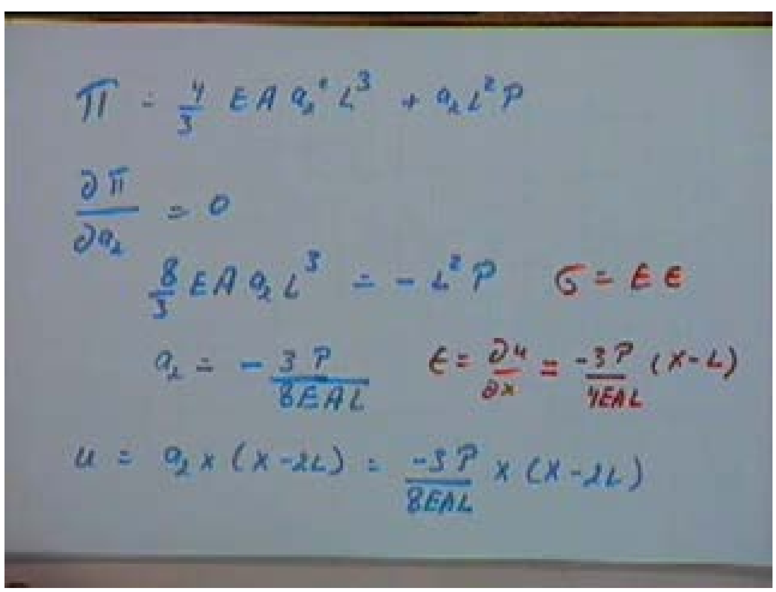

So equal to 4 by 3 EA a_2 squared L cube plus a_2 L squared P. Is that okay? Fine, so this is my expression for potential energy. Now we will say del pi by, the only parameter we have here is a_2. So we say delta pi by del a_2 should be equal to 0 which will give us 8 by 3 EA a_2 L cube will be equal to minus L squared P which means a_2 will be equal to 3 P by 8 EAL minus which means that our deformation u, our deformation u is given by a_2 x into x minus 2 L so this will become minus 3 P by 8 EAL into x into x minus 2 L. So this will be our expression for the deformation and similarly if we find out epsilon, epsilon is equal to del u by del x that will be 2 a_2 into x minus L that will become minus 3 P by 4 EAL into x minus L, so we have an expression for epsilon as well as an expression for u. If we know epsilon, we also know sigma as sigma is equal to E times epsilon. So this is how we can find out all the stresses and strains in the body.

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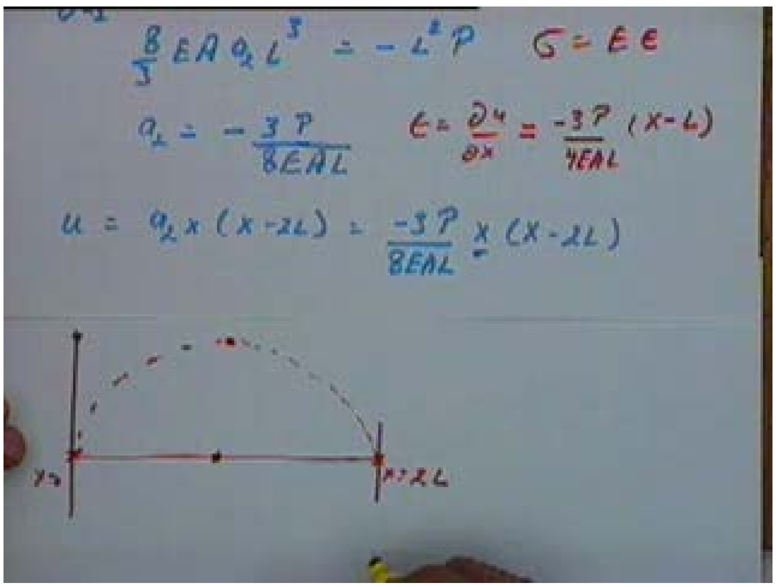

But if you look at this solution, in this solution this deformation u, how do that vary with the distance? This is x equal to 0 and this is x equal to 2 L. This deformation u is equal to 0 at x equal to 0 and is equal to 0 at x equal to 2 L. Again? It is maximum at x is equal to L. It is maximum at x equal to L. So it will be maximum at some value here at x equal to L and it is a quadratic relation. Why should it be zero because that is the part of the boundary condition. Look at this figure.

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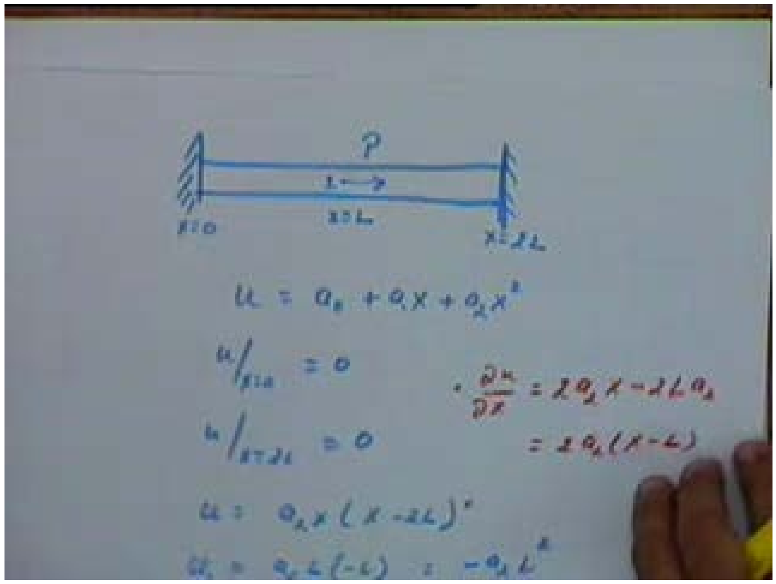

It is fixed here, there is a no deformation here, it is fixed there is no deformation here. That is why we gave the boundary condition as at x equal to 0, u is equal to 0. What was the equation for that? Equation for, this is the equation you have, this equation. Sir how come from this equation you put x equal to 0? You have x over here, this is the x it's all multiplication it's an x nevertheless. So when x equal to 0, deformation becomes 0, at x equal to 2 L this term is 0 so deformation comes here is 0. If I plot the strains at x equal to 0, it is going to be positive. At x equal to 2 L it's going to be negative and at x equal to L it is going to be 0. So the strains would follow a pattern something like this. Now if you look at this equation the strain is a linear term, it is not a quadratic term, strain is linear. So this is how the strain would like?

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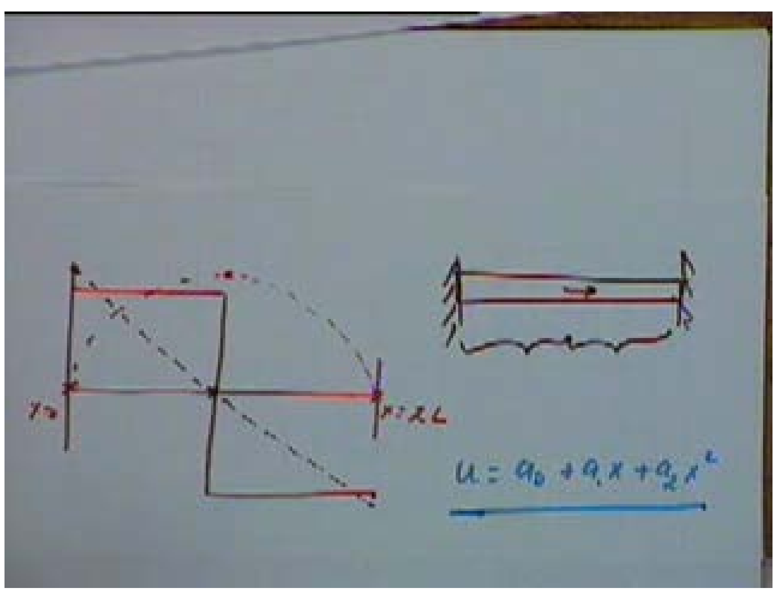

And this is how the deformation would look like but if you look at the actual problem where we have this rod and the force acting here. This whole portion is going to have a uniform tensile strain and this whole portion is going to have a uniform compressive strain, we are not getting that by this method. We are getting a straight linear variation in strain as well as in stress. The actual variation would probably look something like this. This is how the actual strain is going to look but we are getting a strain which is like this that is basically because the assumption that we started with that our deformation u is given by u equal to a_0 plus a_1 x plus a_2 x squared. This assumption is not correct. This assumption that the deformation will be a quadratic function is not a correct assumption because in this case we know that there will be a discontinuity here, so we will probably have a different mathematical description for u in this part and in this part. So since this assumption itself is wrong, we are not getting the correct result and there is some error between the actual and the result that we have found out.

The finite element method will give us a systematic way of choosing an expression for u and then solving out the system of equations. So this Rayleigh Ritz method we are going to use for most of our finite element applications. So we will start of from this point in the next class. If you have anything then we will discuss it next time.